ON BABINET'S PRINCIPLE

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1. Introduction. In acoustic and electromagnetic theory much use is made of what is called the Principle of Babinet **(1)**. The principle states that the problems of diffraction by an aperture, S, in a plane screen and by a plane obstacle occupying the position S, are equivalent. It is the intent of this paper to indicate how this notion of equivalent boundary problems, for partial differential equations, can be extended to other situations. One may alter the underlying differential equation or the boundary conditions but the word plane is essential, that is, data must always be given on a plane.

It was observed by Rubin **(4)** that a kind of Babinet Principle holds in the diffraction of water waves by a "dock." The principle we propose is somewhat different in the water wave problem and of wider applicability. Its emphasis is on the integral equation formulation of boundary problems and its principal aim is to simplify, as much as possible, these equations.

In § 2 we shall prove the classical Babinet principle in slightly different form in order to introduce the method. In §§ 3 and 4, we state and prove the extended principle and finally in § 5, we give two simple examples.

2. The Classical Babinet Principle. In order to have a point of departure, we state and prove the classical principle in a slightly different form for two dimensions.

THEOREM 1. *Consider the following two problems:* $u(x, y)$, $v(x, y)$ *solutions of*

$$w_{xx} + w_{yy} + w = 0, \qquad y < 0, \tag{2.1}$$

continuous in $y \leqslant 0$, *satisfying a radiation condition*

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial w}{\partial r} - iw\right) = 0$$

uniformly in the polar angle, $r^2 = x^2 + y^2$, *and with*

(I) $u(x, 0) = 0$ on $|x| > a$, $\qquad u_y(x, 0) = g(x)$ on $|x| < a$,
(II) $v_y(x, 0) = 0$ on $|x| > a$, $\qquad v(x, 0) = h(x)$ on $|x| < a$;

$g(x)$ *and* $h(x)$ *given functions which we assume analytic on* $|x| < a$. *Then the solution of problem* (II) *yields simultaneously the solution of problem* (I).

Received September 19, 1957. This work was sponsored in part by the Office of Naval Research contract N onr-222(25), University of California at Berkeley and in part by Air Force Office of Scientific Research contract AF-18(600)-1138, Carnegie Institute of Technology.

Assuming we can solve (II), let v^0, v^1, v^2 be solutions for $h(x) = h_0(x)$, cos x and sin x respectively, $h_0(x)$ any (fixed) particular solution of the differential equation,

$$\frac{d^2h}{dx^2} + h = -g(x), \qquad -a < x < +a. \tag{2.2}$$

Now set

$$u(x, y) = v_y^0(x, y) + Av_y'(x, y) + Bv_y^2(x, y).$$

A and B any constants. Then clearly $u(x, 0) = 0$ on $|x| > a$. Now $g(x)$ and hence $h_0(x)$ are analytic on $|x| < a$ and thus $v^i(x, y)$ can be continued analytically across $y = 0$ $|x| < a$ as solutions of (2.1). Thus we can write for $y = 0$, $|x| < a$,

$$v_{yy}^i(x, 0) = -v_{xx}^i(x, 0) - v^i(x, 0)$$

and hence by (2.2)

$$u_y(x, 0) = -\left(\frac{d^2}{dx^2} + 1\right)(h_0(x) + A\cos x + B\sin x) = g(x).$$

There remains the question of determining A and B. It is well known that solutions of (II), continuous at $(\pm a, 0)$ have the form,

$$v(x, 0) = c^{\pm}[(x \mp a)^2 + y^2]^{\frac{1}{2}} + \dots.$$

The dots indicate terms with continuous first derivatives at $(\pm a, 0)$. Since u derives from v by differentiation, we have accordingly,

$$u(x, 0) = (C_0^{\pm} + C_1^{\pm}A + C_2^{\pm}B)[(x \mp a)^2 + y^2]^{-\frac{1}{2}} + \dots$$

near $x = \pm a$. But $u(x, y)$ is to be continuous in $y \leqslant 0$, hence we make

