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SMOOTH FAMILIES OF FIBRATIONS AND ANALYTIC SELECTIONS OF POLYNOMIAL HULLS

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Constructed are strictly increasing smooth families $\Sigma^t \subseteq \partial D \times \mathbf{C}^2$, $t \in [0,1]$, of fibrations over the unit circle with strongly pseudoconvex fibers all diffeomorphic to the ball $\overline{B^4}$ such that there is no analytic selection of the polynomial hull of Σ^0 and which end at the product fibration $\Sigma^1 = \partial D \times \overline{B^4}$. In particular these examples show that the continuity method for describing the polynomial hull of a fibration over ∂D fails even if the complex geometry of the fibers is relatively simple.

1. INTRODUCTION

Let $\mathcal{P}_n$ be the algebra of holomorphic polynomials in n complex variables and let $X \subseteq \mathbf{C}^n$ be a compact subset of the complex space $\mathbf{C}^n$. The polynomial hull $\widehat{X}$ of X is defined as

$$\widehat{X} := \{z_o \in \mathbf{C}^n; |p(z_o)| \leqslant \sup_{z\in X} |p(z)|, p \in \mathcal{P}_n\} .$$

Let $D \subseteq \mathbf{C}$ be the unit disc in the complex plane $\mathbf{C}$ and let ∂D be its boundary, the unit circle in $\mathbf{C}$. An H^∞ analytic disc with boundary in X is an H^∞ mapping $h : D \to \mathbf{C}^n$ such that

$$h(\xi) \in X \quad \text{almost everywhere} \quad dm(\xi) ,$$

where $dm(\xi)$ stands for the Lebesgue measure on ∂D. By the maximum principle it follows immediately that if h is an H^∞ analytic disc with boundary in X, then the whole disc $h(D)$ lies in the polynomial hull $\widehat{X}$ of X, that is, $h(D) \subseteq \widehat{X}$. It is a classical result by Stolzenberg, [**10**], that it is not always the case that the set $\widehat{X} \setminus X$ can be given as the union of the H^∞ analytic discs with boundaries in X. Later Wermer, [**11**], refined Stolzenberg's example and constructed a fibration over the unit circle ∂D with fibers in $\mathbf{C}$ with the same property.

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On the other hand there is a series of papers [**1, 4, 5, 7, 8, 9**] on the polynomial hull of a fibration

$$X := \bigcup_{\xi \in \partial D} \{\xi\} \times X_\xi \subseteq \partial D \times \mathbf{C}^n$$

over ∂D, which show that in the case the geometry of the fibers X_ξ, $\xi \in \partial D$, is tame, that is, arbitrary dimension n and all fibers are geometrically convex [**1, 4, 7, 9**] or $n = 1$ and the fibers are only connected and simply connected [**5, 8**], one can describe the polynomial hull of X as the union of the graphs $\{(z, h(z)); z \in D\}$ of the H^∞ analytic discs h in $\mathbf{C}^n$ for which

$$h(\xi) \in X_\xi \ \text{ almost everywhere } \ dm(\xi) .$$

A disc h of this kind is called an *analytic selection* of the polynomial hull of X. An example by Helton and Merino, [**6**], shows that the condition on the fibers to be only connected and simply connected is not enough for the same result to hold for $n \geqslant 2$. Namely, they found an example of a fibration X over ∂D with connected and simply connected fibers in $\mathbf{C}^2$, whose polynomial hull $\widehat{X}$ is nontrivial, but there is no graph of an H^∞ analytic disc whose boundary lies in X.

All proofs of the above positive results for $n \geqslant 2$ are essentially based on a very clever use of the Hanh-Banach theorem and are, therefore, linear (convex) in their nature. One could hope that exploiting the complex geometry of the fibers X_ξ, $\xi \in \partial D$, one could still get some positive results on the description of the polynomial hull of X as Forstnerič did in [**5**] in the case of one dimensional fibers. See also [**8**]. In this paper we give two examples, inspired by the example by Helton and Merino, [**6**], which show that the so called continuity method for describing the polynomial hull of a fibration over ∂D, which was so successfully used by Forstnerič for $n = 1$, [**5**], fails even in the case the complex geometry of the fibers is simple. See also [**2**].

