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HEAT KERNEL BOUNDS, POINCARÉ SERIES, AND L^2 SPECTRUM FOR LOCALLY SYMMETRIC SPACES

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Abstract

We derive upper Gaussian bounds for the heat kernel on complete, noncompact locally symmetric spaces $M = \Gamma\backslash X$ with nonpositive curvature. Our bounds contain the Poincaré series of the discrete group Γ and therefore we also provide upper bounds for this series.

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1. Introduction

Let $X = G/K$ denote a Riemannian symmetric space of nonpositive curvature. In a series of papers sharp upper and lower bounds for the heat kernel $K(t, x, y)$ on X were obtained: Davies and Mandouvalos [6] derived sharp upper and lower bounds in the hyperbolic setting, that is, $X = \mathbb{H}^n$. Anker and Ji [1] generalized this result to Riemannian symmetric spaces of nonpositive curvature for those $t > 0$ and $x, y \in X$ such that $1 + t \geq cd(x, y)$ for a constant $c > 0$. Finally, Anker and Ostellari [2, 15] were able to give a proof without this additional assumption.

In this paper, we are concerned with upper Gaussian bounds for the heat kernel on complete locally symmetric spaces $M = \Gamma\backslash X$ where $X = G/K$ denotes a Riemannian symmetric space of nonpositive curvature and $\Gamma \subset G$ a discrete subgroup of G that acts freely by isometries on X. Our methods are inspired by similar results due to Davies and Mandouvalos for hyperbolic manifolds $M = \Gamma\backslash\mathbb{H}^n$ in [6] but, because of nonconstant sectional curvature, the proofs in the more general case of locally symmetric spaces are a little more involved. We also want to emphasize that we make no restrictions concerning the rank of X, that is, the dimension of a maximal flat in X.

This paper is organized as follows. We use the precise heat kernel bounds due to Anker and Ostellari for the heat kernel on X (see Section 2 for their result) and a formula relating the heat kernels on X and $M = \Gamma\backslash X$ in order to derive in Section 4 upper bounds for the heat kernel on M. These upper bounds contain the so-called Poincaré series

$$P(s; x, y) = \sum_{\gamma\in\Gamma} \exp(-sd(x, \gamma y)), \quad x, y \in X, s > 0.$$

Unfortunately, it seems to be rather difficult to determine upper bounds for the function $(x, y) \mapsto P(s; x, y)$. On the other hand, we are able to give upper bounds for $x \mapsto P(s; x, x)$ if s is larger than $2\|\rho\|$ (for a definition of ρ we refer to Section 2); see Section 3. Hence, we use a theorem due to Davies and Mandouvalos and lower bounds for the bottom of the L^2 spectrum of M derived in Section 4 to get upper bounds for the heat kernel on M where the functions $P(s; x, x)$ appear instead of $P(s; x, y)$ and s can be chosen larger than $2\|\rho\|$ (see Section 5).

2. Preliminaries

We recall some basic material about symmetric spaces and state Anker's and Ostellari's result concerning the heat kernel $K(t, x, y)$ on symmetric spaces mentioned in the introduction.

In the following, $X = G/K$ denotes a symmetric space with nonpositive sectional curvature, where G is a noncompact reductive Lie group in Harish-Chandra's class that acts by isometries on X and K is a maximal compact subgroup of G. The respective Lie algebras are denoted by $\mathfrak{g}$ and $\mathfrak{k}$.

Given a corresponding Cartan involution θ we obtain the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$ into the eigenspaces of θ. The subspace $\mathfrak{p}$ of $\mathfrak{g}$ can be identified with the tangent space $T_{eK}X$.

For any maximal Abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ we refer to $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ as the set of all restricted roots for the pair $(\mathfrak{g}, \mathfrak{a})$, that is, Σ contains all $\alpha \in \mathfrak{a}^* \setminus \{0\}$ such that

$$\mathfrak{g}_\alpha := \{Y \in \mathfrak{g} \mid \mathrm{ad}(H)(Y) = \alpha(H)Y \text{ for all } H \in \mathfrak{a}\} \neq \{0\}.$$

These subspaces $\mathfrak{g}_\alpha$ are called root spaces.

Once a positive Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$ is chosen, we denote by Σ^+ the subset of positive roots, and by Σ_0^+ the subset of indivisible positive roots, where a positive root α is called indivisible if $(1/2)\alpha$ is not a root. Furthermore,

$$\rho = \frac{1}{2} \sum_{\alpha\in\Sigma^+} (\dim \mathfrak{g}_\alpha)\alpha$$

is half the sum of the positive roots (counted according to their multiplicity).

Recall the *Cartan decomposition* of G:

$$G = K \exp \overline{\mathfrak{a}^+} K. \tag{1}$$

More precisely, this means that each $g \in G$ can be written as $g = k_1 \exp(H)k_2$ with $k_1, k_2 \in K$, and a *unique* $H \in \overline{\mathfrak{a}^+}$.

