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Strongly 0-dimensional Modules

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Abstract. In a multiplication module, prime submodules have the following property: if a prime submodule contains a finite intersection of submodules, then one of the submodules is contained in the prime submodule. In this paper, we generalize this property to infinite intersection of submodules and call such prime submodules strongly prime submodules. A multiplication module in which every prime submodule is strongly prime will be called a strongly 0-dimensional module. It is also an extension of strongly 0-dimensional rings. After this we investigate properties of strongly 0-dimensional modules and give relations of von Neumann regular modules, Q-modules and strongly 0-dimensional modules.

1 Introduction

Let R be a commutative ring with identity and M be a unitary R-module. The nilradical of R is defined to be the set of all nilpotent elements of R and denoted by $\mathrm{Nil}(R)$. It is known that the nilradical of R is equal to intersection of all prime ideals of R. For any submodule N of M, the annihilator of M/N, denoted by $(N:M)$, is the set of all elements r in R such that $rM \subseteq N$. A submodule N of M is called prime if $N \neq M$ and $rm \in N$ implies $r \in (N:M)$ or $m \in N$ for $r \in R$, $m \in M$. The set of all prime submodules of M is denoted by $\mathrm{Spec}(M)$. It is known that for any ring R, the set of prime ideals of R is non-empty if and only if $R \neq (0)$. However, some modules have no prime submodules and such modules are called primeless. Among examples of primeless modules are the zero module and $E(p)$, which is a $\mathbb{Z}$-submodule of $\mathbb{Q}/\mathbb{Z}$ [7, Example]. An R-module M is called a multiplication module provided that for each submodule N of M, there exists an ideal I of R such that $N = IM$. Note that for an R-module M, the set of prime submodules is non-empty precisely when M is a multiplication module. It is clear that M is a multiplication R-module if and only if $N = (N:M)M$ for every submodule N of M. By [2, Corollary 2.11], if M is a multiplication module, then the submodule N of M is prime if and only if the ideal $(N:M)$ of R is prime. The radical of a submodule N is defined to be the intersection of all prime submodules of M that contains N, and it is denoted by M-$\mathrm{rad}(N)$. A submodule N of M is called *nilpotent* if $(N:M)^kN = 0$ for some $k \in \mathbb{Z}^+$. We say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M. The nilradical of M is the set of all nilpotent elements of M and denoted by $\mathrm{Nil}(M)$ [1]. If M is a faithful multiplication module, then $\mathrm{Nil}(M) = \mathrm{Nil}(R)M = \bigcap Q$, where the intersection runs over all prime submodules of M [1, Theorem 6]. If M has no nonzero nilpotent elements, then M is called reduced. Also, a submodule N of M is idempotent if $N = (N:M)N$ (see [1]).

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Let S be a multiplicatively closed subset of R and M be an R-module. Then $S^{-1}M = \{\frac{m}{s} \mid m \in M,\ s \in S\}$ is a $S^{-1}R$-module. Note that if M is a multiplication R-module, then $S^{-1}M$ is a multiplication $S^{-1}R$-module. If N is a submodule of M and $\sqrt{(N:M)} \cap S = \varnothing$, then $S^{-1}N$ is a proper submodule of $S^{-1}M$ [8, Theorem 3.1]. Note that every submodule of $S^{-1}M$ is of the form $S^{-1}N$, where N is a submodule of M [8, Theorem 3.3]. Furthermore ([8, Theorem 3.4]),

$$\operatorname{Spec}(S^{-1}M) = \left\{ S^{-1}\mathfrak{P} \mid (\mathfrak{P}:M) \cap S = \varnothing \text{ and } \mathfrak{P} \in \operatorname{Spec}(M) \right\}.$$

