Probability in the Engineering and Informational Sciences (2024), **38**:4 619–631
doi:10.1017/S0269964824000019

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Analyzing the multi-state system under a run shock model

Murat Ozkut[1], Cihangir Kan[2] and Ceki Franko[2]

[1]Department of Mathematics, Izmir University of Economics, Izmir, Turkey
[2]Department of Financial and Actuarial Mathematics, School of Mathematics and Physics, Xi'an Jiaotong-Liverpool University, Suzhou, Jiangsu, China
Corresponding author: Murat Ozkut; Email: murat.ozkut@ieu.edu.tr

Keywords: mean residual life; multi-state system; phase-type distribution; shock model

Abstract
A system experiences random shocks over time, with two critical levels, d_1 and d_2, where $d_1 < d_2$. k consecutive shocks with magnitudes between d_1 and d_2 partially damaging the system, causing it to transition to a lower, partially working state. Shocks with magnitudes above d_2 have a catastrophic effect, resulting in complete failure. This theoretical framework gives rise to a multi-state system characterized by an indeterminate quantity of states. When the time between successive shocks follows a phase-type distribution, a detailed analysis of the system's dynamic reliability properties such as the lifetime of the system, the time it spends in perfect functioning, as well as the total time it spends in partially working states are discussed.

1. Introduction

The time interval between consecutive shocks or the shock-induced damage usually defines system failure in shock models. Various shock models have been proposed and examined in the literature, which can be categorized into five groups: extreme shock model [2, 6], run shock model [19, 22], δ-shock model [14, 15], cumulative shock model [8, 26], and mixed shock model [9, 27]. The mixed shock model combines at least two different shock models. In [7], for instance, system fails upon the occurrence of k_1 consecutive shocks of size between d_1 and d_2 or a single large shock of size at least d_2. Recently, Ozkut [21] extended this model by combining two run shock models. One can see other mixed shock models in [11, 20].

The extreme shock model states that a system will fail if an individual shock surpasses a certain level, denoted by d [24]. The time between shocks is represented by T_i, while the magnitude of the shock is represented by Y_i. The lifetime of the system, S, is calculated as $S = \sum_{i=1}^{N} T_i$, with the stopping random variable N defined as $\{N = n\} \equiv \{Y_1 \leq d, \dots, Y_{n-1} \leq d, Y_n > d\}$. This model has been explored in several studies [4, 10, 28].

Particularly in contemporary real-world scenarios, it is imperative to take into account the extensive array of potential system states. Furthermore, the escalating requisites for system assessment and design pose challenges to the applicability of conventional binary systems. The conceptualization of multi-state systems was initially propounded by the work of [13] in the year 1968. Subsequent to this seminal contribution, investigations into the realm of multi-state systems have been ongoing in fields, such as reliability theory, strategic decision theory, and health-care [1, 16, 17].

Many researchers pay much more attention to the shock models in binary settings, but there are very few studies on their extensions to multi-state systems [5, 29]. Zhao et al. [29] classified shocks into three types , such as the highest, medium and the lowest impact on the system, according to the severity of the impact of the shock event on the environment. Eryilmaz [5] presented an extreme shock model in which a large shock causes complete failure, as in the classical model, for a multi-state system. In such a shock model, the system fails when a shock with large magnitude occurs. If the magnitude of a shock is between two critical levels, it has partial damage and leads the system to a lower state. although the working state gets lower and lower, the system never breaks down, no matter how many shocks with partial damage occur before a large shock. In many real life incidence transitions to lower state occurs due to occurrence of k consecutive shocks which is a more general form of the precedence. More precisely, if k consecutive shocks' magnitude fall between two critical levels d_1 and d_2, $(d_1 < d_2)$, the system transitions to a partially working state with reduced capacity. Therefore, the present paper generalizes the extreme shock models given in [5] to run shock models in multi-state systems. That is, the proposed model becomes extreme shock model when $k = 1$. The new extension is more flexible. Consequently, we will able to model more real life situations. In credit risk modeling, shocks can be related to economical events such as crisis and recessions [18], in network modeling, they can be cyber attacks like distributed denial of service (DDoS) [25], and in insurance, they are natural disasters or accidents. For example, an insurance companies updates credit scores (states) of their customers after random accidents (shocks). Each accident has different costs (shock magnitude). Consecutive accidents having costs between thresholds d_1 and d_2 will lead to a change in the customers' credit scores. But the customer still continues to benefit from advantages of the insurance. If a catastrophic accident which costs more than d_2 occurs, then customer will no longer get insured. The structure of this paper has been designed in the following way: the system design is outlined in Section 2, while a detailed analysis of the system's dynamic reliability properties is provided in Section 3. Finally, a numerical illustration is given in Section 4.

