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PROJECTIVE CHARACTER VALUES ON REAL AND RATIONAL ELEMENTS

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Abstract

Let α be a complex-valued 2-cocycle of a finite group G with α chosen so that the α-characters of G are class functions and analogues of the orthogonality relations for ordinary characters are valid. Then the real or rational elements of G that are also α-regular are characterised by the values that the irreducible α-characters of G take on those respective elements. These new results generalise two known facts concerning such elements and irreducible ordinary characters of G; however, the initial choice of α from its cohomology class is not unique in general and it is shown the results can vary for a different choice.

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1. Introduction

Throughout this paper G will denote a finite group.

DEFINITION 1.1. A 2-*cocycle* of G over $\mathbb{C}$ is a function $\alpha : G \times G \to \mathbb{C}^*$ such that $\alpha(1, 1) = 1$ and $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x, y, z \in G$.

The set of all such 2-cocycles of G forms a group $Z^2(G, \mathbb{C}^*)$ under multiplication. Let $\delta : G \to \mathbb{C}^*$ be any function with $\delta(1) = 1$. Then $t(\delta)(x, y) = \delta(x)\delta(y)/\delta(xy)$ for all $x, y \in G$ is a 2-cocycle of G, which is called a *coboundary*. Two 2-cocycles α and β are *cohomologous* if there exists a coboundary $t(\delta)$ such that $\beta = t(\delta)\alpha$. This defines an equivalence relation on $Z^2(G, \mathbb{C}^*)$ and the *cohomology classes* $[\alpha]$ form a finite abelian group, called the *Schur multiplier* $M(G)$.

DEFINITION 1.2. Let α be a 2-cocycle of G. Then $g \in G$ is α-*regular* if $\alpha(g, h) = \alpha(h, g)$ for all $h \in C_G(g)$.

Setting $y = z = 1$ in Definition 1.1 yields $\alpha(x, 1) = 1$ and similarly $\alpha(1, x) = 1$ for all $x \in G$, hence 1 is α-regular. Let $\beta \in [\alpha]$. Then $g \in G$ is α-regular if and only if it is β-regular. If g is α-regular then any conjugate of g is also α-regular, so one may refer to the α-regular conjugacy classes of G (see [3, Problem 11.4]). Finally, if $m \in \mathbb{N}$ is relatively prime to $o(g)$, then it is easy to show g^m is α-regular.

DEFINITION 1.3. Let α be a 2-cocycle of G. Then an *α-representation* of G of *dimension n* is a function $P : G \to \mathrm{GL}(n, \mathbb{C})$ such that $P(g)P(h) = \alpha(g, h)P(gh)$ for all $g, h \in G$.

To avoid repetition all α-representations of G in this paper are defined over $\mathbb{C}$. An α-representation is also called a *projective* representation of G with 2-cocycle α and its trace function is its *α-character*. Let $\mathrm{Proj}(G, \alpha)$ denote the set of all irreducible α-characters of G. The relationship between $\mathrm{Proj}(G, \alpha)$ and α-representations is much the same as that between $\mathrm{Irr}(G)$ and ordinary representations of G (see [4, page 184] for details). The following known results concerning α-representations and characters may all be found in [3, Problems 11.7 and 11.8] and [1, Sections 1 and 4]. First, $\sum_{\xi \in \mathrm{Proj}(G,\alpha)} \xi(1)^2 = |G|$. Next $g \in G$ is α-regular if and only if $\xi(g) \neq 0$ for some $\xi \in \mathrm{Proj}(G, \alpha)$ and $|\mathrm{Proj}(G, \alpha)|$ is the number of α-regular conjugacy classes of G. For $[\beta] \in M(G)$ there exists $\alpha \in [\beta]$ such that $o(\alpha) = o([\beta])$ and α is *class-preserving*, that is, the elements of $\mathrm{Proj}(G, \alpha)$ are class functions. Henceforward it will be assumed that the initial choice of 2-cocycle α has these two properties, but the choice made within such 2-cocycles will affect the results obtained in Section 2. Under these assumptions the 'standard' inner product $\langle \ , \ \rangle$ may be defined on α-characters of G and the 'normal' orthogonality relations hold.

DEFINITION 1.4. Let $g \in G$. Then g is a *real* element if g is conjugate to g^{-1}, and g is a *rational* element if g is conjugate to g^m for all $m \in \mathbb{N}$ with m relatively prime to $o(g)$.