Let R be a ring, $I_1, I_2, \dots, I_m$ a finite number of ideals of R, and P a prime ideal of R such that $\bigcap_{i=1}^{m} I_i \subseteq P$. Then, $I_j \subseteq P$ for some $j \in \{1, \dots m\}$. In [4], the authors have recently generalized the above statement to infinite intersections. The authors called a prime ideal P of R *strongly prime* if, for any index set S, $\bigcap_{i \in S} I_i \subseteq P$ implies $I_j \subseteq P$ for some $j \in S$. And a ring is called *strongly* 0*-dimensional* if all prime ideals are strongly prime ideals. Following this definition, it was shown that every strongly 0-dimensional ring is 0-dimensional. Further, it was proved that every Artinian ring is a strongly 0-dimensional ring. Moreover, they gave the relations among strongly 0-dimensional rings, von Neumann rings, Q-rings, and compactly packed rings. In a multiplication module, if a prime submodule contains any finite intersection of submodules, then it has to contain at least one of them [6, Proposition 5]. Note that this property is not valid for every module. For example, consider the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z}$. Indeed, $(\mathbb{Z} \times 0) \cap (0 \times \mathbb{Z}) \subseteq (0,0)$, but $(\mathbb{Z} \times 0) \nsubseteq \mathbb{Z} \times \mathbb{Z}$ and $(0 \times \mathbb{Z}) \nsubseteq (0,0)$.

In this paper we will generalize this concept to infinite intersection, and we will call a prime submodule $\mathfrak{P}$ strongly prime if $\bigcap_{\alpha \in S} N_\alpha \subseteq \mathfrak{P}$ implies $N_\beta \subseteq \mathfrak{P}$ for some $\beta \in S$. A module M is said to be *strongly* 0*-dimensional* if all prime submodules are strongly prime submodules. In the following section we give some characterizations of strongly 0-dimensional modules, and in the last section we investigate some properties of von Neumann regular modules. After this we get a relation among von Neumann regular modules, Q-modules, and strongly 0-dimensional modules.

2 Strongly 0-dimensional Modules

In this section we extend strongly 0-dimensional rings to multiplication modules. With the above characterizations we will determine the relation between strongly 0-dimensional modules and strongly 0-dimensional rings.

Definition 2.1 Let M be a multiplication R-module, S be an index set, and N_α's be submodules of M for $\alpha \in S$. A prime submodule $\mathfrak{P}$ of M is said to be *strongly prime* if $\bigcap_{\alpha \in S} N_\alpha \subseteq \mathfrak{P}$ implies $N_\beta \subseteq \mathfrak{P}$ for some $\beta \in S$. A multiplication module M is said to be *strongly* 0*-dimensional* if all prime submodules of M are strongly prime submodules.

Proposition 2.2 *Every homomorphic image of a strongly* 0*-dimensional module is a strongly* 0*-dimensional module.*

Proof Let M be a strongly 0-dimensional R-module and let M' be any R-module. Let $f\colon M \to M'$ be a module epimorphism. Suppose that $\bigcap_{\alpha \in S} N'_\alpha \subseteq \mathfrak{P}'$ for some submodules N_α' and prime submodule $\mathfrak{P}'$ of M'. There exist submodules N_α and a prime submodule $\mathfrak{P}$ of M such that $\ker(f) \subseteq N_\alpha$, $\ker(f) \subseteq \mathfrak{P}$ and $f(N_\alpha) = N_\alpha'$, $f(\mathfrak{P}) = \mathfrak{P}'$. Hence we obtain

$$\bigcap_{\alpha \in S} N_\alpha' = \bigcap_{\alpha \in S} f(N_\alpha) = f\Big(\bigcap_{\alpha \in S} N_\alpha\Big) \subseteq \mathfrak{P}' = f(\mathfrak{P}).$$

Thus $\bigcap_{\alpha \in S} N_\alpha \subseteq \mathfrak{P}$. Since M is strongly 0-dimensional, $N_\alpha \subseteq \mathfrak{P}$ for some $\alpha \in S$. Therefore, $N_\alpha' = f(N_\alpha) \subseteq f(\mathfrak{P}) = \mathfrak{P}'$ for some $\alpha \in S$. ■

Corollary 2.3 *Let M be an R-module and N a submodule of M. If M is a strongly 0-dimensional module, then M/N is a strongly 0-dimensional $R/(N:M)$-module.*

Proof If M is a strongly 0-dimensional module, then so is M/N by Proposition 2.2. And so M/N is a strongly 0-dimensional $R/(N:M)$-module by change of rings. ■