2. System design

Suppose there are two critical levels, d_1 and d_2, with $d_1 < d_2$. If k consecutive shocks $(k \geq 1)$ with a magnitude between these levels, they will partially damage the system and put it into a lower, partially functional state. Each additional k consecutive shocks in this range further reduces the state of the system by one unit. However, if a shock of larger than d_2, it will have a devastating effect and result in complete failure. The number of states of the system is random, represented by $\Psi(s)$ at time s, where $\Psi(s) \in \left\{0, 1, \ldots, N_v^{(k)} + 1\right\}$. Let $N_v^{(k)}$ be a random variable denoting the number of k consecutive shocks, whose magnitude is between d_1 and d_2 until the first extreme shock above d_2, in a total number of shocks v. The states "$N_v^{(k)} + 1$" and "0" correspond to perfect functioning and complete failure, respectively, with a total of $N_v^{(k)} + 1$ working states $\left\{1, \ldots, N_v^{(k)} + 1\right\}$. At $s = 0$, the system works perfectly and will continue to do so until the first k successive shocks within (d_1, d_2) or a shock above d_2. Let S^j be the amount of time that the system has spent in state j, $j = 1, \ldots, N_v^{(k)} + 1$. The length of time the system will operate at its best can also be represented by $S^{N_v^{(k)}+1}$, and the lifetime of the system is $S = S^1 + \ldots + S^{N_v^{(k)}+1}$. Hence, for $N_v^{(k)} = 0$, $S^{N_v^{(k)}+1}$ is equal to the lifetime of the system, S. Clearly, $S - S^{N_v^{(k)}+1}$ is a random variable denoting the time elapsed following the first k successive shocks between critical levels d_1 and d_2 until the occurrence of the catastrophic shock. More precisely, $S - S^{N_v^{(k)}+1}$ is the time elapsed in partially working states. To gain a deeper insight into the model, a possible realization is presented in Figure 1.

According to Figure 1, for $k = 2$, the system is in the perfect state until the fourth shock occurs since Y_3 and Y_4 are the first two consecutive shocks between d_1 and d_2. After the fourth shock, the system transits into a lower state, and continues to function partially. Then Y_7 and Y_8 are the second two consecutive shocks between d_1 and d_2. After the eighth shock, the system transits into a lower state, and continues to function partially. Finally, a catastrophic shock Y_{11} causes the system failure. In this realization, there are totally two two-consecutive shocks until the system failure. Hence, $N_{11}^{(2)} = 2$ and the system lifetime is $S = S^1 + S^2 + S^3$.

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

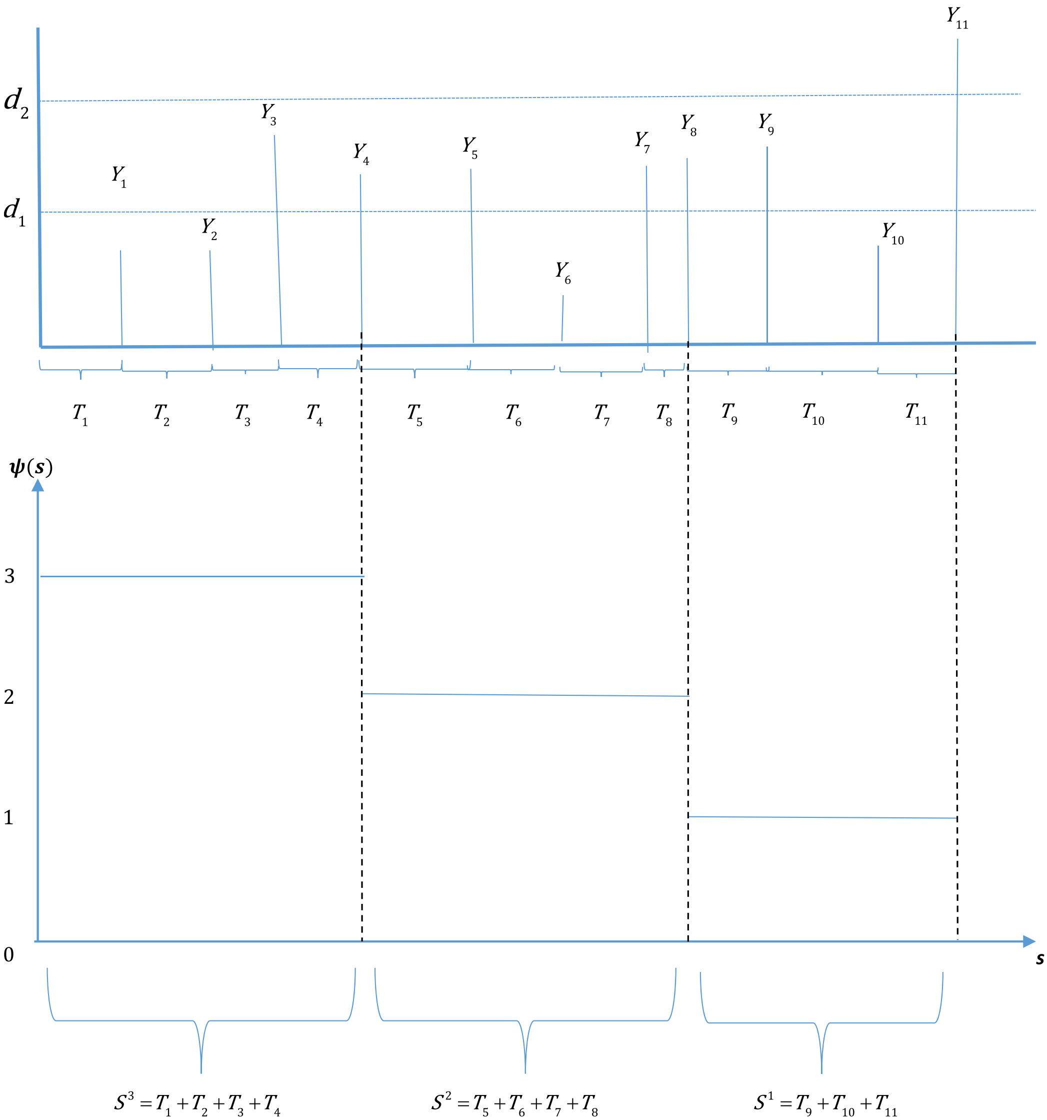

Figure 1. *A potential instance of the system and state variation processes.*

The proposed model can be used in urban planning, power transmission network, network reliability and information technology and cyber security. Security of the system is affected by DDoS attacks (shocks) which makes an online services unavailable to customers by temporarily or permanently cutting the host server. If the network cloud center is subject to consecutive attacks between critical volume levels d_1 and d_2, the cloud will leak information but still operate with a reduced trust index. A complete failure of the server occurs when the volume of the DDoS attack exceeds the volume level d_2. The random variable $S^{N_v^{(k)}+1}$ represents the time that the server has worked with full trust.