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To state Anker's and Ostellari's result, we need a little more preparation. Notice first that G acts by isometries on X and that therefore the Laplace–Beltrami operator is G-invariant and hence, the heat kernel $K(t, x, y)$ is G-invariant, that is, $K(t, gx, gy) = K(t, x, y)$ for all $g \in G$ and $x, y \in X$. If we denote by $x_0 = eK$ the base point of X and if we choose points x, y in the homogeneous space X, there are isometries $g, h \in G$ such that $x = gx_0$ and $y = hx_0$. Because of the Cartan decomposition $G = K \exp \overline{\mathfrak{a}^+} K$ of the Lie group G there are $k, k' \in K$ and $H = H(g^{-1}h) \in \overline{\mathfrak{a}^+}$ with $g^{-1}h = k \exp H(g^{-1}h)k'$. We can therefore write the heat kernel as follows:

$$\begin{aligned} K(t, x, y) &= K(t, gx_0, hx_0) = K(t, x_0, g^{-1}hx_0) \\ &= K(t, x_0, k \exp H(g^{-1}h)k'x_0) = K(t, x_0, \exp Hx_0). \end{aligned}$$

Of course, the isometries g and h are not necessarily uniquely determined. But $H \in \overline{\mathfrak{a}^+}$ is uniquely determined by x and $y \in X$: assuming that the isometries g' and $h' \in G$ satisfy also $x = g'x_0$ and $y = h'x_0$, then clearly $g' = gk_1$ and $h' = hk_2$ with $k_1, k_2 \in K$. On the other hand, this implies $g'^{-1}h' = k_1^{-1}g^{-1}hk_2 = k_1^{-1}k \exp H(g^{-1}h)k'k_2$ and the claim is proven because the $H \in \overline{\mathfrak{a}^+}$ in the Cartan decomposition is unique. For the distance between x and $y \in X$ we obtain by an analogous calculation the formula

$$d(x, y) = d(x_0, \exp Hx_0) = \|H\|.$$

THEOREM 2.1 (see [2, 15]). *For all $H \in \overline{\mathfrak{a}^+}$ and all $t > 0$,*

$$\begin{aligned} K(t, x_0, \exp Hx_0) \asymp t^{-n/2} &\Bigg(\prod_{\alpha \in \Sigma_0^+} (1 + \langle \alpha, H \rangle)(1 + t + \langle \alpha, H \rangle)^{(m_\alpha + m_{2\alpha})/2 - 1} \Bigg) \\ &\times \exp(-\|\rho\|^2 t - \langle \rho, H \rangle - \|H\|^2/4t). \end{aligned}$$

Note that we write $f \asymp h$ for functions f and h if there is a positive constant c such that $(1/c)h \leq f \leq ch$.

REMARK 2.2. If we denote by G_1 the Lie subgroup of G with Lie algebra $\mathfrak{g}_1 := [\mathfrak{g}, \mathfrak{g}]$ and by $K_1 := K \cap G_1$, then K_1 is a maximal compact subgroup of the semisimple Lie group G_1 and the center of G_1 is finite. Furthermore, $X = G/K$ splits as the Riemannian product of the symmetric space $X_1 = G_1/K_1$ of noncompact type and the Euclidean space $\mathfrak{p} \cap Z(\mathfrak{g})$, where $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

3. Poincaré series and the critical exponent

Let us denote by $X = G/K$ a symmetric space of nonpositive sectional curvature and by Γ a discrete subgroup of G that acts freely on X. The resulting locally symmetric space is denoted by $M = \Gamma \backslash X$.

A major role in our estimates is played by the *Poincaré series*

$$P(s; x, y) := \sum_{\gamma \in \Gamma} \exp(-s d(x, \gamma y))$$

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with $s \in (0, \infty)$, $x, y \in X$, and its *critical exponent*

$$\delta(\Gamma) := \inf\{s \in (0, \infty) \mid P(s; x, y) < \infty\}.$$

Since all $\gamma \in \Gamma$ are isometries, an application of the triangle inequality implies that the definition of the critical exponent $\delta(\Gamma)$ does not depend on the choice of the points x and $y \in X$. We further remark that, because of $P(s; \gamma_1 x, \gamma_2 y) = P(s; x, y)$ for all $\gamma_1, \gamma_2 \in \Gamma$, the Poincaré series $P(s; \cdot, \cdot)$ can be considered as a function on $M \times M$. Therefore, the same notation will occasionally be used for the respective function on $M \times M$.

If $N(R; x, y) := \#\{\gamma \in \Gamma \mid d(x, \gamma y) \leq R\}$ denotes the orbit counting function, one can prove the equality

$$\delta(\Gamma) = \limsup_{R \to \infty} \frac{\log N(R; x, y)}{R}; \tag{2}$$

see [14] or [18]. The critical exponent $\delta(\Gamma)$ is therefore a measure of the exponential growth rate of Γ orbits in X.

Before we begin with estimating of the Poincaré series, we give an upper bound for the critical exponent $\delta(\Gamma)$.