Theorem 2.4 *Let M be a torsion-free R-module. If M is a strongly 0-dimensional module, then M is a simple module.*

Proof Suppose that M is a strongly 0-dimensional R-module. Let N be the intersection of all non-zero submodules N_α of M. Suppose for contradiction that $N = 0$. Since M is a strongly 0-dimensional module and (0) is prime, we get that one of the $N_\alpha = 0$, this is a contradiction. Thus $N \neq 0$. Now let $0 \neq m \in N$. Since N is the smallest non-zero submodule of M and $\mathrm{Rm} \subseteq N$, we get $\mathrm{Rm} = N$. Let $0 \neq a \in R$, then $\mathrm{Ram} \subseteq \mathrm{Rm}$. Hence $\mathrm{Rm} = N = \mathrm{Ram}$. So there exists an $r \in R$ such that $m = \mathrm{ram}$. Since M is torsion-free, we get a as a unit element of R. Thus R is a field, and so M is a simple module. ■

Proposition 2.5 *Every strongly 0-dimensional module is 0-dimensional.*

Proof Let M be a strongly 0-dimensional module and $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$ be two prime submodules of M. Then by Corollary 2.3 we get $M/\mathfrak{P}_1$ a strongly 0-dimensional $R/(\mathfrak{P}_1 : M)$-module. Since $\mathfrak{P}_1$ is prime, $M/\mathfrak{P}_1$ is torsion-free. Thus by Theorem 2.4, $M/\mathfrak{P}_1$ is a simple module. Hence $\mathfrak{P}_2/\mathfrak{P}_1 = (0)$ and so $\mathfrak{P}_2 = \mathfrak{P}_1$. ■

Theorem 2.6 *Every Artinian multiplication module is strongly 0-dimensional.*

Proof Let M be an Artinian multiplication module and let $\bigcap_{\alpha \in S} N_\alpha \subseteq \mathfrak{P}$ for some submodules N_α and prime submodule $\mathfrak{P}$ of M. Since M is Artinian, we have $\bigcap_{\alpha \in S} N_\alpha = \bigcap_{i=1}^{n} N_{\alpha_i}$ for some finite subset $\{\alpha_i\}_{i=1}^{n}$ of S, where $n \in \mathbb{Z}^+$. Then

$$\Big(\bigcap_{\alpha \in S} N_\alpha : M\Big) = \Big(\bigcap_{i=1}^{n} N_{\alpha_i} : M\Big) = \bigcap_{i=1}^{n} (N_{\alpha_i} : M) \subseteq (\mathfrak{P} : M).$$

It follows that $(N_{\alpha_k} : M) \subseteq (\mathfrak{P} : M)$ for some $k \in \mathbb{Z}^+$, since $(\mathfrak{P} : M)$ is a prime ideal of R. So $(N_{\alpha_k} : M)M \subseteq (\mathfrak{P} : M)M$. Since M is a multiplication module, $N_{\alpha_k} \subseteq \mathfrak{P}$ for some $\alpha_k \in S$. Hence M is a strongly 0-dimensional module. ■

Theorem 2.7 *Let M be a finitely generated faithful multiplication R-module. Then M is a strongly 0-dimensional module if and only if R is a strongly 0-dimensional ring.*

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Proof Let M be a strongly 0-dimensional module and let $\bigcap_{\alpha\in S} I_\alpha \subseteq P$ for some ideals I_α and prime ideal P of R. Then $(\bigcap_{\alpha\in S} I_\alpha)M \subseteq PM$ (here PM is a prime submodule of M by [2, Corollary 2.11]). By using [2, Theorem 1.6(i)] we have $\bigcap_{\alpha\in S}(I_\alpha M) \subseteq PM$. Since M is a strongly 0-dimensional module, $I_\beta M \subseteq PM$ for some $\beta \in S$. Hence $I_\beta \subseteq P$ from [2, Theorem 3.1]. For the converse, suppose that R is a strongly 0-dimensional ring and $\bigcap_{\alpha\in S} N_\alpha \subseteq \mathfrak{P}$ for some submodules N_α and prime submodule $\mathfrak{P}$. Since M is a multiplication module, there exist ideals I_α and prime ideal P such that $N_\alpha = I_\alpha M$ and $\mathfrak{P} = PM$. Thus we have