$$C_0^{\pm} + C_1^{\pm}A + C_2^{\pm}B = 0.$$

These equations determine A and B unless the homogeneous system, $C_1^{\pm}A + C_2^{\pm}B = 0$ should have a non-zero solution A_0, B_0. But if the latter were so, then clearly,

$$u_0(x, y) = A_0 v_y^{\ 1}(x, 0) + B_0 v_y^{\ 2}(x, 0)$$

is a solution of (I) continuous in $y \leqslant 0$. Moreover, it is a non-trivial solution since it is evident that v^1 and v^2 are even and odd functions of x respectively, and are non-zero. For a proof one uses the uniqueness of the solution of (II). The solution of (I) is known to be unique and thus we would have $u_0(x, y) = 0$ which in turn implies $A_0 = B_0 = 0$.

Problem (II) is easily reduced to an integral equation. For if we set

$$v(x, y) = \int_{-a}^{+a} H_0[(x - t)^2 + y^2)^{\frac{1}{2}}] f(t)\, dt \tag{2.3}$$

where H_0 is the Hankel function of first kind, we have a solution of (II) provided only that f satisfies,

$$\int_{-a}^{+a} H_0(|x-t|)\, f(t)\, dt = h(x) \qquad \text{on } |x| < a. \tag{2.4}$$

On the basis of Theorem 1, we see that in some fashion problem (I) can be reduced to the same equation. However, the technique for doing so without using Theorem 1, requires some rather involved computation and we are led to the conclusion that (II) is in some sense "simpler" than (I).

Various generalizations of the above theorem suggest themselves. Clearly we could apply the same technique with any elliptic differential equation in two variables with constant coefficients. Moreover, the single line segment $|x| < a$, $y = 0$ could be replaced by a series of such segments along the x-axis. The particular extension we wish to discuss concerns a change in the boundary conditions (I) and (II). In particular we aim always to replace one problem by another in which the first of conditions (II) holds, that is, $v_y(x, 0) = 0$ on $|x| > a$. In this manner we always reduce our secondary problem to an integral equation as in (2.3) and (2.4). To minimize notation we restrict ourselves (except in § 5) to Laplace's equation and to the single strip $y = 0$, $|x| < a$.

3. Statement. Let H denote the class of functions $u(x, y)$, continuous in $y \leqslant 0$ and harmonic in $y \leqslant 0$ for $(x, y) \neq (\pm\, a, 0)$. We write

$$D \equiv \frac{d}{dx}, \quad Y \equiv \frac{d}{dy}, \quad D^{-1}f \equiv \int_0^x f(t)dt.$$

We call the following problem (P I): *Find the function $u(x, y) \in H$ such that*

(I) $\quad Lu = 0 \quad \text{on } y = 0 \quad |x| > a; \qquad Mu = g(x) \quad \text{on } y = 0 \quad |x| < a.$

Here $g(x)$ is a given function analytic on $|x| < a$, L is a linear differential operator with constant coefficients,

$$L = L(D, Y) = \sum_{m=0}^{a} a_m D^m Y + \sum_{m=0}^{b} b_m D^m, \quad a_m, b_m \text{ constants,} \tag{3.1}$$

and M has the more general form,

$$M = M(D, Y) = \sum_{m=-\rho}^{r} r_m D^m Y + \sum_{m=-\sigma}^{s} s_m D^m, \quad r_m, s_m \text{ constants.} \tag{3.2}$$

Note that the form (3.1) is the most general differential operator, with constant coefficients, for functions in H since higher order derivatives with respect to y can always be replaced by derivatives with respect to x.

If there exists a growth condition as $x^2 + y^2 \to \infty$ which adjoined to problem (P I) guarantees that the solution is unique, we say (P I) *is unique*. We leave aside the difficult question of whether such conditions always exist. We say (P II) *solves* (P I) if the solution of (P I) can be obtained from that of (P II) by integration, differentiation and algebraic operations.

The product of two operators of form (3.2), when applied to functions of H, is again an operator of the same type since Y^2 can be replaced by $-D^2$.