Theorem 1.1. *There exists a smooth family of fibrations*

$$\Sigma^t := \bigcup_{\xi \in \partial D} \{\xi\} \times \Sigma^t_\xi, \quad (t \in [0,1])$$

in $\partial D \times \mathbf{C}^2$ with the following properties:

1. *for all $t \in [0,1]$ and for all $\xi \in \partial D$ the interiors Ω^t_ξ of the fibers Σ^t_ξ are strongly pseudoconvex domains in $\mathbf{C}^2$ with smooth boundaries, all diffeomorphic to the ball and such that $\overline{\Omega^t_\xi} = \Sigma^t_\xi$,*
2. *all fibers of the fibration Σ^1 are Euclidean balls in $\mathbf{C}^2$,*
3. *the family is strictly increasing in the sense that for all $\xi \in \partial D$ and for all pairs $t, \tau \in [0,1]$, $t < \tau$, the inclusion*

$$\Sigma^t_\xi \subseteq \Omega^\tau_\xi$$

holds,

4. *the fibration Σ^0 has the property that its polynomial hull is nontrivial, but there is no H^∞ analytic selection of the fibration Σ^0.*

Theorem 1.2. *There exists a smooth family of fibrations*

$$\Sigma^t := \bigcup_{\xi \in \partial D} \{\xi\} \times \Sigma^t_\xi, \quad (t \in [0,1])$$

in $\partial D \times \mathbf{C}^2$ with the properties (1), (2) and (3) of Theorem 1 and with the additional properties:

4. *for every $t \in [0,1]$ and for every $\xi \in \partial D$ there is a fixed small open ball B_o included in the interior Ω^t_ξ of all fibers Σ^t_ξ,*
5. *there is a point z_o in the polynomial hull of Σ^0, $z_o \notin \Sigma^0$, through which there is no graph of an H^∞ analytic selection of Σ^0.*

2. Blowing up an arc

In this section we prove the following proposition.

Proposition 2.1.

Let γ be a smooth arc in $\mathbf{R}^2 \subseteq \mathbf{C}^2$. Then there exists a smooth strictly plurisubharmonic function $\widetilde{\rho}$ on $\mathbf{C}^2$ such that

(a) *$\gamma = \{z \in \mathbf{C}^2; \widetilde{\rho}(z) = 0\} = \{z \in \mathbf{C}^2; \nabla\widetilde{\rho}(z) = 0\}$ and*

(b) *there exists $C > 0$ such that for every $c \geqslant C$ the level set $\{z \in \mathbf{C}^2; \widetilde{\rho}(z) = c\}$ is an Euclidean 3-sphere.*

Proof: Let f be any smooth nonnegative function on $\mathbf{R}^2$ such that

(a) the zero set of f and the zero set of the gradient ∇f are both equal to γ and

(b) there exists an $r_o > 0$ such that $f(x_1, x_2) = x_1^2 + x_2^2$ for $x_1^2 + x_2^2 \geqslant r_o^2$.

Here the coordinates in $\mathbf{R}^2 \subseteq \mathbf{C}^2$ are x_1, x_2 and the coordinates in $\mathbf{C}^2$ are $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. For $\lambda > 0$ we define

$$\rho_\lambda(z_1, z_2) = f(x_1, x_2) + \lambda(y_1^2 + y_2^2) .$$

Then

(1) the zero set of ρ_λ and the zero set of $\nabla\rho_\lambda$ are both equal to the arc γ and

(2) the Levi form of the function ρ_λ is

$$L(\rho_\lambda) := \frac{1}{4}\begin{pmatrix} f_{x_1x_1} + 2\lambda & f_{x_1x_2} \\ f_{x_1x_2} & f_{x_2x_2} + 2\lambda \end{pmatrix},$$

where the notation $f_{x_i x_j}$ stands for the second partial derivative of the function f with respect to x_i and x_j, $i,j = 1,2$.

Condition (*b*) on the function f ensures that if λ is large enough, the function ρ_λ is strictly plurisubharmonic on $\mathbb{C}^2$. We fix such a λ and denote the function ρ_λ by ρ.