$$PM = \mathfrak{P} \supseteq \bigcap_{\alpha\in S} N_\alpha = \bigcap_{\alpha\in S}(I_\alpha M) = \Big(\bigcap_{\alpha\in S} I_\alpha\Big) M,$$

and so we get $\bigcap_{\alpha\in S} I_\alpha \subseteq P$. Since R is a strongly 0-dimensional ring, we get $I_\beta \subseteq P$ for some $\beta \in S$. Hence $N_\beta = I_\beta M \subseteq PM = \mathfrak{P}$. Thus M is a strongly 0-dimensional module. ■

Proposition 2.8 *Let M be a multiplication R-module and S a multiplicative subset of R. If M is a strongly 0-dimensional R-module, then $S^{-1}M$ is a strongly 0-dimensional $S^{-1}R$-module.*

Proof Let $\bigcap_{\alpha\in T} S^{-1}N_\alpha \subseteq S^{-1}\mathfrak{P}$ for any submodules N_α and prime submodule $\mathfrak{P}$ of M. Since

$$S^{-1}\Big(\bigcap_{\alpha\in T} N_\alpha\Big) \subseteq \bigcap_{\alpha\in T} S^{-1}N_\alpha \subseteq S^{-1}\mathfrak{P},$$

we have $\bigcap_{\alpha\in T} N_\alpha \subseteq \mathfrak{P}$ by [8, Theorem 3.3(i)]. Since M is a strongly 0-dimensional module, $N_\beta \subseteq \mathfrak{P}$ for some $\beta \in T$. Thus we get $S^{-1}N_\beta \subseteq S^{-1}\mathfrak{P}$ for some $\beta \in T$. ■

3 Von Neumann Regular Modules and Strongly 0-dimensional Modules

In this section we will examine von Neumann regular modules and describe the relation with strongly 0-dimensional modules. First, recall that an R-module M is called *von Neumann regular* if and only if every cyclic submodule of M is a direct summand in M [5]. An element $m \in M$ is called von Neumann regular if there exists $r \in (Rm : M)$ and $a \in R$ such that $m = ram$ [1]. In [1, Proposition 9], it is shown that M is von Neumann regular if and only if every element of M is von Neumann regular.

Proposition 3.1 ([1, Proposition 10]) *Let R be a ring and M a faithful multiplication R-module. If M is von Neumann regular, then $\dim(M) = 0$ and $\mathrm{Nil}(M) = 0$. The converse is true if we assume that M is finitely generated.*

Lemma 3.2 *Let R be a ring, M an R-module, and K, L two submodules of M. For each maximal ideal $\mathfrak{m}$ of R, let $K_\mathfrak{m}$ and $L_\mathfrak{m}$ be considered as $R_\mathfrak{m}$-submodules of $M_\mathfrak{m}$. If $K_\mathfrak{m} = L_\mathfrak{m}$ for every $\mathfrak{m}$, then $K = L$.*

Proof One can look for the proof in [9, p. 164, Corollary]. ■

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Lemma 3.3 *Let M be a multiplication R-module. If $M_{\mathfrak{m}}$ is a strongly* 0*-dimensional module for all maximal ideals $\mathfrak{m}$ of R, then* $\dim(M) = 0$.

Proof Let $\mathfrak{P} \subseteq \mathfrak{Q}$ for prime submodules $\mathfrak{P}$, $\mathfrak{Q}$ of M such that $(\mathfrak{P}:M) \subseteq (\mathfrak{Q}:M) \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. Then we have $\mathfrak{P}_{\mathfrak{m}} \subseteq \mathfrak{Q}_{\mathfrak{m}}$. Since $M_{\mathfrak{m}}$ is strongly 0-dimensional, $\dim(M_{\mathfrak{m}}) = 0$ by Proposition 2.5. So we get $\mathfrak{P}_{\mathfrak{m}} = \mathfrak{Q}_{\mathfrak{m}}$. It follows that $\mathfrak{P} = \mathfrak{Q}$ by Lemma 3.2. ■