LEMMA 3.1. *If ρ denotes (as above) half the sum of the positive roots, then*

$$\delta(\Gamma) \leq 2\|\rho\|.$$

PROOF. We consider the symmetric space $X = X_1 \times \mathbb{R}^m$ with nonpositive sectional curvature where X_1 denotes a symmetric space of noncompact type (see Remark 2.2). We further choose $x, y \in X$ and a ball $B(y, \varepsilon)$ with center y and radius $\varepsilon = \varepsilon(\Gamma) > 0$ such that $B(y, \varepsilon) \cap B(\gamma y, \varepsilon) = \emptyset$ for all $\gamma \neq \mathrm{id}$. It follows that

$$N(R; x, y)\mathrm{vol}\, B(y, \varepsilon) \leq \mathrm{vol}\, B(x, R + \varepsilon).$$

Since

$$\mathrm{vol}\, B(x, R) \asymp R^m R^{((\mathrm{rank}\, X_1)-1)/2} \exp(2\|\rho\| R) \tag{3}$$

(see [11]), we obtain the estimate

$$N(R; x, y) \leq \frac{\mathrm{vol}\, B(x, R+\varepsilon)}{\mathrm{vol}\, B(y, \varepsilon)} \leq C \left(\frac{R+\varepsilon}{\varepsilon}\right)^m \left(\frac{R+\varepsilon}{\varepsilon}\right)^{((\mathrm{rank}\, X_1)-1)/2} \exp(2\|\rho\| R).$$

The claim now follows from formula (2). □

3.1. Estimates of the Poincaré series Since the Poincaré series appears in our heat kernel estimates, we prove in this subsection certain upper bounds for this series.

In the following lemma we denote by $\mathrm{inj}(\widetilde{x})$ the *injectivity radius* of $\widetilde{x} \in M = \Gamma \backslash X$. Recall the formula

$$\mathrm{inj}(\widetilde{x}) = \tfrac{1}{2} \min\{d(x, \gamma x) \mid \gamma \in \Gamma \setminus \{\mathrm{id}\}\},$$

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which is true for all $x \in X$ projecting to $\widetilde{x}$, that is, $\pi(x) = \widetilde{x}$ where $\pi : X \to \Gamma\backslash X$ denotes the covering map. For such points x, we put $\operatorname{inj}(x) := \operatorname{inj}(\widetilde{x})$.

It turns out that, under the assumption $s > 2\|\rho\|$, it is easier to obtain upper bounds for the Poincaré series $P(s; x, x)$.

LEMMA 3.2. *Let* $s > 2\|\rho\|$ *and choose* $0 < 2\varepsilon < s - 2\|\rho\|$. *Then there is a constant* $C = C(s, \varepsilon) > 0$, *such that*

$$1 \le P(s; x, x) \le 1 + C\left(\frac{1}{\operatorname{inj}(x)}\right)^{m+((\operatorname{rank} X_1)-1)/2} \cdot \exp((2\|\rho\| - s + 2\varepsilon)\operatorname{inj}(x)).$$

PROOF. The lower bound is trivial since $\mathrm{id} \in \Gamma$. The upper bound follows essentially from (3). In fact,

$$\begin{aligned} P(s; x, x) \le 1 + \sum_{n=0}^{\infty} &\#\{\gamma \in \Gamma \mid \operatorname{inj}(x) + n \le d(x, \gamma x) \le \operatorname{inj}(x) + n + 1\} \\ &\times \exp((-s(\operatorname{inj}(x) + n))). \end{aligned}$$

Since the open balls $B(\gamma x, \operatorname{inj}(x))$ are pairwise disjoint, we obtain the following estimate for $S_n(x) := \#\{\gamma \in \Gamma \mid \operatorname{inj}(x) + n \le d(x, \gamma x) \le \operatorname{inj}(x) + n + 1\}$:

$$S_n(x) \operatorname{vol} B(x, \operatorname{inj}(x)) \le \operatorname{vol} B(x, 2\operatorname{inj}(x) + n + 1).$$

For all $\varepsilon > 0$, $\operatorname{vol} B(x, R) \le c_\varepsilon \exp((2\|\rho\| + \varepsilon)R)$ with a constant $c_\varepsilon > 0$ that depends on the choice of ε. Using (3), we can conclude that

$$\begin{aligned} S_n(x) &\le \frac{\operatorname{vol} B(x, 2\operatorname{inj}(x) + n + 1)}{\operatorname{vol} B(x, \operatorname{inj}(x))} \\ &\le C_\varepsilon \left(\frac{1}{\operatorname{inj}(x)}\right)^{m+((\operatorname{rank} X_1)-1)/2} \cdot \exp((2\|\rho\| + \varepsilon)(2\operatorname{inj}(x) + n + 1)) \\ &\quad \times \exp(-2\|\rho\|\operatorname{inj}(x)). \end{aligned}$$

This implies the following upper bound for the Poincaré series:

$$\begin{aligned} P(s; x, x) \le 1 + C_\varepsilon &\left(\frac{1}{\operatorname{inj}(x)}\right)^{m+((\operatorname{rank} X_1)-1)/2} \cdot \exp((2\|\rho\| - s + 2\varepsilon)\operatorname{inj}(x)) \\ &\times \exp(2\|\rho\| + \varepsilon) \cdot \sum_{n=0}^{\infty} \exp((2\|\rho\| - s + \varepsilon)n). \end{aligned}$$

Because of our choice of s and ε it follows in particular that $2\|\rho\| - s + \varepsilon < 0$ and that the geometric series $\sum_{n=0}^{\infty} \exp((2\|\rho\| - s + \varepsilon)n)$ equals $(1 - \exp(2\|\rho\| - s + \varepsilon))^{-1}$. The proof is complete. □

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Of course, it would suffice for the proof of the lemma above that ε satisfies the inequality $2\|\rho\| - s + \varepsilon < 0$. But this (weaker) assumption does not guarantee that the term $\exp((2\|\rho\| - s + 2\varepsilon)\mathrm{inj}(x))$ converges (exponentially) to zero as $\mathrm{inj}(x) \to \infty$.