2.1. Phase-type distribution

It is worth mentioning that phase-type distributions are useful in reliability evaluations of the systems since they provide a flexible and versatile framework to model complex failure patterns in systems. They are particularly valuable when analyzing systems with multiple phases of operation, such as start-up,

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

steady-state, and shutdown, as well as when considering repair maintenance activities. They also enable the calculation of important reliability metrics such as mean time to failure, system lifetime and aiding in better decision making for system design, maintenance strategies and optimization.

The cumulative distribution function for a non-negative continuous phase-type random variable T is denoted by:

$$P(T \leq x) = 1 - \zeta \exp(\mathbf{\Lambda} x)\, \mathbf{e}',$$

where the dimension of the non-singular matrix $\mathbf{\Lambda}$ is $m \times m$. Furthermore, diagonal and non-diagonal elements are respectively negative and non-negative, and all row sums are non-positive. Additionally, the order of a substochastic vector $\boldsymbol{\zeta}$ is m, and all elements are non-negative, and $\boldsymbol{\zeta}\mathbf{e}' \leq 1$. We shall use $T \sim PH_c(\boldsymbol{\zeta}, \mathbf{\Lambda})$ to represent the continuous phase-type distribution. The expected value of T is given by:

$$E(T) = -\zeta \mathbf{\Lambda}^{-1}\mathbf{e}'.$$

On the other hand, the distribution of the time it takes for an absorbing Markov chain to reach an absorbing state is called a discrete phase-type distribution. The probability mass function (PMF) of a discrete random variable N which has a discrete phase-type distribution is represented by:

$$P(N = n) = \mathbf{a}\mathbf{\Psi}^{n-1}\mathbf{u}',$$

for $n = 1, 2, \ldots$ and the matrix $\mathbf{\Psi} = (\psi_{ij})_{m \times m}$ consists of the transition probabilities among the m transient states, the elements of the vector $\mathbf{u}' = (\mathbf{I} - \mathbf{\Psi})\, \mathbf{e}'$ are transition probabilities from transient states to the absorbing state, $\mathbf{a} = (a_1, \ldots, a_m)$ with $\sum_{i=1}^{m} a_i = 1$, and $\mathbf{I}$ is the identity matrix. $N \sim PH_d(\mathbf{a}, \mathbf{\Psi})$ will be used to represent the random variable N has a discrete phase-type distribution. The next propositions will be useful for the followings.

Proposition 1. *Let $T_1, T_2, \ldots$ be independent and $T_i \sim PH_c(\boldsymbol{\zeta}, \mathbf{\Lambda})$, and independently $N \sim PH_d(\mathbf{a}, \mathbf{\Psi})$. Then*

$$S = \sum_{i=1}^{N} T_i \sim PH_c\left(\boldsymbol{\zeta} \otimes \mathbf{a}, \mathbf{\Lambda} \otimes \mathbf{I} + \left(\mathbf{a}^0 \boldsymbol{\zeta}\right) \otimes \mathbf{\Psi}\right),$$

where $\otimes$ is the Kronecker product [12].

Proposition 2. *Suppose T is a continuous phase-type random variable represented by $T \sim PH_c(\boldsymbol{\zeta}, \mathbf{\Lambda})$. Then, the distribution of $\{T - x | T > x\}$ can be expressed as $PH\left(\frac{\zeta \exp(\mathbf{\Lambda} t)}{\zeta \exp(\mathbf{\Lambda} t) e'}, \mathbf{\Lambda}\right)$ according to He* [12]. *As a result, the mean residual life (MRL) of T can be calculated from:*

$$E(T - x | T > x) = -\frac{\zeta \exp(\mathbf{\Lambda} t)}{\zeta \exp(\mathbf{\Lambda} t)\, e'} \mathbf{\Lambda}^{-1}\mathbf{e}'. \quad (1)$$

3. System evaluation

Let $p_1 = P(Y_i \leq d_1)$, $p_2 = P(Y_i \in (d_1, d_2))$ and $p_3 = P(Y_i > d_2)$ for $d_1 < d_2$ and $i = 1, 2, \ldots$ We consider that inter-arrival times $T_1, T_2, \ldots$ and the magnitude of shocks $Y_1, Y_2, \ldots$ are independent.