Theorem 3.4 *Let M be a finitely generated faithful multiplication R-module. If M is reduced and $M_{\mathfrak{m}}$ is a strongly* 0*-dimensional module for all maximal ideals $\mathfrak{m}$ of R, then M is a von Neumann regular module.*

Proof This follows from Lemma 3.3 and Proposition 3.1. ■

Theorem 3.5 *Let M be a faithful multiplication R-module. If M is finitely generated, then M is a von Neumann regular module if and only if every submodule N of M is idempotent.*

Proof Let M be a von Neumann regular module and N be a submodule of M. It is clear that $(N:M)N \subseteq (N:M)M = N$. Now let $m \in N$. Then there exists an element $r \in (\mathrm{Rm}:M)$ and $a \in R$ such that $m = \mathrm{ram}$. Since $(\mathrm{Rm}:M) \subseteq (N:M)$, we get $m = \mathrm{ram} \in (N:M)N$. Hence $N = (N:M)N$. Now for the converse, suppose that every submodule is idempotent and let $m \in M$. Then $\mathrm{Rm} = (\mathrm{Rm}:M)\,\mathrm{Rm}$. Thus $m \in (\mathrm{Rm}:M)m$, so we get $m = rm$ for some $r \in (\mathrm{Rm}:M)$. Consequently M is a von Neumann regular module. ■

Proposition 3.6 ([1, Corollary 11]) *Let R be a ring and M a faithful multiplication R-module. If R is von Neumann regular, then M is von Neumann regular, and the converse is true if $M \neq PM$ for all prime ideals P of R.*

Note that if the faithful multiplication R-module M is finitely generated, then $M \neq PM$ for all prime ideal P of R by [2, Theorem 3.1].

Theorem 3.7 *Let M be a faithful multiplication R-module. Consider the following three statements:*

(i) *M is a von Neumann regular module.*
(ii) *Every primary submodule of M is a maximal submodule.*
(iii) *Every primary submodule of M is a minimal prime submodule.*

If M is finitely generated, (i)⇒(ii)⇒(iii) *is always true. And* (iii)⇒(i) *is true if M is a reduced module.*

Proof Let M be a finitely generated module.

(i)⇒(ii) Suppose that M is von Neumann regular and N is a primary submodule of M. It is sufficient to verify that N is a prime submodule, because in a such case we get N is a maximal submodule since M is von Neumann regular. Now let $rm \in N$ for some $r \in R$ and $m \in M$. Then $m \in N$ or $r^k \in (N:M)$ for some positive integer k. If $m \in N$, we are done. If $r^k \in (N:M)$, then we get $r = rxr = \cdots = r^k x^{k-1} \in (N:M)$ for some $x \in R$, since R is a von Neumann regular ring. Thus N is a prime submodule of M.

(ii)⇒(iii) Suppose that every primary submodule of M is a maximal submodule. Since every prime submodule is primary, there is no strict chain of prime submodules. Thus every primary submodule is a minimal prime submodule.

Now assume that M is a reduced module.

(iii)⇒(i) Suppose that every primary submodule of M is a minimal prime submodule. Thus every prime submodule is a minimal prime submodule, and we get $\dim(M) = 0$. Since M is reduced, we have $\mathrm{Nil}(M) = 0$. So by Proposition 3.1, M is a von Neumann regular module. ■

Proposition 3.8 ([1, Proposition 14]) *Let R be a ring and N a submodule of an R-module M. Let $H = \{m \in M : (\mathrm{Rm} : M)^k m \subseteq N$ for some positive integer $k\}$. If M is multiplication, then H is a submodule of M. Assuming further that M is finitely generated, we get $H = M\text{-rad}\,N$.*

Proposition 3.9 *Let M be a finitely generated multiplication R-module and N a submodule of M. If M is a von Neumann regular module, then $N = M\text{-rad}\,N$.*

Proof It is known that $N \subseteq M\text{-rad}\,N$. For the converse, let $m \in M\text{-rad}\,N$. Then $(\mathrm{Rm} : M)^k m \subseteq N$ for some $k \in \mathbb{Z}^+$. Since M is von Neumann regular, there exists $r \in (\mathrm{Rm} : M)$ and $a \in R$ such that $m = \mathrm{ram}$. From this we get $m = \mathrm{ram} = r^2a^2m = \cdots = r^k a^k m$, and thus $m \in N$. ■

Recall that an R-module M is said to be a Q-module if every proper submodule N of M is of the form IM, where I is a finite product of primary ideals of R [3].