Let $\chi : \mathbf{R} \to [0,1]$ be a smooth function whose support is contained in the interval $[-1,(r_o+2)^2]$ and which equals 1 on the closed interval $[0,(r_o+1)^2]$. Also, let g be a smooth nonnegative function on $\mathbf{R}$ such that

(1) $g(x) = 0$ for $x \leqslant r_o^2$,
(2) $g'(x) > 0$ and $g''(x) \geqslant 0$ for $x > r_o^2$,
(3) $\rho(z)\chi'\left(|z|^2\right) + g'\left(|z|^2\right) \geqslant 0$ for every $z \in \mathbf{C}^2$.

For $\varepsilon \in (0,1)$ we define

$$\widetilde{\rho}_\varepsilon(z) = \varepsilon\chi\left(|z|^2\right)\rho(z) + g\left(|z|^2\right) \quad \left(z \in \mathbf{C}^2\right) .$$

If ε is small enough, the function $\widetilde{\rho}$ is strictly plurisubharmonic on $\mathbf{C}^2$ and its zero set is the arc γ. We fix such an ε and denote the corresponding function by $\widetilde{\rho}$. Thus the proposition will be proved once we prove the following lemma.

Lemma 2.1. *The zero set of the gradient $\nabla\widetilde{\rho}$ is the arc γ.*

Proof: Let $z^o = (x_1^o + iy_1^o, x_2^o + iy_2^o)$ be a point where the gradient $\nabla\widetilde{\rho}$ is zero. We consider the following three cases:

1. Case $|z^o| < r_o$. Then $\widetilde{\rho} = \varepsilon\rho$ in a neighbourhood of the point z^o and thus $z^o \in \gamma$.

2. Case $|z^o| > r_o + 2$. Then $\widetilde{\rho}(z) = g\left(|z|^2\right)$ in a neighbourhood of the point z^o. Since $g'(x) > 0$ for $x > r_o^2$, we get a contradiction.

3. Case $r_o \leqslant |z^o| \leqslant r_o + 2$. The y components of the gradient $\nabla\widetilde{\rho}$, that is, the derivatives of $\widetilde{\rho}$ with respect to y_1 and y_2 at the point z are equal to

$$\frac{\partial\widetilde{\rho}}{\partial y_j}(z) = 2\left(\lambda\varepsilon\chi\left(|z|^2\right) + \varepsilon\rho(z)\chi'\left(|z|^2\right) + g'\left(|z|^2\right)\right)y_j \quad (j = 1,2) .$$

Therefore, if $\nabla\widetilde{\rho}(z^o) = 0$, one concludes that since

$$(1) \qquad \lambda\varepsilon\chi\left(|z|^2\right) + \varepsilon\rho(z)\chi'\left(|z|^2\right) + g'\left(|z|^2\right) > \varepsilon\left(\rho(z)\chi'\left(|z|^2\right) + g'\left(|z|^2\right)\right) \geqslant 0$$

on $\mathbf{C}^2$, it follows

$$y_1^o = y_2^o = 0 .$$

Our initial assumption (b) on the function f and the fact that $|z^o| \geqslant r_o$ imply

$$f_{x_1}(x_1^o, x_2^o) = 2x_1^o \quad \text{and} \quad f_{x_2}(x_1^o, x_2^o) = 2x_2^o .$$

The x components, that is, the derivatives with respect to x_1 and x_2 variables, of the equation $\nabla\widetilde{\rho}(z^o) = 0$, together with (1) give

$$x_1^o = x_2^o = 0 \ .$$

Hence also the assumption $r_o \leqslant |z^o| \leqslant r_o + 2$ leads to a contradiction and the lemma, thus also the proposition, is proved. ∎

A more geometric interpretation of the above proposition is that for every simple arc γ in $\mathbf{R}^2 \subseteq \mathbf{C}^2$ there exists a smooth family of strictly pseudoconvex domains

$$\Omega_t := \{z \in \mathbf{C}^2; \widetilde{\rho}(z) < t\} \quad (t \in (0, \infty)) \ ,$$

in $\mathbf{C}^2$ with smooth boundary which starts at γ, is strictly increasing in the sense that for each pair of parameters $t < \tau$ the domain Ω_t is compactly included in the domain Ω_τ and which ends at some large Euclidean ball. Observe also that since the gradient $\nabla\widetilde{\rho}$ is nonzero except on γ all the domains Ω_t, $t \in (0, \infty)$, are topological cells.