Recall that a Riemannian manifold M is said to have *bounded geometry* if its injectivity radius $\mathrm{inj}(M) := \inf_{x \in M} \mathrm{inj}(x)$ is bounded from below by a strictly positive constant and if its Ricci curvature is bounded from below. The second condition is always fulfilled if M is a locally symmetric space.

COROLLARY 3.3. *Let $M = \Gamma \backslash X$ be a locally symmetric space and choose $s > 2\|\rho\|$.*

(a) *Assume that M has bounded geometry. Then the Poincaré series $P(s; x, x)$ is (for fixed s) bounded from above.*

(b) *If $x_n \in X$ is a sequence with $\mathrm{inj}(x_n) \to \infty$, it follows that $P(s; x_n, x_n) \to 1$.*

For the following estimates of the Poincaré series we choose an arbitrary (but fixed) point $x' \in X$.

LEMMA 3.4. *Choose $s > 2\|\rho\|$ and $x' \in X$. Then the following results hold.*

(a) *There is a positive constant $C = C(x', s)$ (depending only on x' and s) such that*

$$P(s; x, x') \leq C$$

for all $x \in X$.

(b) *There is a positive constant $C = C(x', s)$ (depending only on x' and s) such that*

$$P(s; x, x) \leq C \exp(s d_M(\pi(x), \pi(x')))$$

for all $x \in X$.

PROOF. (a) The proof is similar to the preceding one:

$$\begin{aligned}
P(s; x, x') &\leq \sum_{n=1}^{\infty} \#\{\gamma \in \Gamma \mid n - 1 \leq d(x, \gamma x') \leq n\} \cdot \exp(-s(n-1)) \\
&\leq \sum_{n=1}^{\infty} \frac{\mathrm{vol}\, B(x, n + \mathrm{inj}(x'))}{\mathrm{vol}\, B(x', \mathrm{inj}(x'))} \cdot \exp(-s(n-1)) \\
&\leq C_\varepsilon \left(\frac{1}{\mathrm{inj}(x')} \right)^{m + ((\mathrm{rank}\, X_1) - 1)/2} \cdot \exp(-2\|\rho\| \mathrm{inj}(x')) \\
&\quad \cdot \sum_{n=1}^{\infty} \exp((2\|\rho\| + \varepsilon)(n + \mathrm{inj}(x'))) \cdot \exp(-s(n-1)) \\
&= C_{\varepsilon, x', s} \sum_{n=1}^{\infty} \exp((2\|\rho\| + \varepsilon - s)n).
\end{aligned}$$

If we choose ε sufficiently small, the last series converges and the claim follows.

(b) Using the triangle inequality $d(x, \gamma x) + d(x, \gamma' x') \geq d(\gamma x, \gamma' x')$, we can conclude for all $\gamma' \in \Gamma$ that

$$\begin{aligned} P(s; x, x) &\leq \sum_{\gamma \in \Gamma} \exp(-s d(\gamma x, \gamma' x')) \cdot \exp(s d(x, \gamma' x')) \\ &= P(s; x, \gamma' x') \exp(s d(x, \gamma' x')) \\ &= P(s; x, x') \exp(s d(x, \gamma' x')). \end{aligned}$$

We choose an isometry $\gamma' \in \Gamma$ with the property

$$d(x, \gamma' x') = \min_{\gamma \in \Gamma} d(x, \gamma x') = d_M(\pi(x), \pi(x')).$$

Now part (b) follows immediately from part (a). □

4. Heat kernel bounds and L^2 spectrum

In our estimates of the heat kernel k on the locally symmetric space $M = \Gamma \backslash X$ we use the identity

$$k(t, \pi(x), \pi(y)) = \sum_{\gamma \in \Gamma} K(t, x, \gamma y),$$

where K denotes the heat kernel on X and $\pi : X \to M$ the covering map. This follows from a heat kernel upper bound like

$$K(t, x, y) \leq C t^{-n/2} \exp\left(-\frac{d^2(x, y)}{(4 + \delta)t}\right)$$

which is true for all Cartan–Hadamard manifolds (see [10, Section 7.4]) and well known bounds for the volume growth of balls. If X is a symmetric space of noncompact type, a proof can be found in [4].

We define

$$\rho_{\min} := \min\{\langle \rho, H \rangle : H \in \overline{\mathfrak{a}^+}, \|H\| = 1\} \geq 0.$$

In the following we study the cases $\delta(\Gamma) < \rho_{\min}$ and $\delta(\Gamma) \geq \rho_{\min}$ separately.