Lemma 3.10 ([1, Theorem 6]) *Let R be a ring and M a faithful R-module.*

(i) $\mathrm{Nil}(M)$ *is a submodule of M.*
(ii) *Assuming M is multiplication, we get* $\mathrm{Nil}(M) = \mathrm{Nil}(R)M = \bigcap_{Q \in \mathrm{Spec}(M)} Q$.

Theorem 3.11 *Let M be a finitely generated faithful multiplication R-module. Then M is a 0-dimensional Q-module if and only if $M/\mathrm{Nil}(M)$ is a Noetherian von Neumann regular $R/\mathrm{Nil}(R)$-module.*

Proof Let M be a 0-dimensional Q-module, and so $M/\mathrm{Nil}(M)$ is a 0-dimensional $R/\mathrm{Nil}(R)$-module. Furthermore we get $\mathrm{Nil}(M/\mathrm{Nil}(M)) = 0$ from [1, Corollary 7], and so $M/\mathrm{Nil}(M)$ is a von Neumann regular module by Proposition 3.1. Now, since M is a Q-module, we get that R is a Q-ring by [3, Theorem 1]. Since $\dim(M) = 0$, we have $\dim(R) = 0$. Thus $R/\mathrm{Nil}(R)$ is Noetherian from [4, Theorem 2.12], and so $M/\mathrm{Nil}(M)$ is Noetherian. For the converse, let $M/\mathrm{Nil}(M)$ be a Noetherian von Neumann regular $R/\mathrm{Nil}(R)$-module. Then $\dim(M/\mathrm{Nil}(M)) = 0$ and $\mathrm{Nil}(M/\mathrm{Nil}(M)) = 0$ by Proposition 3.1. Since $M/\mathrm{Nil}(M)$ is Noetherian,

$$R/\mathrm{Ann}(M/\mathrm{Nil}(M)) = R/\mathrm{Nil}(R)$$

is a Noetherian ring. Thus $R/\mathrm{Nil}(R)$ is a von Neumann regular ring by Proposition 3.6. Hence R is a 0-dimensional Q-ring by [4, Theorem 2.12]. Consequently, M is a 0-dimensional Q-module by [3, Theorem 1]. ■

Theorem 3.12 *Let M be a faithful multiplication R-module. If M is finitely generated, then M is a reduced strongly 0-dimensional module if and only if M is a Noetherian von Neumann regular module.*

Proof Let M be a reduced strongly 0-dimensional module. Since M is reduced, $\mathrm{Nil}(M) = 0$. Since M is strongly 0-dimensional module, $\dim(M) = 0$ from Proposition 2.5. Now by Proposition 3.1, M is a von Neumann regular module. Now, since $\mathrm{Nil}(M) = \mathrm{Nil}(R)M = 0$, we get $\mathrm{Nil}(R) = 0$. That is, R is a reduced ring. Since M is a strongly 0-dimensional module, R is a strongly 0-dimensional ring, and so by [4, Theorem 2.10] R is Noetherian. Hence M is a Noetherian module. Conversely, let M be a Noetherian von Neumann regular module. Then $\dim(M) = 0$ and $\mathrm{Nil}(M) = 0$. So M is reduced. Since M is Noetherian, M is an Artinian module [9, p. 180, Theorem 2]. Thus M is a strongly 0-dimensional module by Theorem 2.6. ∎

Corollary 3.13 *Let M be a finitely generated faithful multiplication R-module. If M is reduced, then the following statements are equivalent:*

(i) *M is a strongly 0-dimensional module.*
(ii) *M is a Noetherian von Neumann regular module.*
(iii) *M is a 0-dimensional Q-module.*

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