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

Let V represent the number of shocks preceding the initial shock exceeding d_2 occurs. Then,

$$P(V = v) = (1 - p_3)^{v-1} p_3,$$

$v = 1, 2, \ldots$

Given that the random variable V follows a geometric distribution, Proposition 1 can be applied to show that the lifetime random variable $S = \sum_{i=1}^{V} T_i$ exhibits a phase-type distribution with a corresponding survival function:

$$P(S > s) = \boldsymbol{\zeta} \exp\left(\left(\boldsymbol{\Lambda} + (1 - p_3)\, \boldsymbol{\Lambda}^0 \boldsymbol{\zeta}\right) s\right) \mathbf{e}'. \tag{2}$$

Let the random variable $N_v^{(k)}$ represents the number of k consecutive shocks whose magnitude is between d_1 and d_2 until the occurrence of the first extreme shock over d_2 in a total number of shocks v. Since the probability that a shock whose magnitude is above d_2 given that it is greater than d_1 is $\frac{p_3}{1-p_1}$, where $p_3 = 1 - p_1 - p_2$, then using Theorem 2.1 in [23],

$$P\left(N_v^{(k)} = x\right) = \left(\sum_{i=0}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \binom{x_1 + \cdots + x_k + x}{x_1, \cdots, x_k, x} p^{v-1} \left(\frac{q}{p}\right)^{x_1 + \cdots + x_k}\right)(1 - p),$$
$$x = 0, 1, \ldots, \left\lfloor \frac{v-1}{k} \right\rfloor,$$

where $p = \frac{p_2}{1-p_1}$ and $q = 1 - \frac{p_2}{1-p_1}$. In this equation, $x = 0, 1, \ldots, \lfloor \frac{v-1}{k} \rfloor$ are the possible states of the system, $x = 0$ indicates there is no k consecutive shocks in total v number of shocks while $x = \lfloor \frac{v-1}{k} \rfloor$ indicates the maximum number of k consecutive shocks in total $v - 1$ number of shocks since the last shock is always the catastrophic one.

In addition, $N_v^{(k)} + 1$ denotes the number of functional states of the system with PMF:

$$P\left(N_v^{(k)} + 1 = x\right) = \left(\sum_{i=0}^{k-1} \sum_{x_1, x_2, \ldots, x_k} \binom{x_1 + \cdots + x_k + x - 1}{x_1, \cdots, x_k, x - 1} p^{v-1} \left(\frac{q}{p}\right)^{x_1 + \cdots + x_k}\right)(1 - p),$$
$$x = 1, \ldots, \left\lfloor \frac{v-1}{k} \right\rfloor + 1,$$

where $p = \frac{p_2}{1-p_1}$ and $q = 1 - \frac{p_2}{1-p_1}$.

Let U_1 be the number of shocks before the first k successive shocks whose magnitude is between d_1 and d_2 occur. Similarly, U_i is the number additional shocks to get the ith k consecutive shocks in (d_1, d_2) after occurrence of the $(i-1)$th k consecutive shocks in (d_1, d_2), $i = 2, \ldots, N_v^{(k)}$ and $U_{N_v^{(k)}+1}$ is the number of additional k consecutive shocks in (d_1, d_2) to get the extreme shock whose magnitude is above d_2.

Lemma 3. *$u_1 + \cdots + u_{x+1} = v, u_1 > 0, \cdots, u_{x+1} > 0, x = 0, 1, \ldots \lfloor \frac{v-1}{k} \rfloor$ and $v = 1, 2, \ldots$*

$$P\left(U_1 = u_1, \ldots, U_{x+1} = u_{x+1}, N_v^{(k)} = x, V = v\right) = \prod_{i=1}^{x+1} P_{U_i,k},$$

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

where

$$P_{U_i,k}=\begin{cases}\displaystyle\sum_{x_2=0}^{U_i-k-1} N(x_2,U_i-k-1,k)p_2^{x_2}p_1^{U_i-k-1-x_2}p_1p_2^k & , \quad U_i>k\\ p_2^k & , \quad U_i=k\end{cases}$$

and

$$P_{U_{x+1},k}=\sum_{x_2=0}^{\min(U_{x+1}-1,k)} N(x_2,U_{x+1}-1-x_2+1,k)p_2^{x_2}p_1^{U_{x+1}-1-x_2}p_3$$

$i=1,\ldots x$, *where*

$$N(d,n,k)=\begin{cases}\binom{n}{d} & \text{if } d<n\\ 0 & \text{if } n=d\geq k\\ \displaystyle\sum_{i=0}^{k-1}N(d-i,n-1-i,k) & \text{if } n>d\geq k\end{cases}$$

[3].

Corollary 4. *The joint distribution of U_1 and V is:*

$$P(U_1=u,V=v)$$
$$=\begin{cases}\displaystyle\sum_{x_2=0}^{u-k-1-\lfloor\frac{u-k-1}{k}\rfloor} N(x_2,u-k-1,k)p_1^{u-k-x_2}(p_1+p_2)^{v-1-u}p_2^{k+x_2}p_3 & \text{if } u<v \text{ and } u\geq k\\ \left[\displaystyle\sum_{x_{12}=0}^{v-1-\lfloor\frac{v-1}{k}\rfloor} N(x_{12},v-k-1,k)p_1^{v-1-x_{12}}p_2^{x_{12}}p_3\right]\times & \text{if } u-v\geq k\\ \left[\displaystyle\sum_{x_{22}=0}^{u-v-1-k-\lfloor\frac{u-v-1-k}{k}\rfloor} N(x_{22},u-v-1-k,k)\,(p_1+p_3)\,p_2^{k+x_{22}}\times\right.\\ \left.\displaystyle\sum_{j=0}^{u-v-1-k-x_{22}}\binom{u-v-1-k}{j}p_1^jp_3^{u-v-1-k-x_{22}-j}\right]\\ 0 & \text{otherwise}\end{cases}$$

Corollary 5.