THEOREM 4.1. *Assume $\delta(\Gamma) < \rho_{\min}$. Then there is for any $s \in (\delta(\Gamma), \rho_{\min})$ a constant $C = C(s) > 0$ such that for all $t > 0$ and $\widetilde{x}, \widetilde{y} \in M = \Gamma \backslash X$ the estimate*

$$k(t, \widetilde{x}, \widetilde{y}) \leq C t^{-n/2} (1 + t)^m \exp\left(-\|\rho\|^2 t - \frac{d_M^2(\widetilde{x}, \widetilde{y})}{4t}\right) P(s; \widetilde{x}, \widetilde{y})$$

holds. Here, m is defined by $m := \sum_{\alpha \in \Sigma_0^+} ((m_\alpha + m_{2\alpha})/2 - 1)$.

Proof. We use Theorem 2.1 to estimate $k(t, \widetilde{x}, \widetilde{y}) = \sum_{\gamma\in\Gamma} K(t, x, \gamma y)$, where $x, y \in X$ are chosen such that $\pi(x) = \widetilde{x}$ and $\pi(y) = \widetilde{y}$. For this we denote by $H(\gamma)$ the unique element in $\overline{\mathfrak{a}^+}$ with

$$K(t, x, \gamma y) = K(t, x_0, \exp H(\gamma)x_0)$$

and

$$d(x, \gamma y) = \|H(\gamma)\|$$

(see Section 2). First, we obtain

$$\begin{aligned} k(t, \widetilde{x}, \widetilde{y}) &\leq C_1 t^{-n/2}(1+t)^m \\ &\quad\times \sum_{\gamma\in\Gamma}\bigg(\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha, H(\gamma)\rangle)^{(m_\alpha+m_{2\alpha})/2}\bigg) \\ &\quad\times \exp(-\|\rho\|^2 t - \langle\rho, H(\gamma)\rangle - (\|H(\gamma)\|^2/4t)). \end{aligned}$$

Because of $d_M(\widetilde{x}, \widetilde{y}) = \min_{\gamma\in\Gamma} d(x, \gamma y)$ it further follows that

$$\begin{aligned} k(t, \widetilde{x}, \widetilde{y}) &\leq C_1 t^{-n/2}(1+t)^m \exp\bigg(-\|\rho\|^2 t - \frac{d_M^2(\widetilde{x}, \widetilde{y})}{4t}\bigg) \\ &\quad\times \sum_{\gamma\in\Gamma}\bigg(\prod_{\alpha\in\Sigma_0^+}(1+\langle\alpha, H(\gamma)\rangle)^{(m_\alpha+m_{2\alpha})/2}\bigg)\exp(-\langle\rho, H(\gamma)\rangle) \\ &\leq C_1 t^{-n/2}(1+t)^m \exp\bigg(-\|\rho\|^2 t - \frac{d_M^2(\widetilde{x}, \widetilde{y})}{4t}\bigg) \\ &\quad\times \sum_{\gamma\in\Gamma}\bigg(\prod_{\alpha\in\Sigma_0^+}(1+\|\alpha\|\cdot\|H(\gamma)\|)^{(m_\alpha+m_{2\alpha})/2}\bigg)\exp(-\rho_{\min}\|H(\gamma)\|). \end{aligned}$$

Now we take a closer look at the last sum. Since the term

$$\prod_{\alpha\in\Sigma_0^+}(1+\|\alpha\|\cdot\|H(\gamma)\|)^{(m_\alpha+m_{2\alpha})/2}$$

is the square root of a polynomial in $\|H(\gamma)\|$, we can find for every $s \in (\delta(\Gamma), \rho_{\min})$ a constant $C_2 = C_2(s) > 0$, such that

$$\begin{aligned} \prod_{\alpha\in\Sigma_0^+}(1+\|\alpha\|\cdot\|H(\gamma)\|)^{(m_\alpha+m_{2\alpha})/2}\exp(-\rho_{\min}\|H(\gamma)\|) &\leq C_2\exp(-s\|H(\gamma)\|) \\ &= C_2\exp(-sd(x, \gamma y)). \end{aligned}$$

This concludes the proof. □

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The condition $s < \rho_{\min}$ prevents an application of the results from Section 3.1: Using the triangle inequality, we can conclude that $P(s; x, y) \leq P(s; x, x) \exp(sd(x, y))$. But for the proof of the estimates of the Poincaré series $P(s; x, x)$ in the subsection mentioned we made the assumption that $s > 2\|\rho\|$. In Section 5, we therefore give further heat kernel estimates where this problem does not occur.

In the following we give an estimate of the heat kernel on quotients $M = \Gamma\backslash X$ for larger subgroups Γ, that is, $\delta(\Gamma) \geq \rho_{\min}$. The statement of the next theorem is also true for subgroups with $\delta(\Gamma) < \rho_{\min}$ but the estimate is weaker than the one obtained in Theorem 4.1.

THEOREM 4.2. *Assume that $\delta(\Gamma) \geq \rho_{\min}$. Then there is for all $\varepsilon > 0$ a constant $C = C(\varepsilon) > 0$ such that the following estimate for the heat kernel k on M holds:*

$$k(t, \widetilde{x}, \widetilde{y}) \leq Ct^{-n/2}(1+t)^m \exp(-(\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min} + \varepsilon)^2)t) \times P(\delta(\Gamma) + \varepsilon/2; \widetilde{x}, \widetilde{y}),$$

where m is defined as in Theorem 4.1.