Clearly every rational element of G is real; also G contains a nontrivial real element if and only if $|G|$ is even. The next two theorems are standard results in ordinary character theory concerning real and rational elements (see [3, Problems 2.11 and 2.12] and [6, Exercise XVIII.14]).

THEOREM 1.5. *Let $g \in G$. Then $\chi(g)$ is real for all $\chi \in \mathrm{Irr}(G)$ if and only if g is a real element.*

THEOREM 1.6. *Let $g \in G$. Then the following statements are equivalent:*

(a) *$\chi(g)$ is rational for all $\chi \in \mathrm{Irr}(G)$;*
(b) *g is conjugate to g^m for all $m \in \mathbb{N}$ with m relatively prime to $|G|$;*
(c) *g is a rational element.*

In Section 2, these two results will be generalised to irreducible α-characters and an α-regular real or rational element of G.

2. Values of α-characters

Let P be an α-representation of G of dimension n with α-character ξ. Then $P(g)P(g^{-1}) = \alpha(g, g^{-1})I_n$ for any $g \in G$, and hence $P(g^{-1}) = \alpha(g, g^{-1})P(g)^{-1}$. It follows

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TABLE 1. α-character table of S_4.

	1	(1 2 3)	(1 2 3 4)
ξ_1	2	1	$-\sqrt{2}$
ξ_2	2	1	$\sqrt{2}$
ξ_3	4	-1	0

that $\xi(g^{-1}) = \alpha(g, g^{-1})\overline{\xi(g)}$, where the bar denotes complex conjugation (see [5, Lemma 1.11.11]).

THEOREM 2.1. *Let α be a 2-cocycle of G and let $g \in G$ be α-regular. Then g is a real element if and only if $\xi(g) = \pm|\xi(g)|\omega$ for all $\xi \in \mathrm{Proj}(G, \alpha)$, where $\omega^2 = \alpha(g, g^{-1})$.*

PROOF. Suppose g is real and let $\xi \in \mathrm{Proj}(G, \alpha)$ such that $\xi(g) \neq 0$. Then $\alpha(g, g^{-1})\overline{\xi(g)} = \xi(g)$ and the choice of α from Section 1 implies $\alpha(g, g^{-1})$ is a root of unity. Choose ω such that $\omega^2 = \alpha(g, g^{-1})$. Then $\xi(g)^2 = |\xi(g)|^2\omega^2$ and so $\xi(g) = \pm|\xi(g)|\omega$.

Conversely, suppose $\xi(g) = \pm|\xi(g)|\omega$ for all $\xi \in \mathrm{Proj}(G, \alpha)$, where $\omega^2 = \alpha(g, g^{-1})$. Then

$$\sum_{\xi \in \mathrm{Proj}(G,\alpha)} \xi(g)\overline{\xi(g^{-1})} = \overline{\alpha(g, g^{-1})}\omega^2 \sum_{\xi \in \mathrm{Proj}(G,\alpha)} |\xi(g)|^2 = |C_G(g)|,$$

and hence by the second orthogonality relation for α-characters g is conjugate to g^{-1}. □

Let $g \in G$ be α-regular. From Theorem 2.1, if $\alpha(g, g^{-1}) = 1$ or -1, then g is a real element if and only if $\xi(g)$ is real or purely imaginary, respectively, for all $\xi \in \mathrm{Proj}(G, \alpha)$. It should be noted that the root of unity ω that occurs in Theorem 2.1 depends upon the choice of α, as the next example illustrates.

EXAMPLE 2.2. Every element of the symmetric group S_4 is rational and $M(S_4)$ is cyclic of order 2. Also S_4 has two *Schur representation* groups (also known as *covering* groups) up to isomorphism (see [4, Theorem 12.2.2]). One is the binary octahedral group and an α-character table of S_4 for $o(\alpha) = 2$ constructed from this group is given in Table 1 (see [5, Theorem 5.6.4]). We deduce that $\alpha(g, g^{-1}) = 1$ for all α-regular $g \in S_4$. The other Schur representation group is $\mathrm{GL}(2, 3)$ and it is easy to check that a β-character table of S_4 for $o(\beta) = 2$ constructed from this group is identical to Table 1 except that the three entries in the last column are multiplied by i, so $\beta((1\ 2\ 3\ 4), (1\ 2\ 3\ 4)^{-1}) = -1$.

Two variations of Theorem 2.1 are discussed next, the first of which is easy to see.

