This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to W. O. J. Moser, University of Manitoba, Winnipeg, Manitoba.

A GENERALIZATION OF A PROBLEM OF L. LORCH AND L. MOSER

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(received Sept. 10, 1960)

In connection with problem P33, Canad. Math. Bull. 3(1960), 189, I should like to offer the following solution and remarks which include a generalization.

The problem requests that one show $R_{n+1}R_{n-1} - R_n^2 = x$, $n > 0$, where

$$R_n = R_n(x) = \Sigma_{k=0}^{n} \binom{n+k}{n-k} x^k .$$

Since now

$$R_n = \Sigma_{k=0}^{n} \binom{2n-k}{k} x^{n-k},$$

and since it is well-known that

$$S_n(z) = \Sigma_{k=0}^{[n/2]} \binom{n-k}{k} 2^{n-2k} z^k = \frac{u^{n+1} - v^{n+1}}{u - v},$$

where

$$u = 1 + (z+1)^{\frac{1}{2}}, \quad v = 1 - (z+1)^{\frac{1}{2}},$$

it is immediate that

$$R_n(x) = (x/4)^n S_{2n}(4/x),$$

and by a routine calculation with the explicit summation formula for $S_n(z)$ one readily verifies the proposed result.

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The neat form of the proposal suggests that a more general Turán expression may be obtained explicitly, which we now show.

In general let us define a functional operator T by means of

$$T_x f(x) = f(x + a) f(x + b) - f(x) f(x + a + b).$$

Then the expression to be evaluated in the original problem is $T_n (R_n)$ with $a = +1$, $b = -1$. We shall show how to evaluate $T_n (R_n)$ for arbitrary a and b.

But first we note that the form of R_n given in the original problem is rather special. It is easy to see that we may work with a more general form and not lose any of the elegance of formulation. In fact let us define R_n in general by means of

$$R_n(x) = r^n x^{tn} S_{jn}(w)$$

where w is some function of x, and r, t, j are fixed parameters. It follows readily from this that

$$T_n (R_n) = r^{2n+a+b} x^{t(2n+a+b)} T_n (S_{jn}(w)),$$

and so it is only necessary to be able to evaluate $T_n (S_{jn})$ really. To show that there is no difficulty in doing this with an S_n something like that in the original problem we shall consider the case when $j = 1$, and suppose that

$$S_n(x) = \frac{u^n - v^n}{u - v},$$

whether it may also be possible to write it as a binomial sum being immaterial.

It is again a routine calculation to verify first that

$$T_n(S_n) = \frac{(uv)^n}{(u - v)^2}[u^{a+b} + v^{a+b} - u^a v^b - u^b v^a]$$

but this factors nicely and we find the elegant expression

$$T_n(S_n) = (uv)^n . \frac{u^a - v^a}{u - v} . \frac{u^b - v^b}{u - v} = (uv)^n S_a S_b .$$

Thus one would expect to obtain very neat explicit Turán express-

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ions from such expressions. This result illustrates one other very interesting feature of the operator T when it is applied to many of the ordinary functions considered in elementary analysis. Many times we find instances of $T_x f(x) = g(x) f(a) f(b)$, with $g(x)$ being separated from the terms involving a and b in a neat manner. Thus, in problem E1396 [Amer. Math. Monthly 67(1960), 81] it turns out that for the ordinary Fibonacci numbers (that is $f_1 = f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$) we have, with the notation introduced above

$$T_n (f_n) = (-1)^n f_a f_b .$$

As other specific instances we might mention such as

$$T_x \sin x = \sin a \sin b = T_x \cos x,$$

$$T_x (c + dx) = d^2 a b ,$$

$$T_x \cosh x = T_x \sinh x = - \sinh a \sinh b.$$

Finally, Forsythe and Danese in various papers in the Duke Journal have studied what amounts to $T_n(P_n)$ for various classical polynomials P_n, such as the Legendre polynomials, ultraspherical, etc.

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