PROOF. In order to estimate

$$k(t, \widetilde{x}, \widetilde{y}) = \sum_{\gamma\in\Gamma} K(t, x, \gamma y),$$

we again use Theorem 2.1. First of all, we concentrate on the term

$$\exp\left(-\|\rho\|^2 t - \langle\rho, H\rangle - \frac{\|H\|^2}{4t}\right) \leq \exp\left(-\|\rho\|^2 t - \rho_{\min}\|H\| - \frac{\|H\|^2}{4t}\right).$$

A straightforward calculation shows that for any $\beta \in \mathbb{R}$ the right-hand side of this inequality coincides with the left-hand side of the next inequality:

$$\exp(-(\rho_{\min} + \beta)\|H\| - \|\rho\|^2 t) \exp(-(\|H\|/2\sqrt{t} - \beta\sqrt{t})^2) e^{\beta^2 t} \leq \exp(-(\rho_{\min} + \beta)\|H\| - \|\rho\|^2 t) e^{\beta^2 t}.$$

Choose $\varepsilon > 0$ and define $\beta := \beta(\varepsilon) := \delta(\Gamma) - \rho_{\min} + \varepsilon$.

We obtain the estimate (see the proof of Theorem 4.1)

$$\begin{aligned} k(t, \widetilde{x}, \widetilde{y}) &\leq C_1 t^{-n/2}(1+t)^m \exp(-(\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min} + \varepsilon)^2)t) \cdot \\ &\quad \times \sum_{\gamma\in\Gamma}\left(\prod_{\alpha\in\Sigma_0^+}(1 + \langle\alpha, H(\gamma)\rangle)^{(m_\alpha + m_{2\alpha})/2}\right) \exp(-(\delta(\Gamma) + \varepsilon)\|H(\gamma)\|) \\ &\leq Ct^{-n/2}(1+t)^m \exp(-(\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min} + \varepsilon)^2)t) \\ &\quad \times P(\delta(\Gamma) + \varepsilon/2; x, y). \end{aligned}$$

In the last step we have again used $\langle\alpha, H(\gamma)\rangle \leq \|\alpha\| \cdot \|H(\gamma)\|$ and $\|H(\gamma)\| = d(x, \gamma y)$. □

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REMARK 4.3. Because of $k \to 0$ (if $t \to \infty$) this estimate of the heat kernel k is only of interest if there is an $\varepsilon > 0$ such that $\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min} + \varepsilon)^2$ is positive. But this is equivalent to $\|\rho\| + \rho_{\min} > \delta(\Gamma) \geq \rho_{\min}$.

For symmetric spaces $X = G/K$ of noncompact type with rank 1 the Lie subalgebra $\mathfrak{a}$ has dimension 1 and therefore $\rho_{\min} = \|\rho\|$. The condition from above in this case becomes $\|\rho\| \leq \delta(\Gamma) < 2\|\rho\|$.

The preceding results of this section can be applied in order to give a lower bound for the *bottom of the* L^2 *spectrum*,

$$\lambda_0(M) := \inf\{\lambda \mid \lambda \in \sigma(\Delta_M)\} \geq 0,$$

for locally symmetric spaces $M := \Gamma\backslash X$. The basis for our estimates is the following lemma.

LEMMA 4.4 [3, 13, 17]. *Let M be a Riemannian manifold with Laplace–Beltrami operator Δ_M and heat kernel K. Then, for any pair $(x, y) \in M \times M$,*

$$\lim_{t\to\infty} \frac{1}{t} \log K(t, x, y) = -\lambda_0(M).$$

THEOREM 4.5. *Let $\Gamma\backslash X$ be a noncompact locally symmetric space. Then, the following holds for the bottom $\lambda_0(\Gamma\backslash X)$ of the L^2 spectrum:*

(a) $\lambda_0(\Gamma\backslash X) = \|\rho\|^2$ *if* $\delta(\Gamma) < \rho_{\min}$;
(b) $\|\rho\|^2 \geq \lambda_0(\Gamma\backslash X) \geq \|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2$ *if* $\rho_{\min} \leq \delta(\Gamma) \leq \|\rho\| + \rho_{\min}$.

PROOF. The upper bound $\|\rho\|^2$ for $\lambda_0(\Gamma\backslash X)$ follows from Lemma 4.4, Theorem 2.1, and the fact that

$$k(t, \pi(x), \pi(y)) \geq K(t, x, y).$$

For the lower bounds we apply Lemma 4.4 and Theorems 4.1 or 4.2: The function $h : [1, \infty) \to \mathbb{R}$, $t \to t^{-n/2}(1 + t)^m$ is monotone decreasing since $m < n/2$. Therefore, we obtain in the first case the estimate

$$k(t, \widetilde{x}, \widetilde{x}) \leq b(\widetilde{x}) \exp(-\|\rho\|^2 t)$$

for all $t \geq 1$ with a positive function b on $\Gamma\backslash X$. In the second case, analogous considerations lead for any $\epsilon > 0$ to

$$k(t, \widetilde{x}, \widetilde{x}) \leq b_\varepsilon(\widetilde{x}) \exp(-(\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min} + \varepsilon)^2)t)$$

for all $t \geq 1$. □

In the case of $\delta(\Gamma) > \|\rho\| + \rho_{\min}$ the term $\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2$ is negative. Thus, the lower bound for $\lambda_0(\Gamma\backslash X)$ is still zero in this case.