We call the highest positive order of differentiation in an operator (3.2) the *order* of the operator. For an operator of form (3.1), we write L^* for $L(D, -Y)$ so that L^*L involves only x-differentiation on functions of H.

Consider the special class of problems (P II),

$$\text{(II)}\quad Yw = 0 \quad \text{on } y = 0,\quad |x| > a,\quad Mw = h(x) \quad \text{on } y = 0,\quad |x| < a.$$

Let F denote the class of functions $f(x)$, continuous and satisfying a Hölder condition on $|x| < a$ with $(a^2 - x^2)^\alpha f(x)$ continuous at $x = \pm a$ for some $\alpha < 1$. Define the linear transformation $W(x, y; f)$ over F by

$$w(x, y; f) = -(\pi)^{-1} \int_{-a}^{+a} f(t) \log [(x-t)^2 + y^2]^{\frac{1}{2}} \, dt. \tag{3.3}$$

Now if $w(x, y) = w(x, y; f)$, $f \in F$, it is clear that w is harmonic in $y < 0$, continuous on $y \leqslant 0$, and satisfies $Yw = 0$ on $y = 0$, $|x| > a$. We can accordingly make $w(x, y)$ a solution of (P II) by choosing an $f \in F$ such that,

$$MW(x, 0; f) = g(x), \qquad |x| < a. \tag{3.4}$$

If the solution, w, of (P II) has the representation, (3.3), with an $f \in F$ satisfying (3.4), we call (P II) *representable.* We remark that if instead of Laplace's equation we had started with the equation

$$u_{xx} + u_{yy} - k^2 u = 0,$$

all bounded solutions of (P II) would vanish exponentially as $x^2 + y^2 \to \infty$ and their representability would be an immediate consequence of Green's theorem (see § 5). For Laplace's equation representability is complicated by the presence of the logarithmic term.

THEOREM 2. *Suppose problem* (P I) (*as above*) *is unique with order* $L = k$, *and that problem* (P II):

$$\text{(II)}\quad Yv = 0 \quad \text{on } y = 0,\ |x| > a,\quad D^{-2k}L^*MYv = h(x) \quad \text{on } y = 0,\ |x| < a.$$

is representable. Then (P II) *solves* (P I).

4. Verification. Suppose (P II) can be solved. Let $v_i(x, y)$, $i = 0, 1, \ldots, 2k$ be solutions of (P II) for

$$h(x) = 1, x, \ldots, x^{2k-1}, h(x) = D^{-2k}L^*Lg(x),$$

respectively. Set

$$v(x, y) = \sum_{i=0}^{2k-1} A_i v_i + v_{2k}, \tag{4.1}$$

with constants A_i to be determined. Thus

$$D^{-2k}ML^*Yv = \sum_{i=0}^{2k-1} A_i x^i + D^{-2k}L^*Lg(x) \qquad \text{on } y = 0,\quad |x| < a. \tag{4.2}$$

https://doi.org/10.4153/CJM-1958-064-1 Published online by Cambridge University Press

Now suppose we can find $u(x, y)$ such that

$$Lu = Yv \qquad \text{in } y < 0 \tag{4.3}$$

with u harmonic in $y < 0$. Then we would have $Lu = 0$ on $y = 0$, $|x| > a$. Also

$$L^*LMu = L^*MYv = D^{2k}(D^{-2k}L^*MYv). \tag{4.4}$$

The relation (4.4) holds first in $y < 0$. However, with $g(x)$ and hence the right side of (4.2) analytic for $|x| \leqslant a$, one sees that $v(x, y)$ can be continued across $y = 0$, $|x| < a$ as a harmonic function. Thus by (4.3) the harmonic function $u(x, y)$ can be continued across $y = 0$, $|x| < a$ so that (4.4) continues to hold for $y = 0$, $|x| < a$, and by (4.2),

