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# GLOBAL EXISTENCE FOR THE GENERALISED 2D GINZBURG-LANDAU EQUATION

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**Abstract**

Ginzburg-Landau type complex partial differential equations are simplified mathematical models for various pattern formation systems in mechanics, physics and chemistry. Most work so far has concentrated on Ginzburg-Landau type equations with one spatial variable (1D). In this paper, the authors study a complex generalised Ginzburg-Landau equation with two spatial variables (2D) and fifth-order and cubic terms containing derivatives. Based on detail analysis, sufficient conditions for the existence and uniqueness of global solutions are obtained.

## 1. Introduction

There have been many discussions on Ginzburg-Landau type equations (GLEs) and generalised Ginzburg-Landau type equations (GGLEs).

Ghidaglia and Heron [16] and Doering *et al.* [6] studied the finite-dimensional global attractor and related dynamical issues for the following 1D or 2D (namely, with one or two spatial variables) GLE:

$$u_t - (1 + i\nu)\Delta u + (1 + i\mu)|u|^2 u - au = 0,$$

where $i = \sqrt{-1}$, $a > 0$, $\nu$ and $\mu$ are given real numbers. Levermore and Oliver [21] studied the 1D GLE as a model problem. In [3], Bu considered the global existence of the Cauchy problem of the following 2D GLE:

$$u_t - (\nu + i\alpha)\Delta u + (\mu + i\beta)|u|^{2q} u + \gamma u = 0, \tag{1.1}$$

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with $q = 1$ or 2 and $\alpha\beta > 0$ or $|\beta| \le \sqrt{5}/2$. Bartuccelli *et al.* [1, 2] and Doering *et al.* [7] investigated turbulence, weak or strong solutions and length scales for (1.1) in higher dimensions. Mielke and Schneider [23–25] studied sharper results for (1.1) on bounded and unbounded domains.

For the 1D generalised Ginzburg-Landau equation, derived by Doelman [5],

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x - \alpha_5 |u|^4 u, \tag{1.2}$$

where $\alpha_0 > 0$, $\alpha_j = a_j + ib_j$, $j = 1, \dots, 5$, $a_1 > 0$, $a_5 > 0$, Duan *et al.* and Gao *et al.* ([9–13, 17]) studied the global existence of solutions, the finite-dimensional global attractor, Gevery regularity of solutions, the exponential attractor, the number of determining nodes and inertial forms. In [8], Duan and Holmes obtained global existence for the Cauchy problem of (1.2) under the condition of $4a_1a_5 > (b_3 - b_4)^2$ for the global existence of the initial value problem. Guo and Wang [18] considered the special 2D generalised Ginzburg-Landau equation

$$u_t = \rho u + (1 + i\nu)\Delta u - (1 + i\mu)|u|^{2\sigma} u + \alpha \boldsymbol{\lambda}_1 \nabla(|u|^2 u) + \beta(\boldsymbol{\lambda}_2 \nabla u)|u|^2, \tag{1.3}$$

where $\rho > 0$, $\alpha$, $\beta$, $\nu$, $\mu$ are real numbers, and $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are real constant vectors. They studied the existence of a finite-dimensional global attractor of (1.3) with periodic boundary conditions, assuming that there exists a positive number $\delta > 0$ so that

$$\frac{1}{\sqrt{1 + ((\mu - \nu\delta^2)/(1 + \delta^2))^2} - 1} \ge \sigma \ge 3.$$

In all the above mentioned papers, appropriate boundary conditions are assumed for the existence of finite-dimensional global attractors, determining nodes and inertial forms.

