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LETTER TO THE EDITOR

RUIN PROBABILITIES EXPRESSED IN TERMS OF STORAGE PROCESSES

SØREN ASMUSSEN, **Aalborg University*
SØREN SCHOCK PETERSEN, ***The Danish State Life Insurance Company*

Abstract

It is shown by a simple sample path argument that the ruin probabilities for a risk reserve process with premium rate $p(r)$ depending on the reserve r and finite or infinite horizon are related in a simple way to the state probabilities of a compound Poisson dam with the same release rate $p(r)$ at content r. In the infinite horizon case, this result has been established by Harrison and Resnick (1978), and in the finite horizon case with constant p it extends well-known relations to the $M/G/1$ virtual waiting time.

FINITE HORIZON; SAMPLE PATH COMPARISON

We consider a risk reserve process $\{R_t\}_{t \geqq 0}$ with i.i.d. claims at the epochs of a Poisson process and a premium rate $p(r)$ which depends on the current reserve $R_t = r$. We are interested in the probabilities

$$(1) \qquad \psi(u, T) = \mathbb{P}\Big(\inf_{0 \leqq t \leqq T} R(t) < 0 \mid R_0 = u\Big), \quad \psi(u) = \mathbb{P}\Big(\inf_{0 \leqq t < \infty} R(t) < 0 \mid R_0 = u\Big)$$

of ruin before time T, or of ultimate ruin.

By far the most prominent case in the literature is the case of a constant premium rate, $p(r) = p$ independently of r, for which a considerable body of theory has been developed. The study was initiated in risk theory, but more recently, the relation to queueing theory (expressing the ruin probabilities as the state probabilities of an $M/G/1$ queue with the same arrival intensity and service times distributed as the claims; e.g. [8], [9], [11] or [1] XIII.1.1) has become more generally appreciated and makes a number of known facts about queues available to risk theory (the converse is also true!). Our purpose here is to establish a similar result for a general premium rate p, only expressed in terms of the content process of a dam (with the same input and with release rate $p(r)$ in state r) rather than in terms of queues.

We first make the notation and basic conditions more precise. We assume that the claim sizes $U_1, U_2, \cdots$ are i.i.d., say with distribution B, and independent of the Poisson arrival process $\{N_t\}_{t \geqq 0}$, the intensity of which is denoted by α. The premium rate $p(r)$ is assumed to be in $D[0, \infty)$ with $\inf_{0 \leqq r \leqq R} p(r) > 0$ for each $R < \infty$ (this last condition could to some extent be weakened to include also some cases where $p(r) \to 0$, $r \to 0$; such a premium policy seems, however, to lack sense from a practical point of view). It is convenient to assign some

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* Postal address: Institute of Electronic Systems, Aalborg University, Strandvejen 19, DK-9000 Aalborg, Denmark.
Research partly carried out while this author was visiting the University of California, Santa Barbara.
** Postal address: Statsanstalten for Livsforsikring, Kampmannsgade 4, DK-1645 København V, Denmark.

arbitrary value, say $p(0)=0$, to $p(r)$ also if $r<0$; this does not affect the definition (1) of the ruin probabilities and their interpretation. The risk process $\{R_t\}_{t\geqq 0}$ may then be described as moving according to the differential equation

$$\dot{R}=p(R) \tag{2}$$

in between jumps of $\{N_t\}_{t\geqq 0}$ and to have a downwards jump of size U_n at the nth jump of $\{N_t\}_{t\geqq 0}$. Similarly, the corresponding dam content process $\{V_t\}_{t\geqq 0}$ moves according to $\dot{V}=-p(V)$ in between jumps of $\{N_t\}_{t\geqq 0}$ and has an upwards jump of size U_n at the nth jump of $\{N_t\}_{t\geqq 0}$ (note that $p(r)=0$, $r\leqq 0$, ensures $V_t\geqq 0$). More formal constructions can be found in [6], [7]. Examples of the sample paths are given in Figure 1, corresponding to the linear case $p(r)=p+\beta r$ which is of particular interest in risk theory since, e.g., one possible interpretation is the reserve being invested at interest $i\%$ p.a. so that $\beta=\ln(1+i/100)$ is the continuous force of interest. See for example [4], [5] and references there. Here p is thus increasing, but decreasing premium rates p could occur, for example, if the insurance company reduces the premium or pays out dividend once the reserve has grown sufficiently large.

We define $\tau=\inf\{t\geqq 0\colon R_t<0\}$ and let $\mathbb{P}_0$ refer to the initial condition $V_0=0$. Here is our main result.

Theorem. $\psi(u,T)=\mathbb{P}_0(V_T>y)$ *for all* $u\geqq 0$.