$$P(V<U_1)=\sum_{v=1}\sum_{u=v+1}P(U_1=u,V=v)$$

and

$$P(U_1<V)=\sum_{u=1}\sum_{v=u+1}P(U_1=u,V=v).$$

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

Theorem 6. *Let T_i denote the inter-arrival time between the $(i-1)$th and ith shocks, $i \geq 2$. The survival function of $S^{N_v^{(k)}+1}$ can be calculated by:*

$$P\left(S^{N_v^{(k)}+1} > s\right) = (\boldsymbol{\zeta} \otimes \mathbf{a}) \exp\left(\left(\boldsymbol{\Lambda} \otimes \mathbf{I} + \left(\mathbf{a}^0 \boldsymbol{\zeta}\right) \otimes \boldsymbol{\Psi}\right) s\right) \mathbf{e}', \tag{3}$$

where $\mathbf{a} = (1, 0, \ldots, 0)$ *and* $\boldsymbol{\Psi} = \begin{bmatrix} 1-p_2-p_3 & p_2 & 0 & \cdots & 0 \\ 1-p_2-p_3 & 0 & & \cdots & 0 \\ \vdots & & & \ddots & \\ 1-p_2-p_3 & 0 & 0 & \cdots & p_2 \\ 1-p_2-p_3 & 0 & 0 & \cdots & 0 \end{bmatrix}_{k \times k}.$

Proof. $S^{N_v^{(k)}+1} = \sum_{i=1}^{N} T_i$ denotes the amount of time the system operates at its best performance, where N is the number of shocks until k consecutive shocks whose magnitude is between d_1 and d_2 or a single shock whose magnitude is greater than d_2, that is $N = \min(U_1, V)$. The proof is completed by using proposition 1 and Lemma 3 in [7]. □

Theorem 7.

$$P\left(S - S^{N_v^{(k)}+1} > s\right) = P(S > s)\, P(V > U_1), \tag{4}$$

and

$$P\left(S - S^{N_v^{(k)}+1} = 0\right) = P(V < U_1).$$

Proof. From Corollary 4, we have:

$$P\left(S - S^{N_v^{(k)}+1} > s\right) = P\left(\sum_{i=1}^{V-U_1} T_i > s | V > U_1\right) P(V > U_1).$$

When $V > U_1$, $V - U_1$ is geometrically distributed having mean $1/p_3$. Therefore,

$$P\left(\sum_{i=1}^{V-U_1} T_i > s | V > U_1\right) = P\left(\sum_{i=1}^{V^*} T_i > s\right) = P(S > s),$$

where $P(V^* = v) = p_3(1-p_3)^{v-1}, v = 1, 2, \ldots$ Thus,

$$P\left(S - S^{N_v^{(k)}+1} > s\right) = P(S > s)\, P(V > U_1).$$

□

Theorem 8.

$$P\left(S^{N_v^{(k)}+1} > s_1, S > s_2\right) = P(S > s_1) - P(V > U_1)P(S > s_2 - s_1)\left(1 - P(S^{N_v^{(k)}+1} > s_1)\right), \tag{5}$$

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

Proof. When $s_1 < s_2$, the joint distribution of S and $S^{N_v^{(k)}+1}$ can be expressed as:

$$\begin{aligned} P\left(S^{N_v^{(k)}+1} \leq s_1, S \leq s_2\right) &= P\left(S^{N_v^{(k)}+1} \leq s_1, S \leq s_2, N_v^{(k)} = 0\right) + \\ &\quad P\left(S^{N_v^{(k)}+1} \leq s_1, S \leq s_2, N_v^{(k)} \neq 0\right) \\ &= P\left(S \leq s_1, N_v^{(k)} = 0\right) + P\left(S^{N_v^{(k)}+1} \leq s_1, S - S^{N_v^{(k)}+1} \leq s_2 - s_1, N_v^{(k)} \neq 0\right) \\ &= P\left(S \leq s_1 | V < U_1\right) P\left(V < U_1\right) + \\ &\quad P\left(S^{N_v^{(k)}+1} \leq s_1, S - S^{N_v^{(k)}+1} \leq s_2 - s_1 | V > U_1\right) P\left(V > U_1\right). \end{aligned} \tag{6}$$

Clearly,

$$P\left(S \leq s_1 | V < U_1\right) = P\left(S^{N_v^{(k)}+1} \leq s_1\right). \tag{7}$$

Conversely, for $V > U_1$, $V - U_1$ and U_1 are independent random variables and

$$P\left(U_1 = u, V - U_1 = a | V > U_1\right) = p_1^{u-1}(1-p_1)(1-p_3)^{a-1}p_3.$$

Thus, from equation (4),

$$\begin{aligned} &P\left(S^{N_v^{(k)}+1} \leq s_1, S - S^{N_v^{(k)}+1} \leq s_2 - s_1 | V > U_1\right) \\ &= P\left(\sum_{i=1}^{U_1} T_i \leq t, \sum_{i=1}^{V-U_1} T_i \leq s_2 - s_1 | V > U_1\right) \\ &= P\left(\sum_{i=1}^{U_1} T_i \leq s_1 | V > U_1\right) P\left(\sum_{i=1}^{V-U_1} T_i \leq s_2 - s_1 | V > U_1\right) \\ &= P\left(S^{N_v^{(k)}+1} \leq s_1\right) P\left(S \leq s_2 - s_1\right). \end{aligned} \tag{8}$$

Using (7) and (8) in (6) one obtains,

$$\begin{aligned} &P\left(S^{N_v^{(k)}+1} \leq s_1, S \leq s_2\right) \\ &= P\left(S^{N_v^{(k)}+1} \leq s_1\right) \left[P\left(V < U_1\right) + P\left(S \leq s_2 - s_1\right)\left(1 - P\left(V < U_1\right)\right)\right]. \end{aligned}$$