Since the 2D generalised Ginzburg-Landau equation

$$u_t = \alpha_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u_x + \alpha_3 |u|^2 u_y + \alpha_4 u^2 \bar{u}_x + \alpha_5 u^2 \bar{u}_y - \alpha_6 |u|^{2\sigma} u, \tag{1.4}$$

is more closed than the equation which Doleman [5] derived in two dimensions (that is, $\sigma = 2$), and can be regarded as a perturbation of the nonlinear derivative Schrödinger equation, it is worth considering for the case when $\alpha_0 > 0$, $\alpha_j = a_j + ib_j$, $j = 1, \dots, 6$, $a_1 > 0$, $a_6 > 0$, $\sigma > 0$. In [14], Gao and Duan considered the initial value problem for the more general 2D generalised Ginzburg-Landau equation (see [5]) with the suitable initial condition

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \mathbb{R}^2.$$

Here, for simplicity, we omit the term $|u|^2 u$ but the proof is essentially the same.

The main result of [14] is the existence and uniqueness of the global (in time) solution for the Cauchy form of the above problem with initial condition belonging to $H^2(\mathbb{R}^2)$, under the following assumptions on $\sigma$:

**A1** If $b_1 b_6 > 0$, then $\sigma \geq \left(1+\sqrt{10}\right)/2$; if $b_6 = 0$ or $b_1 b_6 < 0$, then there exists a positive number $\delta > 0$ such that

$$\frac{1}{\sqrt{1+(b_1\delta - b_6)^2/(a_1\delta + a_6)^2} - 1} \geq \sigma \geq \frac{1+\sqrt{10}}{2}.$$

In [15], Gao *et al.* obtained the existence and uniqueness of the global (in time) solution and attractor for (1.4) with initial value belonging to $H^2_{\mathrm{per}}(\Omega)$ under the following condition:

**A2** If $b_1 b_6 > 0$, then $\sigma \geq 2$; if $b_6 = 0$ or $b_1 b_6 < 0$, then there exists a positive number $\delta > 0$ such that

$$\frac{1}{\sqrt{1+(b_1\delta - b_6)^2/(a_1\delta + a_6)^2} - 1} > \sigma > 2.$$

REMARK. Here and after, the positive number $\delta$ is a variant constant which depends on $\mu$ and $\nu$ in different regions. It can be seen in the proof.

The initial value is

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \tag{1.5}$$

where $\Omega = (0, L_1) \times (0, L_2)$, $u$ is spatially periodic.

In the present paper, we consider a periodic problem for (1.4) with $\sigma = 2$, that is,

$$u_t = \alpha_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u_x + \alpha_3 |u|^2 u_y + \alpha_4 u^2 \bar{u}_x + \alpha_5 u^2 \bar{u}_y - \alpha_6 |u|^4 u. \tag{1.6}$$

In [4], Bu *et al.* obtained global existence under the condition

**A3** $(|b_4 - b_2| + |b_5 - b_3|)^2 < 4a_1 a_6$, $a_1 > 0$, $a_6 > 0$,

$$\sum_{j=2}^{j=5} |\alpha_j|^2 < a_1 \left(\frac{3}{2} - \sqrt{1+(b_6/a_6)^2}\right) \quad \text{and} \quad |b_6/a_6| < \sqrt{5}/2.$$

Meantime, a numerical example for the blow-up phenomenon of the solution was given.

Using scaling for space variables and the function $u$, we may normalise $a_1$, $a_6$ in (1.6) to 1. Therefore (1.4) can be written as

$$\begin{aligned} u_t = {} & \alpha_0 u + (1+i\nu)\Delta u + \alpha_1 |u|^2 u_x + \alpha_2 |u|^2 u_y + \alpha_3 u^2 \bar{u}_x \\ & + \alpha_4 u^2 \bar{u}_y - (1+i\mu)|u|^4 u. \end{aligned} \tag{1.7}$$

We will give a condition sharper than **A3** to guarantee the global existence of (1.7) with a periodic boundary condition and initial condition.

For the local existence of (1.7) with (1.5) and periodic boundary condition in $H^2(\Omega)$, we refer to [4] and [14].