The bounds for the bottom of the L^2 spectrum from above generalize numerous former achievements: If Γ is a Fuchsian group, the results can be found in [7–9, 16]. For hyperbolic spaces $X = \mathbb{H}^n$ with $n \geq 3$ these results are contained in [19]. Corlette proved these results for rank-1 symmetric spaces X of noncompact type (see [4]).

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A generalization to symmetric spaces of noncompact type with arbitrary rank is due to Leuzinger (see [12]).

5. Gaussian bounds

In this section we wish to apply a theorem due to Davies and Mandouvalos in order to obtain Gaussian bounds for the heat kernel on locally symmetric spaces. These bounds will be suitable for applying the estimates of the Poincaré series from Section 3.1.

THEOREM 5.1 (Davies and Mandouvalos; see [5]). *Let M denote a noncompact Riemannian manifold with dimension $n \geq 3$. We further denote by $\sigma : M \to (0, \infty)$ a C^∞ function and by $F \in \mathbb{R}$ a constant such that*

$$-\frac{\Delta_M \sigma}{\sigma} \geq F.$$

Assume that for all $t \in (0, 1]$ and $x \in M$ the following (on-diagonal) estimate for the heat kernel K on M holds:

$$K(t, x, x) \leq C t^{-n/2} \sigma^2(x).$$

Then for all $\mu \in (0, 1)$, $t > 0$, and $x, y \in M$ the Gaussian estimate

$$K(t, x, y) \leq C_\mu t^{-n/2} \sigma(x)\sigma(y) \exp\left((2\mu - \lambda_0(M))t - \frac{d^2(x, y)}{4(1+\mu)t}\right)$$

holds.

We begin with the definition of a function σ on our symmetric space $X = G/K$ which descends to a suitable function on the quotient space $M = \Gamma\backslash X$ for a discrete subgroup $\Gamma \subset G$ that acts freely on X.

DEFINITION 5.2. Choose a nonnegative function $f \in C_c^\infty([0, \infty))$ with $f(0) \neq 0$ and put $h : X \times X \to [0, \infty)$, $(x, y) \mapsto f(d^2(x, y))$. For $s > \delta(\Gamma)$ we define

$$\sigma : X \to (0, \infty), \ x \mapsto \int_X h(x, y)\sqrt{P(s; y, y)}\, d\mathrm{vol}(y).$$

To show that this function has the properties we need (in view of Theorem 5.1) we prove the next lemma.

LEMMA 5.3. *The function σ is differentiable, Γ-invariant, and therefore defines a function on the quotient space $\Gamma\backslash X$. Furthermore:*

(a) *there is a constant $c > 1$ such that*

$$\frac{1}{c}\sqrt{P(s; x, x)} \leq \sigma(x) \leq c\sqrt{P(s; x, x)};$$

(b) *there is a constant F with*

$$|\Delta_X \sigma(x)| \leq F\sigma(x).$$

In particular, $-\Delta_X \sigma/\sigma \geq -F$.

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PROOF. The differentiability of σ and the Γ-invariance are evident. For the proof of the remaining assertions we first remark that the triangle inequality implies that

$$P(s;\,y,\,y)\le \exp(2sd(x,\,y))P(s;\,x,\,x).$$

(a) The definition of the function σ implies the existence of a constant $\beta>0$, such that $h(x,\,y)=0$ for all points $x,\,y\in X$ with $d(x,\,y)>\beta$. We therefore obtain

$$\begin{aligned}\sigma(x)&=\int_{d(x,y)\le\beta} h(x,\,y)\sqrt{P(s;\,y,\,y)}\,d\mathrm{vol}(y)\\&\le\int_{d(x,y)\le\beta} h(x,\,y)\exp(sd(x,\,y))\sqrt{P(s;\,x,\,x)}\,d\mathrm{vol}(y)\\&\le(\max h)e^{s\beta}\sqrt{P(s;\,x,\,x)}\mathrm{vol}\,B(x,\,\beta)\le c_1(s,\,h)\sqrt{P(s;\,x,\,x)},\end{aligned}$$

with a constant $c_1>0$ that depends only on s and the function h. Notice that in the last step we have used the fact that the volume of a ball $B(x,\,\beta)$ in X is smaller than the volume of a corresponding ball in some hyperbolic space $\mathbb{H}^n$ of constant curvature.