The proof follows from:

$$P\left(S^{N_v^{(k)}+1} > s_1, S > s_2\right) = P\left(S^{N_v^{(k)}+1} > s_1\right) + P\left(S > s_2\right) - 1 + P\left(S^{N_v^{(k)}+1} \leq s_1, S \leq s_2\right).$$

□

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

Table 1. *Expectation of random variables* S, $S^{N_\nu^{(k)}+1}$, *and* $S - S^{N_\nu^{(k)}+1}$.

p_1	p_2	k	$E(S)$	$E(S^{N_\nu^{(k)}+1})$	$E(S - S^{N_\nu^{(k)}+1})$
0.4	0.5	1	10	1.667	8.333
		2	10	3.75	6.25
		3	10	5.83	4.17
0.3	0.6	1	10	1.429	8.571
		2	10	3.077	6.923
		3	10	4.757	5.243
0.3	0.5	1	5	1.429	3.571
		2	5	2.727	2.273
		3	5	3.684	1.316

In Table 1, the expectation of random variables S, $S^{N_\nu^{(k)}+1}$, and $S - S^{N_\nu^{(k)}+1}$ are calculated. It is worth noting that as the value of k increases, so does the duration of the system in a perfect state.

3.1. Mean residual life functions

The system's MRL is defined by:

$$E\left(S - s|S > s\right) = \frac{1}{P\left(S > s\right)} \int_0^\infty P\left(S > s + x\right) dx,$$

since the continuous random variable S is non-negative.

The event $\{S > s\}$ denotes the survival of the system beyond time s, which is appropriate for estimating binary systems' the MRL. Conversely, the expression:

$$E(S - s|S^{N_\nu^{(k)}+1} > s),$$

captures the system's mean residual lifetime, conditional on its optimal performance at time s. This expression reveals the expected duration until failure while the system is functioning at its maximum capacity at time s.

Proposition 9.

$$\begin{aligned} E(S - s|S^{N_\nu^{(k)}+1} > s) = &\frac{1}{P\left(S^{N_\nu^{(k)}+1} > s\right)} \int_0^\infty P(S > s + x)dx - P(V > U_1) \int_0^\infty P(S > x)dx \\ &+ P(V > U_1)P(S^{N_\nu^{(k)}+1} > s) \int_0^\infty P(S > x)dx \\ &= \frac{1}{P\left(S^{N_\nu^{(k)}+1} > s\right)} \left[E\left(S - s|S > s\right) P\left(S > s\right) - P(V > U_1)E(S) \left[1 - P(S^{N_\nu^{(k)}+1} > s)\right]\right]. \end{aligned} \tag{9}$$

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

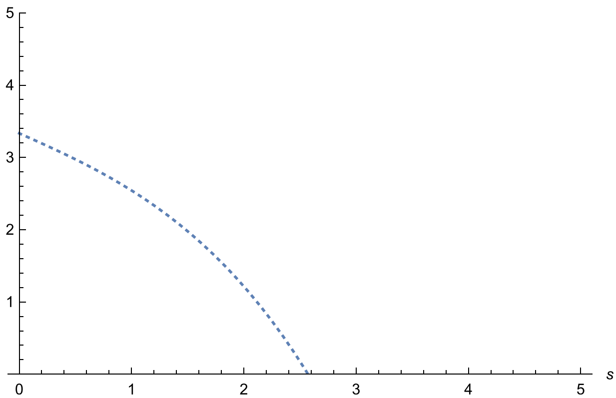

Figure 2. *The mean residual life of the network traffic when it is known that the network has full bandwidth at time* s.

Proof. Using previous theorem,

$$E(S-s|S^{N_\nu^{(k)}+1}>s)=\frac{1}{P\left(S^{N_\nu^{(k)}+1}>s\right)}\int_0^\infty P\left(S>s+x,S^{N_\nu^{(k)}+1}>s\right)dx$$

$$=\frac{1}{P\left(S^{N_\nu^{(k)}+1}>s\right)}\int_0^\infty P\left(S>s+x,S^{N_\nu^{(k)}+1}>s\right)dx.$$

From equation (5), we have:

$$=\frac{1}{P\left(S^{N_\nu^{(k)}+1}>s\right)}\int_0^\infty [P(S>s+x)-P(V>U_1)P(S>x)$$
$$\times\left(1-P(S^{N_\nu^{(k)}+1}>s)\right)\Big]dx$$
$$=\frac{1}{P\left(S^{N_\nu^{(k)}+1}>s\right)}\left[\int_0^\infty P(S>s+x)dx-P(V>U_1)\int_0^\infty P(S>x)dx\right.$$
$$\left.+P(V>U_1)P(S^{N_\nu^{(k)}+1}>s)\int_0^\infty P(S>x)dx\right].$$

□

Remark 10. Using (2) and (3) in (9), we have,

$$E(S-s|S^{N_\nu^{(k)}+1}>s)=\frac{1}{(\zeta\otimes\mathbf{a})\exp\left(\left(\mathbf{\Lambda}\otimes\mathbf{I}+\left(\mathbf{a}^0\zeta\right)\otimes\mathbf{\Psi}\right)s\right)\mathbf{e}'}\left[-\zeta\exp\left(\left(\mathbf{\Lambda}+(1-p_3)\,\mathbf{\Lambda}^0\zeta\right)s\right)\right.$$
$$\times\left(\mathbf{\Lambda}+(1-p_3)\,\mathbf{\Lambda}^0\zeta\right)^{-1}\mathbf{e}'- \tag{10}$$
$$P(V>U_1)\left((\zeta\otimes\mathbf{a})\left(\mathbf{\Lambda}+(1-p_3)\,\mathbf{\Lambda}^0\zeta\right)^{-1}\mathbf{e}'\right)$$
$$\left.\times\left[1-(\zeta\otimes\mathbf{a})\exp\left(\left(\mathbf{\Lambda}\otimes\mathbf{I}+\left(\mathbf{a}^0\zeta\right)\otimes\mathbf{\Psi}\right)s\right)\mathbf{e}'\right]\right].$$