$$L^*L(Mu - g) = 0 \quad \text{on } y = 0, \quad |x| < a. \tag{4.5}$$

Now (4.5) is an ordinary differential equation for $Mu - g$, with constant coefficients and of order $2k$ in x on $y = 0$, $|x| < a$. If we require the vanishing of $Mu - g$ and its first $2k - 1$ derivatives at the one point $x = 0$, then we guarantee that $Mu - g = 0$, on $y = 0$, $|x| < a$, that is, that u is a solution of (P I). Define $u_i(x, y)$ $(i = 0, 1, \ldots, 2k)$ as harmonic functions satisfying

$$Lu_i = Yv_i \qquad \text{in } y < 0.$$

Then the solution of (4.3) can be written

$$u(x, y) = \sum_{i=1}^{2k-1} A_i u_i + u_{2k}, \tag{4.6}$$

and the conditions $(Mu - g)^{(m)}$, $m = 0, 1, \ldots, 2k - 1$, at $x = 0$, which guarantee that $Mu = g$, become

$$\sum_{i=1}^{2k-1} A_i(Mu_i)^m = g^{(m)} - (Mu_{2k})^{(m)} \qquad \text{at } y = 0, x = 0. \tag{4.7}$$

This is a system of linear equations for the determination of A_i, $i = 0, 1, \ldots, 2k - 1$. It has a solution unless the homogeneous system,

$$\sum_{i=0}^{2k-1} A_i(Mu_i)^m = 0 \qquad \text{at } y = 0, x = 0. \tag{4.8}$$

has a non-trivial solution. Suppose (4.8) has a solution A_i^0 and set

$$u^0 = \sum_{i=0}^{2k-1} A_i^0 u_i .$$

Then by (4.4) and the definition of the v_i,

$$L^*L\, Mu^0 = D^{2k}\, D^{-2k} L^*MY\left[\sum_{i=0}^{2k-1} A_i^0 v_i\right] = 0 \qquad \text{on } y = 0\ |x| < a.$$

But now (4.8) implies this ordinary differential equation has the solution $Mu^0 = 0$ on $y = 0$, $|x| < a$. Accordingly u^0 is a solution of (P I) with $g(x) = 0$

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and since (P I) is unique we infer $u^0 = 0$ in $y \leqslant 0$. Now we apply (4.3) and learn that

$$Lu^0 = 0 = Y\left(\sum_{i=0}^{2k-1} A_i^0 v_i\right), \qquad \text{in } y \leqslant 0$$

and therefore,

$$D^{-2k}L^*MY\left(\sum_{i=0}^{2k-1} A_i^0 v_i\right) = 0 = \sum_{i=0}^{2k-1} A_i^0 x^i,$$

so that $A_i^0 = 0$. Thus we conclude (4.7) always has a solution.

We must now indicate the construction (4.3). Since (P II) is representable, we can write (4.3) in the form

$$Lu = Y\,W(x, y; f) \tag{4.9}$$

for some $f \in F$. Now set

$$G(x, y, t) = \int_{p_1} [L(iw, -w)]^{-1} \exp[-wy + iw(x - t)]\, dw + \int_{p_2} [L(iw, w)]^{-1} \exp[wy + i(x - t)]\, dw \tag{4.10}$$

where $p_1(-\infty, 0)$ and $p_2(0, \infty)$ are any paths in the w-plane avoiding the poles of the integrand and ultimately running along the real axis. $G(x, y, t)$ is clearly harmonic in $y < 0$ and

$$LG(x, y, t) = \int_{p_1} \exp[-wy + iw(x - t)]\, dw + \int_{p_2} \exp[wy + i(x - t)]\, dw$$
$$= 2y[y^2 + (x - t)^2]^{-1} = 2\frac{\partial}{\partial y}\log[(x - t)^2 + y^2]^{\frac{1}{2}}.$$

Thus if we define another transformation, $U(x, y; f)$, over F by

$$U(x, y; f) = -(2\pi)^{-1}\int_{-a}^{+a} f(t)G(x, y, t)\, dt \tag{4.11}$$

we have

$$LU = YW$$

and (4.9) is solved by setting $u(x, y) = U(x, y; f)$.

