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BANACH ALGEBRAS WITH ONE DIMENSIONAL RADICAL

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A Banach algebra A with radical R is said to have property (S) if the natural mapping from the algebraic tensor product $A \otimes A$ onto A^2 is open, when $A \otimes A$ is given the projective norm. The purpose of this note is to provide a counterexample to Zinde's claim that when A is commutative and R is one dimensional the fulfillment of property (S) in A implies its fulfillment in the quotient algebra A/R.

Let A be a Banach algebra with radical R and let A^2 denote the linear span of products of elements of A. A is said to have property (S) if the natural map π from the algebraic tensor product $A \otimes A$ onto A^2 is open, when $A \otimes A$ is given the projective norm.

Thus A will have (S) if there is a constant K such that

$$\|z\|_\pi = \inf\left\{\sum \|x_i\| \cdot \|y_i\| \ : \ \sum x_i y_i = z\right\} \le K\|z\|$$

whenever $z \in A^2$.

In [2] Zinde proved that if $\dim R = 1$ then property (S) will hold in A if it holds in the quotient algebra A/R, and stated the converse as obvious. However Loy [1] showed that if $\dim R = 1$ and A/R has (S) then $R \cap A^2 = 0$ implies $R \cap \overline{A^2} = 0$.

We provide an example of a commutative separable Banach algebra A

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with one dimensional radical R such that A has (S) and $R \cap A^2 = 0$ while $R \cap \overline{A^2} \neq 0$, thus showing that the converse to Zinde's result does not hold.

Let A_0 be the complex commutative algebra generated by the formal symbols $\{r, a_i, x_i, z_i : i \in N\}$ subject to

$$r^2 = rx_i = rz_i = ra_i = 0 \quad \text{for all } i,$$

$$x_i y_i = a_i a_j = a_i x_j = 0 \quad \text{whenever } i \neq j,$$

$$x_i^2 = r + z_i \quad \text{for all } i,$$

$$x_i^2 - x_{i+1}^2 = a_i^2 \quad \text{for all } i.$$

Thus an element $y \in A_0$ may be uniquely expressed as

$$(1) \quad y = r + \sum \alpha_i a_i + \sum \beta_i x_i + \sum \gamma_{ij} z_i^j + \sum \delta_i a_i x_i + \sum \mu_{ij} a_i x_i z_i^j + \sum \nu_{ij} x_i z_i^j + \sum \pi_{ij} a_i z_i^j$$

where $\lambda, \alpha_i, \nu_i, \gamma_{ij}, \delta_i, \mu_{ij}, \nu_{ij}, \pi_{ij} \in C$ for all i, j and the sums are finite.

Define a norm on A_0 by

$$\|y\| = |\lambda| + 2\sum |a_i| 2^{-i} + 2\sum |\beta_i| + \sum |\gamma_{ij}| 2^{-2ij} + \sum |\delta_i| 2^{-i} + \sum |\mu_{ij}| 2^{-i(2j+1)} + \sum |\nu_{ij}| 2^{-2ij} + \sum |\pi_{ij}| 2^{-i(2j+1)}$$

It is easily checked that this norm is submultiplicative. Let A be the completion of A_0 with respect to $\|\cdot\|$, then A is commutative and separable and each element of A is uniquely expressible as in (1) with possibly infinite sums.

Clearly $R = \text{Rad } A = \mathbb{C}r$, $R \cap A^2 = 0$ and, since $\lim z_i = 0$, $R \cap \overline{A^2} = R$.

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To show that A has (S) we first consider $z \in A^2 \cap A_0$, so

$$z = \sum_{i=1}^{n} \alpha_i x_i^2 + \sum_{j\geq 2} \beta_{ij} z_i^j + \sum \gamma_{ij} x_i z_i^j + \sum \delta_i a_i x_i + \sum \pi_{ij} a_i z_i^j + \sum \mu_{ij} a_i x_i z_i^j$$

where the sums are finite. Now

$$\begin{aligned}\sum_{i=1}^{n} a_i x_i^2 &= \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} \alpha_k\right)\left(x_i^2 - x_{i+1}^2\right) + \left(\sum_{i=1}^{n} \alpha_i\right) x_n^2 \\ &= \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} \alpha_k\right) a_i^2 + \left(\sum_{i=1}^{n} \alpha_i\right) x_n^2 ,\end{aligned}$$