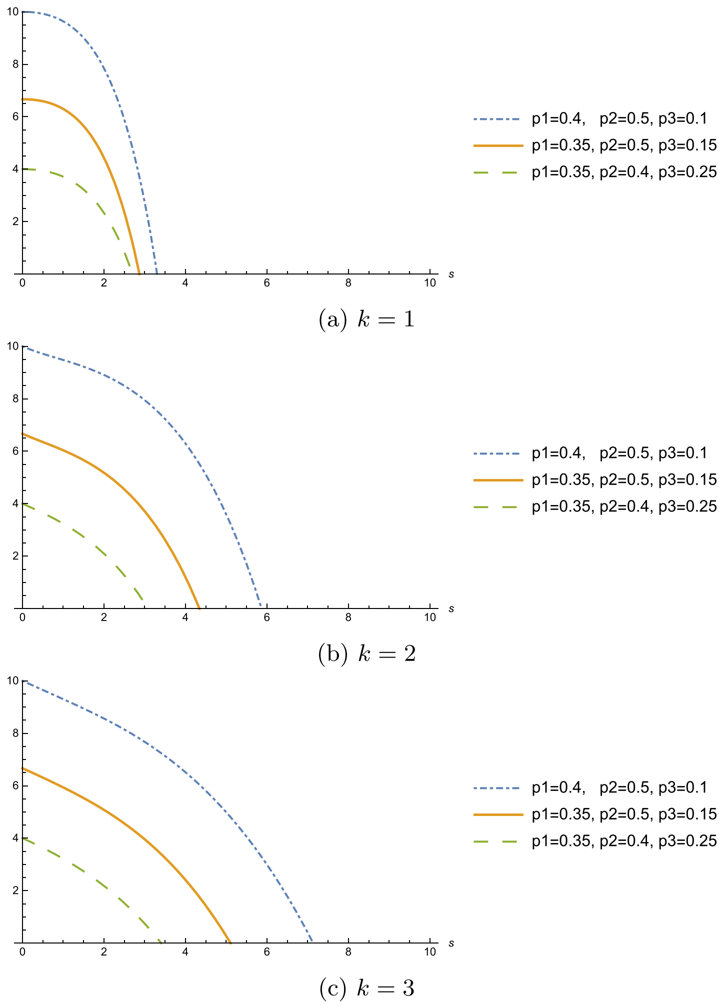

Figure 3. *Effect and sensitivity of* k, p_1 *and* p_2 *on the mean residual life when it is known that the system is in the best performance at time* s.

4. Numerical illustrations

In this section, we consider the realistic example of DDoS attacks in [25]. Consider a network client which stores private data of their clients. Such servers are vulnerable to threats. In our model, system states expressed as bandwidths of the server. The volume of DDoS attacks are defined as the magnitude of the shocks. d_1 and d_2 denote the thresholds for the volume of the DDoS attacks. When the network is exposed to k consecutive DDoS attacks with volumes between d_1 and d_2, the bandwidth of the network will decrease which will affect the response time of the system that leads to a decrease in the functionality of the server. A DDoS attack with a volume of greater than d_2 will cause the server to shut down. This can be considered as a fatal shock.

Let $p_1 = 0.25, p_2 = 0.45$ and $p_3 = 0.3$ be the probabilities that the volume of the DDoS attacks is less than $d_1 = 100$ megabits per sec (Mbps), between $d_1 = 100$ Mbps and $d_2 = 1$ gigabits per sec (Gbps), and greater than $d_2 = 1$ Gbps, respectively. We assume that the bandwidth of the network decreases when two consecutive DDoS attacks having volumes between $d_1 = 100$ Mbps and $d_2 = 1$ Gbps occur. Assuming that the inter-arrival times of DDoS attacks $T_1, T_2, \ldots$ are exponentially distributed with unit

https://doi.org/10.1017/S0269964824000019 Published online by Cambridge University Press

mean, in Figure 2, we plot the MRL of the network traffic when it is known that the network has full bandwidth at time s.

By using equation (10), the effect and sensitivity of k, p_1 and p_2 on the MRL when it is known that the system is in the best performance at time s are discussed in Figure 3.

According to Figure 3, for any value of k, an increase in p_1 or p_2 causes $E(S - s|S^{N_v^{(k)}+1} > s)$ to increase, while an increase in p_3 causes $E(S - s|S^{N_v^{(k)}+1} > s)$ to decrease.

5. Conclusion

We consider a system that experiences random shocks over time, with two critical levels, d_1 and d_2, where $d_1 < d_2$. This system is partially damaged if k successive shocks whose magnitude is between d_1 and d_2 cause the system to fall to a lower partially functional state. However, a shock with a higher magnitude d_2 leads to a complete failure of the system. According to our assumption, the time between successive shocks follows a phase-like distribution, which allows us to evaluate the dynamic reliability properties of the multi-state system. This study has generalized the extreme shock model to run a shock model for a multi-state system. According to the results, an increase in the number of consecutive shocks (k) leads to an increase in the MRL when it is known that the system is in the best performance at time s. In addition, for any value of k, a change in p_1 or p_2 causes $E(S - s|S^{N_v^{(k)}+1} > s)$ in the same direction, while a change in p_3 causes $E(S - s|S^{N_v^{(k)}+1} > s)$ to change in the opposite direction. Throughout this paper, only the impact of the external shock damage has been considered on the failure of the system. The internal damage degradation of the system has not been considered and modeled. For future studies, the impact from internal degradation may be considered. Also, the multi-state system can be re-modeled with different shock models, such as δ shock models, cumulative shock models or combinations of other types of shock models.