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## 2. Global existence

In order to show that the solution exists for all $t > 0$, we only need some conditions such that

$$\|u\|_{H^2} < \infty, \quad \text{for all} \quad t > 0. \tag{2.1}$$

This can be archived by the following *a priori* estimates, that is, $\|u(t)\|_{H^2} < K(T, \|u_0\|)$, $t \in [0, T]$, $T > 0$. This means $\|u(t)\|_{H^2}$ cannot go to infinity at any finite time. In the following, $\int \equiv \int_\Omega dx\, dy$, $\|\cdot\|_B$ denotes the norm in a Banach space $B$ and $\|\cdot\|_p$ denotes $\|\cdot\|_{L^p}$. In order to establish (2.1), we derive *a priori* estimates for the solution of (1.7) (with (1.5) and periodic boundary condition) in the following lemmas.

LEMMA 2.1. *Assume that* $u_0 \in L^2_{\text{per}}(\Omega)$ *and* $(|b_1 - b_3| + |b_2 - b_4|)^2 < 4$, *then for the solution* $u(t)$ *of problem* (1.7) *with* (1.5) *and a periodic boundary condition, we have* $\|u\|_2^2 \le K_1(T; \|u_0\|_2)$, $\int_0^t \|\nabla u\|_2^2 \le K_2(T; \|u_0\|_2)$ *and* $\int_0^t \|u\|_6^6 \le K_3(T; \|u_0\|_2)$, $\forall\, t \in [0, T]$, *where* $K_i$ $(i = 1, 2, 3)$ *are constants which depend only on* $T$, $\|u_0\|_2$ *and* $\alpha_0$, $b_j$ $(j = 1, 2, 3, 4)$.

The proof is similar to that in [4] and [14]; we omit it here.

LEMMA 2.2. *Under the assumptions of Lemma* 2.1, $u_0 \in H^1_{\text{per}}$ *and one of the following conditions holds.*

(I) $|\nu| < \sqrt{5}/2$, $\mu$ *arbitrary and* $M^2 < \left(3/2 - \sqrt{1+\nu^2}\right)$ (*here* $M = \max\{|\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4|\}$).

(II) $|\nu| = \sqrt{5}/2$, $\mu$ *arbitrary and* $M^2 < (3/2 - \sqrt{1+\alpha^2})$, $\alpha = -\mu/(1+\delta+A) + \delta\nu/(1+\delta+A)$, $(A = 2/\sqrt{5})$.

(III) $\mu\nu > 0$, $|\nu| > \sqrt{5}/2$, $\mu$ *arbitrary and* $M^2 < 1/2$.

(IV) $\mu\nu < 0$, $|\mu| < \sqrt{5}/2$, $\nu > \sqrt{5}/2$ *and* $M^2 < (3/2 - \sqrt{1+\mu^2})$.

(V) $\mu\nu < 0$, $|\mu| \ge \sqrt{5}/2$, $|\nu| > \sqrt{5}/2$, $|\alpha| < \sqrt{5}/2$, $-(1+\mu\nu) \le |\alpha||\mu - \nu|$ *and*

$$M^2 < \frac{1}{1+\alpha^2}\left(1 + \frac{|\alpha(\delta\nu - \mu)|}{1+\delta}\right)\left(\frac{3}{2} - \sqrt{1+\alpha^2}\right).$$

*Then the solution* $u(t)$ *of problem* (1.7) *with* (1.5) *and periodic boundary condition is* $\|\nabla u\|_2 \le K_4(T, \|u_0\|_{H^1})$, *for* $0 \le t \le T$.

PROOF. In order to estimate the boundedness of $H^1$-norm, we need the functional $E_\delta(u(t)) = \int \left[|\nabla u(t)|^2/2 + \delta|u(t)|^6/6\right]$, where $\delta > 0$ will be suitably chosen later.