so that

$$\begin{aligned}\|z\|_\pi &\leq 4 \sum_{i=1}^{n-1} \left|\sum_{k=1}^{i} \alpha_k\right| 2^{-2i} + 4 \left|\sum_{i=1}^{n} \alpha_i\right| + \sum_{j\geq 2} |\beta_{ij}| \left\|z_i^{j-1}\right\| \cdot \|z_i\| \\ &\qquad + \sum |\gamma_{ij}| \left\|z_i^j\right\| \cdot \|x_i\| + \sum |\delta_i| \|a_i\| \cdot \|x_i\| + \sum |\pi_{ij}| \|a_i\| \cdot \left\|z_i^j\right\| \\ &\qquad + \sum |\mu_{ij}| \left\|a_i z_i^j\right\| \cdot \|x_i\| \\ &\leq 4\left[\sum_{i=1}^{n-1} \left(|\alpha_i| \sum_{k=i+1}^{n} 2^{-2k}\right) + \left|\sum_{i=1}^{n} \alpha_i\right| + \sum_{j\geq 2} |\beta_{ij}| 2^{-2ij}\right. \\ &\qquad \left. + \sum |\gamma_{ij}| 2^{-2ij} + \sum |\delta_i| 2^{-i} + \sum |\pi_{ij}| 2^{-i} 2^{-2ij} + \sum |\gamma_{ij}| 2^{-i(2j+1)}\right] \\ &\leq 8\left[\sum_{i=1}^{n-1} |\alpha_i| 2^{-2i} + \left|\sum_{i=1}^{n} \alpha_i\right| + \sum_{j\geq 2} |\beta_{ij}| 2^{-2ij} + \sum |\gamma_{ij}| 2^{-2ij}\right. \\ &\qquad \left. + \sum |\delta_i| 2^{-i} + \sum |\pi_{ij}| 2^{-i(2j+1)} + \sum |\mu_{ij}| 2^{-i(2j+1)}\right] \\ &\leq 8\|z\| .\end{aligned}$$

If $y \in A$ is written as in (1) with infinite sums, we denote by y_k the element of A_0 obtained by summing all indices from 1 to k only.

Now consider an arbitrary $a = \sum_{i=1}^{n} t_i s_i \in A^2$. Then

$$t_i = (t_i)_k + \delta_{t_{ik}} , \quad s_i = (s_i)_k + \delta_{s_{ik}} ,$$

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where $\delta_{t_{ik}}, \delta_{s_{ik}} \to 0$ as $k \to \infty$. So given any $p \in \mathbb{N}$ we may choose k sufficiently large to ensure that

$$\max_{i=1,\ldots,n} \{\|\delta_{t_{ik}}\|, \|\delta_{s_{ik}}\|\} < \frac{1}{p} .$$

Then $a = a_k + \Delta_k$ where

$$a_k = \sum_{i=1}^{n} (t_i)_k (s_i)_k ,$$

$$\Delta_k = \sum_{i=1}^{n} \left(\delta_{t_{ik}} \delta_{s_{ik}} + \delta_{t_{ik}} (s_i)_k + \delta_{s_{ik}} (t_i)_k\right) .$$

Then $\|\Delta_k\|_\pi \le n\left(p^{-2}+(M+N)p^{-1}\right)$ where

$$M = \max_{i=1,\ldots,n} \|s_i\| , \quad N = \max_{i=1,\ldots,n} \|t_i\| ,$$

and so $\|\Delta_k\|_\pi \to 0$ as $k \to \infty$. Now

$$\begin{aligned} \|a\|_\pi &\le \|a_k\|_\pi + \|\Delta_k\|_\pi \\ &\le 8\|a_k\| + \|\Delta_k\|_\pi , \end{aligned}$$

since $a_k \in A^2 \cap A_0$. Letting $k \to \infty$ we obtain

$$\|a\|_\pi \le 8\|a\|$$

whenever $a \in A^2$, so that A has property (S).

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