We choose $0<a<\infty$ with $f(\tau)>0$ for all $\tau\in[0,\,a^2]$. It follows that

$$\begin{aligned}\sigma(x)&\ge\int_{d(x,y)\le a} h(x,\,y)\exp(-sd(x,\,y))\sqrt{P(s;\,x,\,x)}\,d\mathrm{vol}(y)\\&\ge \mathrm{e}^{-sa}\sqrt{P(s;\,x,\,x)}\int_{d(x,y)\le a} h(x,\,y)\,d\mathrm{vol}(y)\\&\ge \mathrm{e}^{-sa}\sqrt{P(s;\,x,\,x)}\min_{\tau\in[0,a^2]} f(\tau)\int_{d(x,y)\le a} d\mathrm{vol}(y)\\&\ge c_2\sqrt{P(s;\,x,\,x)},\end{aligned}$$

with a positive constant c_2. In the last step we have again applied a volume comparison theorem in order to find a positive lower bound of the integral. More precisely, we compare the volume of the ball $B(x,\,a)\subset X$ with the volume of a Euclidean comparison ball.

(b) Using (a), we obtain

$$\begin{aligned}|\Delta_X\sigma(x)|&=\left|\int_{d(x,y)\le\beta}(\Delta_X h)(x,\,y)\sqrt{P(s;\,y,\,y)}\,d\mathrm{vol}(y)\right|\\&\le \mathrm{e}^{s\beta}\sqrt{P(s;\,x,\,x)}\max(|\Delta_X h|)=c_3\sqrt{P(s;\,x,\,x)}\\&\le F\sigma(x),\end{aligned}$$

and in particular $F\ge|\Delta_X\sigma(x)|/\sigma(x)\ge\Delta_X\sigma(x)/\sigma(x)$. □

This yields the following result.

COROLLARY 5.4. *Let* $\dim X\ge 3$ *and* $\mu\in(0,\,1)$. *Then we obtain the following upper bounds for the heat kernel* k *on* $M=\Gamma\backslash X$.

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(a) *If* $\delta(\Gamma) < \rho_{\min}$ *and* $s > \delta(\Gamma)$,

$$k(t, \widetilde{x}, \widetilde{y}) \le C_\mu t^{-n/2} \exp\Big((2\mu - \|\rho\|^2)t - \frac{d_M^2(\widetilde{x}, \widetilde{y})}{4(1+\mu)t}\Big) \times \sqrt{P(s; \widetilde{x}, \widetilde{x})}\sqrt{P(s; \widetilde{y}, \widetilde{y})}.$$

(b) *If* $\delta(\Gamma) \ge \rho_{\min}$ *and* $\varepsilon > 0$,

$$k(t, \widetilde{x}, \widetilde{y}) \le C_{\varepsilon,\mu} t^{-n/2} \exp\Big((2\mu - \lambda_0(M))t - \frac{d_M^2(\widetilde{x}, \widetilde{y})}{4(1+\mu)t}\Big) \times \sqrt{P(\delta(\Gamma)+\varepsilon; \widetilde{x}, \widetilde{x})}\sqrt{P(\delta(\Gamma)+\varepsilon; \widetilde{y}, \widetilde{y})}.$$

PROOF. The assertions follow from Theorem 5.1 and Lemma 5.3 since the results in Section 4 imply in both cases a heat kernel estimate of the form

$$k(t, \widetilde{x}, \widetilde{x}) \le C t^{-n/2} \sigma^2(x), \quad t \in (0, 1],$$

with some $x \in X$ such that $\pi(x) = \widetilde{x}$.

We provide some details for case (b). Using Theorem 4.2 and Lemma 5.3, we conclude for $t \in (0, 1]$ that

$$k(t, \widetilde{x}, \widetilde{x}) \le C_\varepsilon t^{-n/2} P(\delta(\Gamma) + \varepsilon/2; \widetilde{x}, \widetilde{x}) \le C'_\varepsilon t^{-n/2} \sigma^2(x),$$

where we put $s := \delta(\Gamma) + \varepsilon/2$ in the definition of σ. The claim follows since $\sigma^2(x) \le c P(\delta(\Gamma) + \varepsilon/2; x, x)$. □

REMARK 5.5. These bounds contain the functions $P(s; x, x)$ instead of $P(s; x, y)$ and s can be chosen as large as one wishes. Since we have estimated the functions $P(s; x, x)$ in Section 3.1 for $s > 2\|\rho\|$, we now have 'complete' upper bounds for the heat kernels.

Using the estimate $\lambda_0(M) \ge \|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2$ (see Theorem 4.5), we obtain the following result.

COROLLARY 5.6. *Let* $\rho_{\min} \le \delta(\Gamma) < \rho_{\min} + \|\rho\|$. *Then for all* $\varepsilon > 0$ *and* $\mu \in (0, 1)$ *there is a constant* $C_{\varepsilon,\mu}$ *such that*

$$k(t, \widetilde{x}, \widetilde{y}) \le C_{\varepsilon,\mu} t^{-n/2} \exp\Big((2\mu - (\|\rho\|^2 - (\delta(\Gamma) - \rho_{\min})^2))t - \frac{d_M^2(\widetilde{x}, \widetilde{y})}{4(1+\mu)t}\Big) \times \sqrt{P(\delta(\Gamma)+\varepsilon; \widetilde{x}, \widetilde{x})}\sqrt{P(\delta(\Gamma)+\varepsilon; \widetilde{y}, \widetilde{y})}.$$

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