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Using (1.7), we get

$$\frac{d}{dt}E_\delta(u(t)) = \alpha_0(\|\nabla u\|_2^2 + \delta\|u\|_6^6) - (\|\Delta u\|^2 + \delta\|u\|_{10}^{10}) + \operatorname{Re}\int(1+i\mu)|u|^4 u\Delta\bar{u} + \delta\int(1+i\nu)|u|^4\bar{u}\Delta u + I_1 + I_2, \quad (2.2)$$

where

$$I_1 = \operatorname{Re}\int\left[\alpha_1|u|^2 u_x + \alpha_2|u|^2 u_y + \alpha_3 u^2\bar{u}_x + \alpha_4 u^2\bar{u}_y\right]\Delta\bar{u},$$

$$I_2 = \operatorname{Re}\int\left[\alpha_1|u|^2 u_x + \alpha_2|u|^2 u_y + \alpha_3 u^2\bar{u}_x + \alpha_4 u^2\bar{u}_y\right]|u|^4\bar{u}.$$

Since $|I_1| \le 2M\int\left[|u|^2|u_x| + |u|^2|u_y|\right]|\Delta u|$ and $|I_2| \le 2\delta M\int|u|^7\left[|u_x| + |u_y|\right]$, then (2.2) can be written as

$$\begin{aligned}\frac{d}{dt}E_\delta(u(t)) \le{}& \alpha_0\left(\|\nabla u\|_2^2 + \delta\|u\|_6^6\right) - \left(\|\Delta u\|^2 + \delta\|u\|_{10}^{10}\right)\\ &+ \frac{1}{2}\operatorname{Re}\int(|u|^4 u, \Delta u)N_0(|u|^4\bar{u}, \Delta\bar{u})'\\ &+ 2M\int\left[|u|^2|u_x| + |u|^2|u_y|\right]|\Delta u| + 2\delta M\int|u|^7\left[|u_x| + |u_y|\right], \quad (2.3)\end{aligned}$$

where $(|u|^4\bar{u}, \Delta\bar{u})'$ denotes the tranpose of $(|u|^4\bar{u}, \Delta\bar{u})$ and

$$N_0 = \bar{N}_0' = \begin{pmatrix} 0 & 1+\delta-i(\nu\delta-\mu) \\ \star & 0 \end{pmatrix}.$$

Here $\star$ denotes the complex conjugate of $1+\delta-i(\nu\delta-\mu)$. Since any $\alpha$ satisfies $|\alpha| < \sqrt{5}/2$, by the standard method (see [15, 23, 24]), we have

$$-\operatorname{Re}\int(1+i\alpha)|u|^4\bar{u}\Delta u \ge \left(3 - 2\sqrt{1+\alpha^2}\right)\int|u|^4|\nabla u|^2. \quad (2.4)$$

Multiplying (2.4) by $-\eta$ ($\eta > 0$ to be chosen later) and adding it to (2.3), we obtain

$$\begin{aligned}\frac{d}{dt}E_\delta(u(t)) \le{}& \alpha_0(\|\nabla u\|_2^2 + \delta\|u\|_6^6) - (1-\kappa)(\|\Delta u\|^2 + \delta\|u\|_{10}^{10})\\ &- \eta\left(3 - 2\sqrt{1+\alpha^2}\right)\int|u|^4|\nabla u|^2\\ &+ \frac{1}{2}\operatorname{Re}\int\left(|u|^4 u, \Delta u\right)N\left(|u|^4\bar{u}, \Delta\bar{u}\right)'\\ &+ 2M\int\left[|u|^2|u_x| + |u|^2|u_y|\right]|\Delta u|\\ &+ 2\delta M\int|u|^7\left[|u_x| + |u_y|\right], \quad (2.5)\end{aligned}$$

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where $0 < \kappa < 1$ is to be determined and

$$N = \bar{N}' = \begin{pmatrix} -2\delta\kappa & 1+\delta-\eta-i(\nu\delta-\mu-\alpha\eta) \\ \star & -2\kappa \end{pmatrix}.$$

Here $\star$ denotes the complex conjugate of $1+\delta-\eta-i(\nu\delta-\mu-\alpha\eta)$. In order to prove $N$ is negative semidefinite, we need the following lemma.

