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On a sufficient optimality condition over convex feasible regions

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In this note a sufficient optimality condition is established for nonlinear programming problems over arbitrary cone domains. A Kuhn-Tucker type sufficient condition is established if the programming problem has a pseudoconvex objective function and a convex feasible region.

1. Introduction

Recent articles in the literature, [1], [2], [3], [4], [6], and [8] have established various sufficient optimality theorems. These extensions have involved replacing orthant domains by cone domains and sometimes allowing the constraints to be quasiconvex. This note establishes a Kuhn-Tucker type sufficient condition, [5] and [7], for programming problems with a pseudoconvex objective function and a convex feasible region.

2. Definitions

ASSUMPTION 1. P is an open set in E^n . θ is a numerical function, g is an m-dimensional vector function, and h is a k-dimensional vector function, each defined on P . Also θ, g , and h are differentiable at $\bar{x}$.

DEFINITION 1. C will denote any arbitrary cone in E^m .

DEFINITION 2. C^* will denote the polar cone of C ; that is,

$$C^* = \{y^* \in E^m : y^{*\prime}y \geq 0 \text{ for all } y \in C\} .$$

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DEFINITION 3 (Pseudoconvex). Let θ be a numerical function defined on an open set $P \subset E^n$ and let Q_1^- denote the negative orthant in E^1. θ is pseudoconvex at $\bar{x}$ with respect to Q_1^- on P if θ is differentiable at $\bar{x}$ and

$$\left.\begin{array}{r} x \in P \\ \nabla_x'\theta(\bar{x})(x-\bar{x}) \notin \text{int } Q_1^- \end{array}\right\} \Rightarrow \theta(x) - \theta(\bar{x}) \notin \text{int } Q_1^- ,$$

or, equivalently,

$$\left.\begin{array}{r} x \in P \\ \theta(x) - \theta(\bar{x}) \in \text{int } Q_1^- \end{array}\right\} \Rightarrow \nabla_x'\theta(\bar{x})(x-\bar{x}) \in \text{int } Q_1^- .$$

DEFINITION 4 (Minimization problem). The minimization problem is to find $\bar{x} \in E^n$, if it exists, such that

$$\theta(\bar{x}) = \min \theta(x) ,$$

$$\bar{x} \in X ,$$

where

$$X = \{x : x \in P \subset E^n, g(x) \in C \subset E^m, h(x) = \{0\} \subset E^k\} .$$

DEFINITION 5 (Kuhn-Tucker problem) (see [5], pp. 94, 162-163). The following is a modified Kuhn-Tucker stationary point problem over cone domains.

Find an $\bar{x} \in P \subset E^n$, $\bar{r} \in -C^* \subset E^m$, and $\bar{s} \in E^k$ such that

$$\nabla_x'\theta(\bar{x}) + \bar{r}'\nabla_x g(\bar{x}) + \bar{s}'\nabla_x h(\bar{x}) = 0 ,$$

$$\bar{r}'g(\bar{x}) = 0 ,$$

$$g(\bar{x}) \in C ,$$

$$h(\bar{x}) = \{0\} ,$$

$$(\bar{r}, \bar{s}) \in -C^* x E^k .$$

3. Sufficient optimality condition

LEMMA 1. *Let* X *be a convex set contained in an open set* P *in* E^n. *Let* f *be a numerical function defined on* P *and differentiable at* $\bar{x}$. *Also let* $f(\bar{x}) = 0$, *where* $\bar{x} \in X$. *If* $f(x) > 0$ *has no solution* $x \in X$, *then* $\nabla' f(\bar{x})(x-\bar{x}) > 0$ *has no solution* $x \in X$.

Proof. We will prove the contrapositive, that is, if $\nabla' f(\bar{x})(x-\bar{x}) > 0$ has a solution $\hat{x} \in X$, then $f(x) > 0$ has a solution $\tilde{x} \in X$.

Since X is convex, $\bar{x} + t(\hat{x}-\bar{x}) \in X$, for some $t \in (0, 1)$ and since f is differentiable at $\bar{x}$, we have that

$$f\big(\bar{x}+t(\hat{x}-\bar{x})\big) = f(\bar{x}) + t\nabla' f(\bar{x})(\hat{x}-\bar{x}) + o(t) .$$

Now with $f(\bar{x}) = 0$ and $\nabla' f(\bar{x})(\hat{x}-\bar{x}) > 0$ we have, for sufficiently small t, $f\big(\bar{x}+t(\hat{x}-\bar{x})\big) > 0$. So letting $\tilde{x} = \bar{x} + t(\hat{x}-\bar{x})$, the result follows. The case where $X = \{\bar{x}\}$ follows immediately.

THEOREM 1. *Let* P, θ, g, *and* h *satisfy Assumption* 1 *with* θ *pseudoconvex with respect to* Q_1^- *at* $\bar{x}$ *on* P; C *be an arbitrary cone; and assume that the feasible region* X *of the minimization problem is convex. If there exists a solution to the Kuhn-Tucker problem, then* $\bar{x}$ *is a solution to the minimization problem.*

Proof. Assume there exists a solution to the Kuhn-Tucker problem. Since X is convex, consider Lemma 1 with $f(x) = \bar{r}'g(x) + \bar{s}'h(x)$ where $\bar{r} \in -C^*$, $g(x) \in C$, and $h(x) = \{0\}$ for any $x \in X$. Now if there exists a solution to the Kuhn-Tucker problem, then

$$[\nabla'\theta(\bar{x})+\bar{r}'\nabla g(\bar{x})+\bar{s}'\nabla h(x)](x-\bar{x}) = 0 \quad \text{for all} \quad x \in X .$$

Now appealing to Lemma 1, we have that

$$[\bar{r}'\nabla g(\bar{x})+\bar{s}'\nabla h(\bar{x})](x-\bar{x}) \le 0 \quad \text{for all} \quad x \in X .$$

Hence

$$\nabla'\theta(\bar{x})(x-\bar{x}) \ge 0 \quad \text{for all} \quad x \in X ,$$

and since θ is pseudoconvex, it follows that $\bar{x}$ is optimal.

References

[1] J.M. Borwein, "A note on Fritz John sufficiency", *Bull. Austral. Math. Soc.* 15 (1976), 293-296.

[2] B.D. Craven, "Sufficient Fritz John optimality conditions", *Bull. Austral. Math. Soc.* 13 (1975), 411-419.

[3] B.D. Craven and B. Mond, "A Fritz John theorem in complex space", *Bull. Austral. Math. Soc.* 8 (1973), 215-220.

[4] T.R. Gulati, "A Fritz John type sufficient optimality theorem in complex space", *Bull. Austral. Math. Soc.* 11 (1974), 219-224.

[5] Olvi L. Mangasarian, *Nonlinear programming* (McGraw-Hill, New York, St. Louis, San Francisco, London, Sydney, Toronto, Mexico, 1969).

[6] V.A. Sposito, "Modified regularity conditions for nonlinear programming problems over mixed cone domains", *Math. Programming* 6 (1974), 167-179.

[7] V.A. Sposito, *Linear and nonlinear programming* (Iowa State University Press, Ames, 1975).

[8] Vince A. Sposito and H.T. David, "Saddle-point optimality criteria of nonlinear programming problems over cones without differentiability", *SIAM J. Appl. Math.* 20 (1971), 698-702.

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