The paths P_1 and P_2 for a particular problem (P I) are dictated by the desired behaviour as $x^2 + y^2 \to \infty$. In this connection the poles of $L(iw, \pm w)$ are of some heuristic value in guessing what this behaviour should be.

Finally, we wish to make some remarks concerning the derivatives of Mu or now of $MU(x, y; f)$ at $x = 0$, $y = 0$. The integrands in (4.9) involve reciprocals of polynomials of degree k (or $k + 1$) in w. Hence, $k - 2$ (or $k - 1$) differentiations can be carried out directly leaving absolutely convergent integrals for $y = 0$. It is not very difficult to see that $k - 1$ (or k) differentiations lead to a function which becomes infinite like $\log|x - t|$ for

$y = 0$, $x \to t$. Further differentiations require some care. From (4.10) it is clear that $k - 1$ (or k) differentiations of G lead to terms of the form

$$C\left\{\int_{-\infty}^{0} \exp[-wy + iw(x-t)]\,dw \pm \int_{0}^{\infty} \exp[wy + iw(x-t)]\,dw\right\} + \dots$$

the dots indicating terms continuous or at most logarithmically infinite for $y = 0$, $x = t$. The first terms here are

$$C' \frac{\partial}{\partial y} \log[(x-t)^2 + y^2]^{\frac{1}{2}} \text{ and } C'' \frac{\partial}{\partial x} \log[(x-t)^2 + y^2]^{\frac{1}{2}},$$

corresponding to $+$ and $-$ respectively. Thus $k - 1$ (or k) differentiations of $U(x, y; f)$ yield $DW(x, y; f)$ or $YW(x, y; f)$. From (3.3) we see that $YW(x, 0; f) = f(x)$ on $|x| < a$; hence further derivatives of $U(x, y; f)$ may be computed from derivatives of the function $f(x)$ or of $W(x, 0; f)$. In particular the functions $v_i(x, y)$ are analytic on $y = 0$, $|x| < a$. The associated functions $f_i(x)$ and $v_i(x, y) = W(x, y; f_i)$ are in turn given by $f_i(x) = Yv_i(x, 0)$ on $|x| < a$, hence are also analytic in $|x| < a$ and have derivatives of all orders at $x = 0$.

5. Examples. Let (P I) be the problem of the "oblique derivative," that is,

(I) $(\alpha D + \beta Y)u = 0$ on $y = 0$, $|x| > a$; $(\gamma D + \delta Y)u = g(x)$ on $y = 0$, $|x| < a$.

Here we have $L^*L = (\beta^2 + \alpha^2)D^2$ and

$$L^*MY = (\alpha\gamma + \beta\delta)D^2Y + (\beta\gamma - \alpha\delta)D^3.$$

The "solving" problem (P II) has then the form

(II) $Yv = 0$ on $y = 0$ $|x| > a$; $[(\alpha\gamma + \beta\delta)Y + (\beta\gamma - \alpha\delta)D]v = h$ on $y = 0$, $|x| < a$.

An explicit solution of (P II) can be obtained which becomes logarithmically infinite as $x^2 + y^2 \to \infty$. For setting $v(x, y) = W(x, y; f)$ we see that the first of conditions (II) is satisfied while the second requires

$$(\alpha\gamma + \beta\delta)YW(x, 0; f) + (\beta\gamma - \alpha\delta)DW(x, 0; f) = h(x) \qquad \text{on } |x| < a,$$

or,

$$(5.1) \quad (\alpha\gamma + \beta\delta)(\beta\gamma - \alpha\delta)^{-1}f(x) - \int_{-a}^{+a} \frac{f(t)}{x - t}\,dt = (\beta\gamma - \alpha\delta)^{-1}h(x) \qquad \text{on } |x| < a,$$

if $\beta\gamma - \alpha\delta \neq 0$. It is easy to see that $\beta\gamma - \alpha\delta = 0$ implies the two derivatives in (I) are in the same direction. Equation (5.1) was solved explicitly by Carleman **(2)**.