LEMMA 2.3. *Let $\kappa \in (0,1)$ be fixed. Then there exists $\delta > 0$, $\eta > 0$, $|\alpha| < \sqrt{5}/2$ and*

$$-(1+\mu\nu) < |\alpha||\mu-\nu|, \tag{2.6}$$

*such that $N$ is negative semidefinite. Hence,* $\operatorname{Re}\int \left(|u|^4 u, \Delta u\right) N \left(|u|^4 \bar{u}, \Delta \bar{u}\right)' \leq 0.$

PROOF. We use the method of [23] and [24] to prove this lemma. We give the complete proof here in order to determine parameters $\delta$, $\eta$ and $\alpha$ in detail (these are a little different from [23] and [24]).

It is clear that $N$ is negative semidefinite if and only if

$$(1+\delta-\eta)^2 + (\delta\nu-\mu-\alpha\eta)^2 \leq 4\delta\kappa^2, \tag{2.7}$$

which means, $\operatorname{dist}(z(\delta), \Sigma) \leq 2\sqrt{\delta}\kappa$, where $z(\delta) = 1 + i\mu + \delta(1-i\nu)$ and $\Sigma = \left\{\eta(1+i\alpha) \mid \eta \geq 0, |\alpha| < \sqrt{5}/2\right\}$.

We prove (2.7) by five cases.

Case (I). When $|\nu| < \sqrt{5}/2$, it suffices to take $\eta = 1+\delta$, $\alpha = \nu$, $0 < \kappa < 1$ and $\delta$ large enough; then (2.7) is satisfied.

Case (II). When $|\nu| = \sqrt{5}/2$, $\mu$ arbitrary, it suffices to take $\eta = 1+\delta$, $\alpha = -\mu/(1+\delta+A) + \delta\nu/(1+\delta+A)$, $(A = 2/\sqrt{5})$, $0 < \kappa < 1$ and $\delta$ large enough; then (2.7) is satisfied.

Case (III). When $\mu\nu > 0$, $|\nu| > \sqrt{5}/2$, $\mu$ arbitrary, it suffice to take $\eta = 1+\delta$, $\alpha = 0$, $0 < \kappa < 1$ and $\delta = \mu/\nu$; then (2.7) is satisfied.

Case (IV). When $\mu\nu < 0$, $|\mu| < \sqrt{5}/2$, $\nu > \sqrt{5}/2$, it suffices to take $\eta = 1+\delta$, $\alpha = -\mu$, $0 < \kappa < 1$ and $\delta$ small enough; then (2.7) is satisfied.

Case (V). When $\mu\nu < 0$, $|\mu| \geq \sqrt{5}/2$, $\nu > \sqrt{5}/2$, $|\alpha| < \sqrt{5}/2$, we may restrict attention to the case $\nu > 0 \geq \mu$, $\alpha > 0$, as $\nu < 0 \leq \mu$, $\alpha < 0$ is similar by conjugation. So we consider the case of $\nu > \sqrt{5}/2$, $\mu \leq -\sqrt{5}/2$ and $\alpha > 0$. Let $\tau = \sqrt{\delta}$, then

$$h(\tau) = \operatorname{dist}(z(\tau^2), \Sigma) - 2\kappa\tau = \frac{\nu-\alpha}{\sqrt{1+\alpha^2}}\tau^2 - 2\kappa\tau - \frac{\mu+\tau}{\sqrt{1+\alpha^2}},$$

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where $\tau = \tau_0 = 3\kappa/\big(2\nu - \sqrt{5}\big)$, $h(\tau_0) = \min_{\tau\in\mathbb{R}} h(\tau)$. We can see $h(\tau_0) \le 0$ if and only if

$$-\kappa^2 \le \frac{1}{1+\alpha^2}(\mu+\alpha)(\nu-\alpha). \tag{2.8}$$

Once

$$-1 < \frac{1}{1+\alpha^2}(\mu+\alpha)(\nu-\alpha) \tag{2.9}$$

holds, we can find $\kappa \in (0,1)$ such that (2.8) is satisfied, thus $h(\tau_0) \le 0$ and therefore $\operatorname{dist}(z(\tau_0^2), \Sigma) \le \kappa\tau_0$.