REMARK 2.1. If one is given a smooth family of simple arcs γ_ξ, $\xi \in \partial D$, in $\mathbf{R}^2 \subseteq \mathbf{C}^2$, then one can choose a smooth family of smooth functions f_ξ, $\xi \in \partial D$, satisfying conditions (a) and (b) for each $\xi \in \partial D$. Since the set of parameters is compact, the functions χ and g and the constants λ and ε can be chosen uniformly, that is, independent of the parameter $\xi \in \partial D$, and the corresponding strictly plurisubharmonic functions $\widetilde{\rho}_\xi(z)$ vary smoothly in both variables ξ and z.

REMARK 2.2. The above construction can be applied to any arc γ in $\mathbf{C}^2$ for which there exists a holomorphic automorphism Φ of $\mathbf{C}^2$ such that $\Phi(\gamma) \subseteq \mathbf{R}^2$.

3. FIRST FAMILY OF FIBRATIONS

We consider now the following family of arcs in $\mathbf{R}^2 \subseteq \mathbf{C}^2$. Let γ_1 be the semicircle in $\mathbf{R}^2$ given by the equation

$$x_1^2 + x_2^2 = 1 \ , \quad x_2 \geqslant 0 \ .$$

For $\xi \in \partial D$ we denote by R_ξ the map

$$R_\xi : \mathbf{C}^2 \longrightarrow \mathbf{C}^2$$

defined by

$$R_\xi(z_1, z_2) := (\xi z_1, z_2) \ .$$

Observe that R_ξ is a linear isomorphism of $\mathbf{C}^2$. For $\xi \in \partial D$ such that $0 \leqslant \arg(\xi) \leqslant \pi/2$ or $(3\pi)/2 \leqslant \arg(\xi) \leqslant 2\pi$ let

$$\gamma_\xi := \gamma_1$$

and for the parameters $\xi \in \partial D$ such that $\pi/2 < \arg(\xi) < (3\pi)/2$ we smoothly perturb the initial arc γ_1 to get arcs γ_ξ which do not pass through the point $(0,1)$ but they still pass through the points $(1,0)$ and $(-1,0)$. For instance, for $\xi = e^{is}$ one may take γ_ξ to be defined by the equation

$$(1-\varrho(s))^2 x_1^2 + x_2^2 = (1-\varrho(s))^2 , \quad x_2 \geqslant 0 ,$$

where $\varrho : \mathbf{R} \to [0,1)$ is any smooth function whose support is the interval $[\pi/2,(3\pi)/2]$. We define

$$\widetilde{\gamma}_\xi := R_{\sqrt{\xi}}(\gamma_\xi) .$$

Here by $\sqrt{\xi}$ we mean the principal branch of the square root, that is, $\sqrt{-1} = i$. Since we have $\gamma_\xi = \gamma_1$ in a neighbourhood of $\xi = 1$ and since the arc γ_1 is symmetric with respect to the x_2-axis, the family of arcs $\widetilde{\gamma}_\xi$, $\xi \in \partial D$, is smooth. Using our initial construction for an arc $\gamma \subseteq \mathbf{R}^2$ and Remarks 2.1 and 2.2, one gets a smooth family of fibrations Σ^t, $t > 0$, in $\partial D \times \mathbf{C}^2$ such that for each t the interiors of all fibers are strongly pseudoconvex domains with smooth boundaries and for t large enough all fibers of Σ^t are Euclidean balls centred at the point $(0,0)$ with the fixed radius $\sqrt{t}$. Also, for every pair $t,\tau \in (0,\infty)$, $t < \tau$, all fibers of the fibration Σ^t are included in the interiors of the corresponding fibers of Σ^τ.

REMARK 3.1. Observe that by a theorem of Docquier and Grauert [**3**] the above properties of the family of fibrations Σ^t, $t > 0$, assure that the fibers of Σ^t remain polynomially convex for each parameter $t > 0$.

To finish our example we first observe that since

$$\left(\sqrt{\xi},0\right), \left(-\sqrt{\xi},0\right) \in \widetilde{\gamma}_\xi \quad (\xi \in \partial D) ,$$

the polynomial hull of Σ^t contains the point $(0,0,0)$ for all $t > 0$. Finally we prove the following lemma.