Proof. Assuming that the claims of $\{R_t\}_{0\leqq t\leqq T}$ occur at times $0<t_1<\cdots<t_N<T$ and are of sizes $U_1,\cdots,U_N$, we represent $\{V_t\}_{0\leqq t\leqq T}$ by letting the Poisson arrival process be given by the epochs $0<T-t_N<\cdots<T-t_1<T$ corresponding to the jump sizes $U_N,\cdots,U_1$. By an obvious reversibility argument, this does not affect the distributions, and since the probability of a jump at time T is 0, it is sufficient to show that if $R_0=u$, $V_0=0$ then the events $\{\tau\leqq T\}$ and $\{V_{T-0}>u\}$ coincide. Let $x_t(u)$ denote the solution of (2) corresponding to $x_0(u)=u$. Then $x_t(u)>x_t(v)$ for all t when $u>v$. Suppose first $V_{T-0}>u$ (this situation corresponds to the solid path of R_t in Figure 1 with $R_0=u=u_1$). Then

$$V_{T-t_1-0}=V_{T-t_1}-U_1=x_{t_1}(V_{T-0})-U_1>x_{t_1}(u)-U_1=R_{t_1-0}-U_1=R_{t_1}.$$

If $V_{T-t_1-0}>0$, we can repeat the argument and get $V_{T-t_2-0}>R_{t_2}$ and so on. Hence if k satisfies $V_{T-t_k-0}=0$ (such a k exists, if nothing else $k=N$), we have $R_{t_k}<0$ so that indeed $\tau\leqq T$. Suppose conversely $V_{T-0}\leqq u$ (this situation corresponds to the broken path of R_t in Figure 1 with $R_0=u=u_2$). Then similarly

$$V_{T-t_1-0}=V_{T-t_1}-U_1=x_{t_1}(V_{T-0})-U_1\leqq x_{t_1}(u)-U_1=R_{t_1-0}-U_1=R_{t_1},\ V_{T-t_2-0}\leqq R_{t_2}$$

and so on. Hence $R_{t_k}\geqq V_{T-t_k-0}\geqq 0$ for all k, and since ruin can only occur at the times of claims, we have $\tau>T$.

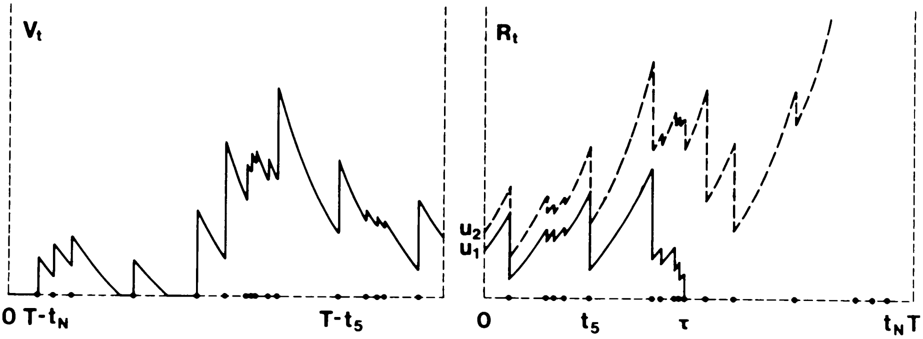

Figure 1

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As is well known ([1], XII.3; see also [7], [3]), the limiting behaviour of $\{V_t\}_{t \geqq 0}$ can be classified into an ergodic case and a non-ergodic (transient or null recurrent) one. In the non-ergodic case, $\mathbb{P}_0(V_T > u) \to 1$ for all $u \geqq 0$, whereas in the ergodic case a limiting random variable V exists (in the sense of convergence in distribution or total variation convergence). The distribution of V then has an atom of size $\pi_0 > 0$ at 0 and a density $g(x)$ on $(0, \infty)$ which are determined by the equations

$$g(x) = \pi_0 Q(x, 0) + \int_0^x Q(x, y) g(y)\, dy \tag{3}$$

$$1 = \pi_0 + \int_0^\infty g(y)\, dy \tag{4}$$

where $Q(x, y) = \alpha(1 - B(x - y))/p(x)$. More generally, for each $\pi_0 > 0$ (3) has a unique solution g, and ergodicity is equivalent to $\int g < \infty$ (in which case (4) is just a normalization to get mass 1). We call the risk process $\{R_t\}_{t \geqq 0}$ *terminating* if $\psi(u) = 1$ for all $u \geqq 0$, and *proper* if $\psi(u) < 1$ for all $u \geqq 0$. Letting $T \to \infty$ in the theorem above, we obtain the following.

Corollary. The risk process $\{R_t\}_{t \geqq 0}$ is either terminating or proper. The proper case arises if and only if the dam process is ergodic, or equivalently if and only if the solutions of (3) *are integrable. In that case, the probability of ultimate ruin is given by*

$$\psi(u) = \mathbb{P}(V > u) = \int_u^\infty g(y)\, dy \tag{5}$$

where g and π_0 satisfy (3), (4).

In the case of a constant p, say $p = 1$, we may identify $\{V_t\}_{t \geqq 0}$ by the $M/G/1$ virtual waiting time, and the theorem is then classical. In the present more general case the corollary is contained in [7], but we feel that the present approach is substantially easier and more elegant, not least combined with the approach to ergodic theory for dams which is given in [1] XIII.3 and which is somewhat different from [6], [3].

From the practical point of view of computing ruin probabilities, one may note that (3) is a linear Volterra integral equation of the second kind and a variety of standard numerical methods are therefore available, see for example [2]. Some worked-out examples will be presented elsewhere [10].

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