From (2.9), we get (2.6). In this case, that is, $\delta = \tau_0^2 = (3\kappa/(2\nu - \sqrt{5}))^2$, we have

$$\eta = \frac{1}{\sqrt{1+\alpha^2}}\operatorname{Re}(z(\delta)\cdot(1+i\alpha)) = \frac{1}{\sqrt{1+\alpha^2}}(1+\delta+\alpha(\delta\nu-\mu)).$$

The proof of Lemma 2.3 is therefore complete.

Using Cauchy's inequality and Lemma 2.3, (2.5) can be written as

$$\frac{d}{dt}E_\delta(u(t)) \le \alpha_0(\|\nabla u\|_2^2 + \delta\|u\|_6^6) - \eta\left(3 - 2\sqrt{1+\alpha^2}\right)\int |u|^4|\nabla u|^2 + \frac{2(1+\delta)M^2}{1-\kappa}\int |u|^4|\nabla u|^2. \tag{2.10}$$

In order to guarantee

$$-\eta\left(3 - 2\sqrt{1+\alpha^2}\right)\int |u|^4|\nabla u|^2 + \frac{2(1+\delta)M^2}{1-\kappa}\int |u|^4|\nabla u|^2 \le 0, \tag{2.11}$$

we use Lemma 2.3 again and obtain teh following cases:

Case (I). When $M^2 \le \big(3/2 - \sqrt{1+\nu^2}\big)(1-\kappa)$, then (2.11) holds. Since the above inequality holds for every $\kappa \in (0,1)$, we know that when $M^2 < \big(3/2 - \sqrt{1+\nu^2}\big)$, then (2.11) holds.

Case (II). When $M^2 < \big(3/2 - \sqrt{1+\alpha^2}\big)$, $\alpha$ as in the proof of Lemma 2.3, then (2.11) holds.

Case (III). When $M^2 < 1/2$, then (2.11) holds.

Case (IV). When $M^2 < \big(3/2 - \sqrt{1+\mu^2}\big)$, then (2.11) holds.

Case (V). When

$$M^2 < \frac{1}{\sqrt{1+\alpha^2}}\left(1 + \frac{|\alpha(\delta\nu-\mu)|}{1+\delta}\right)\left(\frac{3}{2} - \sqrt{1+\alpha^2}\right), \quad |\alpha| < \frac{\sqrt{5}}{2},$$

then (2.11) holds.

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So, under the condition of Lemma 2.2, (2.10) can be written as

$$\frac{d}{dt}E_\delta(u(t)) \le 6\alpha_0 E_\delta(u(t)).$$

By Gronwall's inequality, Lemma 2.2 is proved.

LEMMA 2.4. *Under the assumptions of Lemma* 2.2, *we have*

$$\|\Delta u\|_2 \le K_5(T, \|u_0\|_{H^2}), \quad \text{for } 0 \le t \le T.$$

PROOF. Taking the real part of the inner product of (1.7) with $\Delta^2 u$, we find that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 &= a_0\|\Delta u\|_2^2 - \|\nabla\Delta u\|_2^2 - \operatorname{Re}\alpha_6\int |u|^{2\sigma}u\Delta^2\bar{u} \\ &\quad + \operatorname{Re}\left[\alpha_2\int |u|^2 u_x\Delta^2\bar{u} + \alpha_3\int |u|^2 u_y\Delta^2\bar{u}\right. \\ &\quad \left. +\alpha_4\int u^2\bar{u}_x\Delta^2\bar{u} + \alpha_5\int u^2\bar{u}_y\Delta^2\bar{u}\right].\end{aligned}$$

After integration by parts and using Lemma 2.2 and some elementary manipulation, we get $d(\|\Delta u\|_2^2)/dt \le C_1\|\Delta u\|_2^4 - \|\nabla\Delta u\|_2^2 + C_2$, where $C_1$, $C_2$ depend on $\alpha_0$, $|\alpha_i|$ $(i = 1, \dots, 4)$, $T$, $\mu$ and $\nu$. Thus, by Gronwall's inequality, $\|\Delta u\|_2 \le K_5(T, \|u_0\|_{H^2})$.