COROLLARY 2.3. *Let α be a 2-cocycle of G and let $g \in G$ be α-regular. Then g is a real element if and only if $\xi(g)^2\alpha^{-1}(g, g^{-1}) \in \mathbb{R}_{\geq 0}$ for all $\xi \in \mathrm{Proj}(G, \alpha)$.*

PROOF. Let $\xi \in \mathrm{Proj}(G,\alpha)$ and suppose $\xi(g)^2\alpha^{-1}(g,g^{-1}) = r$ for $r \in \mathbb{R}_{\geq 0}$. Then $r = |\xi(g)|^2$ and the result follows from Theorem 2.1. □

Suppose g is an α-regular real element of G. Then it was shown in Theorem 2.1 that $\xi(g)$ lies on a line in the complex plane of the form $\{r\omega : r \in \mathbb{R}\}$ for all $\xi \in \mathrm{Proj}(G,\alpha)$, where $|\omega| = 1$. Conversely, this latter condition is sufficient to guarantee that an α-regular element g of G is a real element.

COROLLARY 2.4. *Let α be a 2-cocycle of G and let $g \in G$ be α-regular. Then g is a real element if and only if there exists an $\omega \in \mathbb{C}$ such that $\xi(g) = \pm|\xi(g)|\omega$ for all $\xi \in \mathrm{Proj}(G,\alpha)$.*

PROOF. Suppose the second condition holds. Then, using the same argument as that at the end of the proof of Theorem 2.1, it must be the case that the product of ω^2 and the root of unity $\overline{\alpha(g,g^{-1})}$ is 1 and so g is a real element from Theorem 2.1. The converse obviously holds from Theorem 2.1. □

Note that $\omega^2 = \alpha(g,g^{-1})$ from Theorem 2.1 or the proof of Corollary 2.4. So ω is a $|G|$th root of unity if $|G|$ is even (see [4, Theorem 10.11.1]). If $|G|$ is odd, then just one of ω and $-\omega$ is a $|G|$th root of unity.

Rational elements are now considered. Continuing with the notation at the start of this section, an easy proof by induction shows $P(g)^m = f_\alpha(g,m)P(g^m)$ for any $g \in G$ and any $m \in \mathbb{N}$, where $f_\alpha(g,1) = 1$ and

$$f_\alpha(g,m) = \alpha(g,g)\cdots\alpha(g,g^{m-1}) \quad \text{for } m > 1.$$

Let ζ be a primitive $|G|$th root of unity. Then $\xi(g) \in \mathbb{Q}[\zeta]$ and is an algebraic integer for any $g \in G$ (see [5, Corollary 1.2.7]). If $(m,|G|) = 1$ then, as shown in the proof of [2, Theorem 2],

$$\xi(g^m) = f_\alpha^{-1}(g,m)\sigma_m(\xi(g)),$$

where σ_m is the automorphism of $\mathbb{Q}[\zeta]$ over $\mathbb{Q}$ that maps ζ to ζ^m. The Galois group of $\mathbb{Q}[\zeta]$ over $\mathbb{Q}$ is abelian and σ_{-1} represents the restriction of complex conjugation to $\mathbb{Q}[\zeta]$. Thus for all $z \in \mathbb{Q}[\zeta]$, $\sigma_m(\bar{z}) = \overline{\sigma_m(z)}$ and $\sigma_m(|z|^2) = |\sigma_m(z)|^2$. So $|\xi(g^m)|^2 = \sigma_m(|\xi(g)|^2)$.

THEOREM 2.5. *Let α be a 2-cocycle of G and let $g \in G$ be α-regular. Then g is conjugate to g^m for all $m \in \mathbb{N}$ that are relatively prime to $|G|$ if and only, if for all $\xi \in \mathrm{Proj}(G,\alpha)$,*

(a) *there exists a $|G|$th root of unity ω with $\omega^2 = \alpha(g,g^{-1})$ such that $\xi(g) = \pm|\xi(g)|\omega$ and*
(b) *either $\sigma_m(|\xi(g)|) = |\xi(g)|$ and $f_\alpha(g,m) = \omega^{m-1}$, or $\sigma_m(|\xi(g)|) = -|\xi(g)|$ and $f_\alpha(g,m) = -\omega^{m-1}$.*