Acknowledgments. The authors thank the anonymous referees for their helpful comments and suggestions, which were useful in improving the paper. Part of this work was carried out during the second and third authors' visits to Izmir University of Economics, Izmir, Turkey.

Competing interest. The authors state that there is no conflict of interest.

References

[1] Baggio, M. & Perrings, C. (2015). Modeling adaptation in multi-state resource systems. *Ecological Economics* 116: 378–386.
[2] Cha, J. & Finkelstein, M. (2011). On new classes of extreme shock models and some generalizations. *Journal of Applied Probability* 48: 258–270.
[3] Chang, J.C. & Hwang, F.K. (2003). Reliabilities of consecutive-k systems. London: Springer.
[4] Chen, J. & Li, Z. (2008). An extended extreme shock maintenance model for a deteriorating system. *Reliability Engineering and System Safety* 93: 1123–1129.
[5] Eryilmaz, S. (2015). Assessment of a multi-state system under a shock model. *Applied Mathematics and Computation* 269: 1–8.
[6] Eryilmaz, S. & Kan, C. (2019). Reliability and optimal replacement policy for an extreme shock model with a change point. *Reliability Engineering and System Safety* 190: 106513.
[7] Eryilmaz, S. & Tekin, M. (2019). Reliability evaluation of a system under a mixed shock model. *Journal of Computational and Applied Mathematics* 352: 255–261.
[8] Gut, A. (1990). Cumulative shock models. *Advances in Applied Probability* 22: 504–507.
[9] Gut, A. (2001). Mixed shock models. *Bernoulli* 7: 541–555.
[10] Gut, A. & Hüsler, J. (1999). Extreme shock models. *Extremes* 2: 293–305.
[11] Gut, A. & Hüsler, J. (2005). Realistic variation of shock models. *Statistics & Probability Letters* 74: 187–204.
[12] He, Q.M. (2014). Fundamentals of matrix-analytic methods. New York: Springer.
[13] Hirsch, W.M., Meisner, M., & Boll, C. (1968). Cannibalization in multicomponent systems and theory of reliability. *Naval Research Logistics* 15(3): 331–360.
[14] Li, Z.H., Chan, L.Y., & Yuan, Z.X. (1999). Failure time distribution under a δ-shock model and its application to economic design of system. *International Journal of Reliability, Quality and Safety Engineering* 3(6): 237–247.

[15] Li, Z.H., Huang, B.S., & Wang, G.J. (1999). Life distribution and its properties of shock models under random shocks. *Journal of Lanzhou University* 35(4): 1–7.

[16] Lisnianski, A., Frenkel, I., & Ding, Y. (2010). Multi-state system reliability analysis and optimization for engineers and industrial managers. London: Springer.

[17] Lisnianski, A. & Levitin, G. (2003). Multi-state system reliability: Assessment, optimization and applications. Singapore: World Scientific.

[18] Lindskog, F. & McNeil, A.J. (2003). Common Poisson shock models: applications to insurance and credit risk modelling. *ASTIN Bulletin* 33(2): 209–238.

[19] Mallor, F. & Omey, E. (2001). Shocks, runs and random sums. *Journal of Applied Probability* 38: 438–448.

[20] Mallor, F., Omey, E., & Santos, J. (2006). Asymptotic results for a run and cumulative mixed shock model. *Journal of Mathematical Sciences* 138: 5410–5414.

[21] Ozkut, M. (2023). Reliability and optimal replacement policy for a generalized mixed shock model. *TEST* 1–17.

[22] Ozkut, M. & Eryilmaz, S. (2019). Reliability analysis under Marshall–Olkin run shock model. *Journal of Computational and Applied Mathematics* 349: 52–59.

[23] Philippou, A.N. & Makri, F.S. (1986). Success, runs and longest runs. *Statistics & Probability Letters* 4: 211–215.

[24] Shanthikumar, J.G. & Sumita, U. (1983). General shock models associated with correlated renewal sequences. *Journal of Applied Probability* 20: 600–614.

[25] Singh, K.J. & De, T. (2017). Mathematical modelling of DDoS attack and detection using correlation. *Journal of Cyber Security Technology* 1(3-4): 175–186.

[26] Sumita, U. & Shanthikumar, J.G. (1985). A class of correlated cumulative shock models. *Advances in Applied Probability* 17: 347–366.

[27] Wang, G.J. & Zhang, Y.L. (2005). A shock model with two-type failures and optimal replacement policy. *International Journal of Systems Science* 36: 209–214.

[28] Wu, Q. (2012). Reliability analysis of a cold standby system attacked by shocks. *Applied Mathematics and Computation* 218: 11654–11673.

[29] Zhao, X., Wang, S., Wang, X., & Cai, K. (2018). A multi-state shock model with mutative failure patterns. *Reliability Engineering and System Safety* 178: 1–11.

Cite this article: Ozkut M., Kan C. and Franko C. (2024). Analyzing the multi-state system under a run shock model. *Probability in the Engineering and Informational Sciences* 38(4): 619–631. https://doi.org/10.1017/S0269964824000019