By the local existence result, Lemmas 2.1, 2.2 and 2.4, we finally obtain the following global existence result.

THEOREM 2.5 (Global existence). *Under the assumptions of Lemma* 2.2, *there exists a unique global solution of the initial value problem for the 2D generalised Ginzburg-Landau equation* (1.7) *with* (1.5) *in* $H^2_{per}(\Omega)$.

## 3. Some remarks

In this section, we give four remarks on our problem.

REMARK 1. From the conditions in Lemmas 2.1 and 2.2, we find that our results improved the results of [4]. On one hand, we use $M = \max\{|\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4|\}$ instead of $\sum_{j=1}^{j=4}|\alpha_j|^2$, on the other hand, the global result in [4] is just the case of $|\mu| < \sqrt{5}/2$. That is, for Case (IV) and some parts of (I)–(III) in Lemma 2.2 in our paper, our results are sharper and more detailed than [4]. In [4], the authors considered the numerical example for $\nu = 1$, $\mu = -10$, $\alpha_0 = a_1 = a_2 = a_3 = a_4 = 0$ and $b_1 = b_2 = 1$, $b_3 = b_4 = -1$. We note that $M = 1$ and $(|b_1 - b_3| + |b_2 - b_4|)^2 = 16$, so this example does not satisfy some conditions in Lemmas 2.1 and 2.2, but it does satisfy $-(1+\mu\nu) < |\alpha||\mu-\nu|$ for some $|\alpha| < \sqrt{5}/2$. This means by our results, if $\nu$ and $\mu$ satisfy $-(1+\mu\nu) < |\alpha||\mu-\nu|$ for some $|\alpha| < \sqrt{5}/2$, $M$ small enough, then we also have the global solution. In other words, if $\nu$ and $\mu$ satisfy $-(1+\mu\nu) < |\alpha||\mu-\nu|$ for some $|\alpha| < \sqrt{5}/2$, and the nonlinear derivative terms are small perturbations, then the solution will exist for all time.

REMARK 2. Based on our existence result, we can obtain the global attractor by a similar method to [15], using the standard method of [19] and [26].

REMARK 3. We specifically consider a simple example to explain the blow-up of the solution of

$$u_t = \Delta u - |u|^p u + ib|u|^2 u_x, \quad x \in \Omega = (0, L_1) \times (0, L_2),\ t > 0, \tag{3.1}$$

with initial value condition

$$u(x, y, 0) = u_0(x, y), \quad x \in \Omega, \tag{3.2}$$

where $u$ is spatially periodic and $p > 0$.

We use the energy method of [20, 22] to prove the following proposition.

PROPOSITION 3.1 (Blow-up for $p < 2$ and $b \neq 0$). *If* $0 < p < 2$, $u_0 \not\equiv 0$ *and* $E(0) \leq 0$ (*the definition of* $E(t)$ *can be seen in the proof*), *then every smooth solution of* (3.1)–(3.2) *will blow up. That is, there exists* $T < \infty$ *such that* $\lim_{t\nearrow T} \|u\| = +\infty$.