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PROOF. Suppose g is conjugate to g^m for all $m \in \mathbb{N}$ with $(m, |G|) = 1$. Then, in particular, g is a real element of G from Theorem 1.6. Thus $\xi(g) = \pm|\xi(g)|\omega$ for all $\xi \in \mathrm{Proj}(G, \alpha)$, where $\omega^2 = \alpha(g, g^{-1})$ by Theorem 2.1. If $g = 1$, then (a) and (b) hold with $\omega = 1$ and so, as previously noted, in all cases ω is a $|G|$th root of unity. By supposition $\xi(g) = \xi(g^m)$ and so $|\xi(g)|^2 = \sigma_m(|\xi(g)|^2)$ for all such m. Thus $|\xi(g)|^2 \in \mathbb{Q}_{\geq 0}$. Also

$$\pm|\xi(g)|\omega = f_\alpha^{-1}(g, m)\sigma_m(\pm|\xi(g)|\omega) = \pm f_\alpha^{-1}(g, m)\sigma_m(|\xi(g)|)\omega^m,$$

and consequently

$$|\xi(g)| = f_\alpha^{-1}(g, m)\sigma_m(|\xi(g)|)\omega^{m-1}.$$

Now $\sigma_m(|\xi(g)|) = \pm|\xi(g)|$. For the positive sign the conclusion is $f_\alpha(g, m) = \omega^{m-1}$, since $\xi(g) \neq 0$ for some $\xi \in \mathrm{Proj}(G, \alpha)$, and similarly for the negative sign.

Conversely, suppose (a) and (b) are true for all $m \in \mathbb{N}$ with $(m, |G|) = 1$. Then

$$\xi(g^m) = \pm f_\alpha^{-1}(g, m)\sigma_m(|\xi(g)|)\omega^m,$$

with the sign corresponding to that of $\xi(g) = \pm|\xi(g)|\omega$. In either case, using (b),

$$\sum_{\xi \in \mathrm{Proj}(G,\alpha)} \xi(g)\overline{\xi(g^m)} = f_\alpha(g, m)\omega^{1-m} \sum_{\xi \in \mathrm{Proj}(G,\alpha)} |\xi(g)|^2 = |C_G(g)|,$$

and hence by the second orthogonality relation g is conjugate to g^m. □

Suppose α is trivial and g is conjugate to g^m for all $m \in \mathbb{N}$ with $(m, |G|) = 1$. Then with $\omega = 1$, (a) in Theorem 2.5 implies that $\chi(g)$ is real for all $\chi \in \mathrm{Irr}(G)$. In addition, $f_\alpha(g, m) = 1$ for all such m, and so from (b), $|\chi(g)| \in \mathbb{Q}$. Thus $\chi(g) \in \mathbb{Q}$. Conversely, if $\chi(g) \in \mathbb{Q}$ for all $\chi \in \mathrm{Irr}(G)$, then (a) and (b) in Theorem 2.5 obviously hold with $\omega = 1$. So Theorem 2.5 reduces to Theorem 1.6 in this case.

It is possible to replace (a) in Theorem 2.5 by: '(a)′ there exists an $\omega \in \mathbb{C}$ such that $\xi(g) = \pm|\xi(g)|\omega$ and'. Suppose (a)′ and (b) hold. Then $\omega^2 = \alpha(g, g^{-1})$ from the proof of Corollary 2.4. Theorem 2.5 will then still hold using this variation provided ω is a $|G|$th root of unity, which is the case if $|G|$ is even, using the remarks after Corollary 2.4. Suppose $|G|$ is odd and let γ denote the unique $|G|$th root of unity with $\gamma^2 = \alpha(g, g^{-1})$. Now $f_\alpha(g, m)$ is a $|G|$th root of unity, and from (b), $f_\alpha(g, m) = \pm\gamma^{m-1}$ or $\pm(-\gamma)^{m-1}$. Setting $m = 1$ and then 2 shows that $f_\alpha(g, m) = \gamma^{m-1}$, so ω must equal γ in this situation.

Of course, using Theorem 1.6, the conditions in Theorem 2.5 are necessary and sufficient for an α-regular element of G to be a rational element. Also $\mathbb{Q}$ can be replaced by $\mathbb{Z}$ in either formulation of Theorem 2.5, since as previously noted $\xi(g)$ is an algebraic integer for all $\xi \in \mathrm{Proj}(G, \alpha)$ and any $g \in G$. This yields the following useful consequence of Theorem 2.5.

COROLLARY 2.6. *Let α be a 2-cocycle of G and let $g \in G$ be α-regular. If g is a rational element, then $\xi(g)^2\alpha^{-1}(g, g^{-1}) \in \mathbb{Z}_{\geq 0}$ for all $\xi \in \mathrm{Proj}(G, \alpha)$.*

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