LEMMA 3.1. *For $t > 0$ small enough there is no graph of an H^∞ analytic mapping $F : D \to \mathbf{C}^2$ with boundary in the fibration $\Sigma^t \subseteq \partial D \times \mathbf{C}^2$.*

PROOF: We prove the lemma for

$$\Sigma^0 := \bigcup_{\xi \in \partial D} \{\xi\} \times \widetilde{\gamma}_\xi .$$

Once this is proved the normal family argument finishes the proof of the lemma. Namely, assume that there is a sequence $t_n \downarrow 0$, $n \in \mathbf{N}$, such that for all n there exists an H^∞ analytic selection F_n for $\Sigma^n := \Sigma^{t_n}$. By the normal family argument there exists a subsequence of $\{F_n\}_{n\in\mathbf{N}}$, still denoted by F_n, which normally converges to an H^∞ function F_o. Then for every holomorphic polynomial p in three variables and every $z \in D$ we have

$$|p(z, F_o(z))| = \lim_n |p(z, F_n(z))| \leqslant \lim_n \sup_{x\in\Sigma^n} |p(x)| = \sup_{x\in\Sigma^0} |p(x)| \ .$$

The inequality follows because the discs F_n, $n \in \mathbf{N}$, are analytic selections for the fibrations Σ^n, $n \in \mathbf{N}$, and the last equality is true since the family of fibrations Σ^t, $t \geqslant 0$, is continuous in Hausdorff topology of compact sets in $\mathbf{C}^2$. Therefore the graph $\{(z, F_o(z)); z \in D\}$ is contained in the polynomial hull of Σ^0 and so F_o is an analytic selection of Σ^0. Here we used the fact that all fibers of the fibration Σ^0 are polynomialy convex in $\mathbf{C}^2$.

Let us assume now that there is an analytic mapping

$$(f, g) : D \longrightarrow \mathbf{C}^2$$

such that

$$(f(\xi), g(\xi)) \in \widetilde{\gamma}_\xi \quad (\text{ almost everywhere } \xi \in \partial D) \ .$$

Therefore the imaginary part of the function g almost everywhere on ∂D equals to 0 and thus g is a constant function, that is, there is a real number $a \in [0,1]$ such that $g(\xi) = a$ for every $\xi \in \partial D$. Since the arcs $\widetilde{\gamma}_\xi$ for $\pi/2 < \arg(\xi) < (3\pi)/2$ do not pass through the point $(0,1)$ the constant a has to be less than 1. But then for all $\xi \in \partial D$ we have

$$\left(\left(1/\sqrt{\xi}\right) f(\xi), a\right) \in \gamma_\xi$$

and so

$$f(\xi)^2 = \left(1 - a^2\right)\xi \ \text{ almost everywhere } \ dm(\xi) \ ,$$

which leads to a contradiction. □

4. Second family of fibrations

Let $\gamma \subseteq \mathbf{R}^2 \subseteq \mathbf{C}^2$ be the arc

$$x_1^2 + x_2^2 = 1, \quad x_2 \geqslant 0$$

as before. Let $X_1 := \gamma$ and let

$$X_\xi := R_{\sqrt{\xi}} X_1 \ .$$

Since again

$$\left(\sqrt{\xi}, 0\right), \left(-\sqrt{\xi}, 0\right) \in X_\xi \ (\xi \in \partial D) \ ,$$

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it is obvious that the polynomial hull of

$$X := \bigcup_{\xi \in \partial D} \{\xi\} \times X_\xi$$

contains the point $(0,0,0)$.

Lemma 4.1. *There is no H^∞ analytic selection $F: D \to \mathbf{C}^2$ of X which passes through the point* $(0,0)$.

Proof: Let us assume that there is an analytic disc $F = (f,g)$ whose graph has boundary almost everywhere contained in X and is such that $F(0) = (0,0)$. This implies, as in the previous section, that

$$g(\xi) = 0 \quad (\xi \in \overline{D}) .$$

Thus

$$f^2(\xi) = \xi \quad (\text{almost everywhere } \xi \in \partial D) ,$$

a contradiction. □

Since all fibers X_ξ, $\xi \in \partial D$, of X contain the point $(0,1)$, all fibers of the fibrations Σ^t, $t > 0$, constructed similarly as the first family of fibrations, have the point $(0,1)$ in its interior. Finally, repeating the argument from the previous section shows that there exists $t_o > 0$ such that there is no analytic selection for the fibration Σ^{t_o} which passes through the point $(0,0)$. Details are omitted.

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