PROOF. Taking the real part of the inner product of (3.1) with $u$, we get

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\|\nabla u\|^2 - \int |u|^{p+2} - b\,\mathrm{Im}\int |u|^2 u_x \bar{u}, \tag{3.3}$$

where Im denotes the imaginary part of a complex number. Let

$$E(t) = \frac{1}{2}\int |\nabla u|^2 + \frac{1}{p+2}\int |u|^{p+2} + \frac{b}{4}\mathrm{Im}\int |u|^2 u_x \bar{u}. \tag{3.4}$$

Then using (3.1), we have $dE(t)/dt = -\|u_t\|_2^2$, so

$$E(t) \leq E(0), \quad t \geq 0. \tag{3.5}$$

Here $d(\mathrm{Im}\int |u|^2 u_x \bar{u})/dt = 4\,\mathrm{Im}\int |u|^2 u_x \bar{u}_t$ was used. By (3.4) and (3.5), (3.3) can be written as

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u\|^2 &= -4E(t) + \|\nabla u\|^2 + \left(\frac{4}{p+2} - 1\right)\int |u|^{p+2} \\ &\geq -4E(0) + \left(\frac{4}{p+2} - 1\right)\int |u|^{p+2}.\end{aligned}$$

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Since $0 < p < 2$ and $E(0) \le 0$, we obtain

$$\frac{d}{dt}\|u\|^2 \ge 2\left(\frac{4}{p+2} - 1\right)\int |u|^{p+2} \ge K\|u\|^{p+2},$$

where $K = 2(4/(p+2) - 1)|\Omega|^{-p/(p+2)}$ and hence $\|u\|^{p/2} \ge (\|u_0\|^{-p/2} - pKt/2)^{-1}$. Moreover, if $u_0 \not\equiv 0$, then there exist $T \le (2/Kp)\|u_0\|^{-p/2} < +\infty$, such that $\lim_{t \nearrow T}\|u\| = +\infty$. The proof of this proposition is now complete.

Proposition 3.1 shows that if the dissipative term $-|u|^p u$ is not so strong, the solution of (3.1), (3.2) must blow up when $E(0) \le 0$ and $b \ne 0$.

REMARK 4. We will show that the boundedness of $\|u\|_q (q > 4)$ will guarantee global existence for (3.1), (3.2) with $p = 4$. By the discussion of Section 2, we only need to obtain the estimate of $\|\nabla u\|$. We have the following proposition.

PROPOSITION 3.2. *If* $\|u\|_q$ $(q > 4)$ *is bounded then* $\|\nabla u\|$ *is bounded, where* $u$ *is the solution of* (3.1), (3.2) *with* $p = 4$.

PROOF. Taking the real part of the inner part of (3.1) with $-\Delta u$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 = -\|\Delta u\|^2 + \mathrm{Re}\int |u|^4 u\Delta\bar{u} + b\,\mathrm{Im}\int |u|^2 u_x \Delta\bar{u}. \qquad (3.6)$$

Since $\mathrm{Re}\int |u|^4 u \Delta\bar{u} = -\int |\nabla u|^2|u|^4 - \int |\nabla|u|^2|^2$, then (3.6) can be written as

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 \le -\frac{1}{2}\|\Delta u\|^2 + \frac{b^2}{2}\int |u|^4|\nabla u|^2. \qquad (3.7)$$

Since

$$\int |u|^4|\nabla u|^2 \le \left(\int |u|^p\right)^{4/p}\left(\int |\nabla u|^{2p/(p-4)}\right)^{(p-4)/p} \le C\left(\int |\nabla u|^{2p/(p-4)}\right)^{(p-4)/p}$$

and $\|u\|_p$ $(p > 4)$ is bounded, by the Gagliardo-Nirenberg inequality and Young's inequality, we have

$$\begin{aligned}\frac{b^2}{2}\int |u|^4|\nabla u|^2 &\le C\left(\int |\nabla u|^{2p/(p-4)}\right)^{(p-4)/p} \\ &\le C\left(\|\Delta u\|^{10/(p+1)}\|u\|_p^{2(p-4)/(p+1)} + 1\right) \\ &\le C\left(\|\Delta u\|^{10/(p+1)} + 1\right) \le \frac{1}{2}\|\Delta u\|^2 + C.\end{aligned}$$

Then by (3.7), we show that $\|\nabla u\|$ is bounded.

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