As a second example we consider the diffraction of a two dimensional progressive water wave by a rigid dock of finite width. For a physical description see **(4)** or **(3)**. The problem (P I) is

(I) $(Y - K)u = 0$ on $y = 0$, $|x| > a$; $Yu = g(x)$ on $y = 0$, $|x| < a$.

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Here $L^*L = D^2 + K^2$, $L^*MY = D^2(Y + K)$ and hence the solving problem (P II) is

(II) $\quad Yv = 0$ on $y = 0$, $|x| < a$; $(Y + K)v = h(x)$ on $y = 0$, $|x| < a$.

Again (P II) is solved by setting $v(x, y) = W(x, y; f)$ with

$$(Y + K)W(x, 0; f) = h(x) \qquad \text{on } |x| < a$$

or

$$f(x) - K\pi^{-1}\int_{-a}^{+a} f(t) \log |x - t|\, dt = h(x). \tag{5.2}$$

Equation (5.2) is no longer explicitly solvable, but there exists a systematic procedure for its numerical solution as indicated in **(3)**. We remark that the harmonic conjugate, $\nu(x, y)$ of $v(x, y)$ satisfies by (II),

(II)′ $\quad \nu = 0$ on $y = 0$, $|x| < a$; $(- D^2 + KY)\nu = h'(x)$ on $y = 0$, $|x| < a$;

and this is the problem considered in **(4)**.

It is shown in **(3)** that (P I) is unique and that a solution exists. Although we shall not carry through the computation, these facts can be used to establish that (P II) is unique and representable.

As a final example we wish to discuss the same water wave problem when the incident wave strikes the dock at an angle. In this case the boundary conditions for (P I) are the same, but the equation is

$$u_{xx} + u_{yy} - k^2u = 0. \tag{5.3}$$

A minor modification of Theorem 2 shows that (P I) is solved by problem (P II), that is, find a solution of (5.3) such that

(II) $\quad Yv = 0$ *on* $y = 0$, $|x| > a$; $(Y + K)v = h(x)$ on $y = 0$, $|x| < a$.

From the first of conditions (II), $v(x, y)$ may be continued across $y = 0$, $|x| > a$ so as to be a single valued solution of (4.3) in $x^2 + y^2 > a^2$. If it is bounded, it follows that

$$v(x, y) = O\,(\exp[- k(x^2 + y^2)^{\frac{1}{2}}]) \qquad \text{as } x^2 + y^2 \to \infty. \tag{5.4}$$

Problem (P II) can then be reformulated as a positive definite variational problem, namely:

Among all functions, $w(x, y)$, satisfying (5.4) and with

$$F[w] = \iint_{y<0} [w_x^2 + w_y^2 + k^2w^2]\, dx\, dy + \int_{-a}^{+a} (w - h)^2 dx < \infty,$$

find that one minimizing $F[w]$.

The existence of a solution of (PII) under condition (5.4) is easily established. For uniqueness follows immediately from Green's theorem and we can set

$$v(x, y) = - (\pi)^{-1}\int_{-a}^{+a} f(t)\, K_0(k[(x - t)^2 + y^2]^{\frac{1}{2}})\, dt,$$

where K_0 is the singular Bessel function serving as fundamental solution of (5.3). This is representability for equation (5.3). Then (5.3) is satisfied as is (5.4) and the first of condition (II). The second of (II) will also be satisfied if f is a solution of

$$f(x) = K(\pi)^{-1} \int_{-a}^{+a} f(t)\, K_0(k|x - t|)\, dt = h(x) \qquad \text{on } |x| < a.$$

Fredholm theory applies and the uniqueness theorem guarantees the homogeneous equation has no non-trivial solution. It follows that the variation problem also has a solution.

It is rather interesting that (P II) can be formulated as a variational problem for (P I) cannot be so formulated directly, being in reality an oscillation problem with infinite energy. (P II) finds its natural physical prototype in steady state heat flow, an equilibrium phenomena with finite energy. This fundamental difference in the physical character of (P I) and (P II) seems to the author to enhance the interest of the fact that the one can be reduced to the other.

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