DILOGARITHM IDENTITIES IN CLUSTER SCATTERING DIAGRAMS

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Abstract. We extend the notion of y-variables (coefficients) in cluster algebras to cluster scattering diagrams (CSDs). Accordingly, we extend the dilogarithm identity associated with a period in a cluster pattern to the one associated with a loop in a CSD. We show that these identities are constructed from and reduced to trivial ones by applying the pentagon identity possibly infinitely many times.

§1. Introduction

In the seminal paper by Gross, Hacking, Keel, and Kontsevich [14], it was revealed that there is an intimate relation between cluster algebras and scattering diagrams. The underlying cluster pattern for a cluster algebra is constructed by *mutations* [10], while a scattering diagram is constructed based on the *consistency* [16], [21]. In spite of the difference of their principles, the whole information of a given cluster pattern $\mathbf{\Sigma}$ is contained in a certain scattering diagram $\mathfrak{D}$ called a *cluster scattering diagram* (CSD).

Let us briefly summarize the relation between $\mathbf{\Sigma}$ and $\mathfrak{D}$. Let $\Delta(\mathbf{G}^{t_0})$ be the fan (the *G-fan*) spanned by the cones of G-matrices of $\mathbf{\Sigma}$ with the initial vertex t_0. The G-fan $\Delta(\mathbf{G}^{t_0})$ is isomorphic to the *cluster complex* $\Delta(\mathbf{\Sigma})$ of $\mathbf{\Sigma}$ in [11] as a simplicial complex. Then, the union of the codimension one cones of $\Delta(\mathbf{G}^{t_0})$ is embedded into the support $\mathrm{Supp}(\mathfrak{D})$ of $\mathfrak{D}$. Also, it coincides with $\mathrm{Supp}(\mathfrak{D})$ if and only if $\mathbf{\Sigma}$ is of finite type. In general, the structure of $\mathfrak{D}$ outside the support of the G-fan $\Delta(\mathbf{G}^{t_0})$ is very complicated.

So, at least superficially, a cluster pattern carries only a partial information of $\mathfrak{D}$. However, looking at the construction of $\mathfrak{D}$ in [14], we recognize that $\mathfrak{D}$ is uniquely constructed (up to isomorphism) only from the *initial exchange matrix* B_{t_0} of $\mathbf{\Sigma}$. Thus, we may think that a cluster pattern $\mathbf{\Sigma}$ and a CSD $\mathfrak{D}$ are actually *one inseparable object* associated with B_{t_0}.

There is one great advantage of working with $\mathfrak{D}$ rather than only $\mathbf{\Sigma}$; namely, it has an underlying group G which we call the *structure group* for $\mathfrak{D}$. Since G is defined by the inverse limit, the infinite product is well defined. Then, first of all, it is possible to consider an *infinite product of mutations*, which is nothing but a *path-ordered product* in $\mathfrak{D}$. Moreover, the group G controls the whole structure of $\mathfrak{D}$ locally and globally. In particular, the following facts were presented somewhat implicitly in [14], and described more explicitly in [29]:

(a) There are some distinguished elements of G called the *dilogarithm elements*, and they satisfy the *pentagon relation*.
(b) Up to equivalence, $\mathfrak{D}$ consists of walls whose wall elements are dilogarithm elements with positive rational exponents.

(c) Dilogarithm elements act as *mutations* on cluster variables in $\boldsymbol{\Sigma}$.

(d) Every *consistency relation* in $\mathfrak{D}$ is constructed from and reduced to a trivial one (i.e., $g = g$ with $g \in G$) by applying the pentagon relation (and commutative relations) possibly infinitely many times.

Thus, the dilogarithm elements and the pentagon relation are everything for a CSD; moreover, they build a bridge over the gap between the principles of $\boldsymbol{\Sigma}$ and $\mathfrak{D}$, namely, mutations and consistency. We note that the importance of the dilogarithm in scattering diagrams was already noticed in [22]. Also, the construction of scattering diagrams by the pentagon relation appeared in [15] in a geometrical setting.

With this perspective in mind, it is natural to extend basic notions for a cluster pattern to a CSD. The notion of a *seed*, which is most fundamental for a cluster pattern, cannot be extended to the entire CSD, because there is no global chamber structure in a CSD in general [14]. However, the notions of cluster variables (x-variables), coefficients (y-variables), and their mutations can be extended to the entire CSD. In fact, the *theta functions* in [14] is an extension of the notion of *cluster monomials*. In this paper, we extend the notion of *y-variables* to a CSD. Accordingly, we also extend the *dilogarithm identity* (DI) associated with a *period* in a cluster pattern [27] to the one associated with a *loop* in a CSD based on the classical mechanical method in [13]. This is our first main result (Theorem 3.6).

Even though the structure of a CSD is very complicated as already mentioned, it is tractable in some sense thanks to the property (d). As a result, the above DIs are also constructed from and reduced to trivial ones by applying the celebrated *pentagon identity* (the *five-term relation*) of the dilogarithm function [23] *possibly infinitely many times* in a parallel way. (See [32, §2.A] for the background of the reducibility problem of the DIs to the pentagon identity.) This also provides an alternative and independent proof of the DIs. This is our second main result (Theorem 4.6). We also give examples of DIs for rank 2 CSDs of affine type.

We conclude by giving some remarks on related works.

(i) The y-variables are also known as *cluster $\mathcal{X}$-variables* [7]. The corresponding theta functions were studied via the CSDs with principal coefficients in [14] and via the scattering diagrams with the Langlands dual data ($\mathcal{X}$ scattering diagrams) in [2], [3]. Apparently, our y-variables in CSDs are defined in a different way, and we do not know how they are related to the ones in the above works. See also Remark 3.2(c). The results in this paper show that our y-variables behave nicely at least in view of DIs.

(ii) The *quantum analog* of a CSD (QCSD) was constructed and studied by [1], [4], [24] in the skew-symmetric case, and it is naturally extended to the skew-symmetrizable case as well. From their construction, we immediately obtain the *quantum DI (QDI)* (possibly with infinite product) associated with a loop in a QCSD for the quantum dilogarithm of [5], [6], which are the extension of the QDIs for a cluster pattern studied in [8], [18], [19], [26]. They are clearly the counterpart of the DIs in this paper. However, we stress that the main results in this paper are *not* straightforwardly obtained from the quantum case by two reasons. First, the DIs here should be obtained from QDI in their semiclassical limits in the sense of [18]; however, the method therein is only heuristic and not rigorous even for the finite product case. Second, it is not yet known that every constancy relation in a QCSD is reduced to a trivial one by the pentagon relation.

§2. Cluster scattering diagrams

In this section, we quickly recall basic notions for CSDs in [14]. See also [29] for a review, which is closer to the present context.

2.1 Structure group

Let Γ be a *fixed data* consisting of the following:

- a lattice N of rank r,
- a skew-symmetric bilinear form $\{\cdot,\cdot\} : N \times N \to \mathbb{Q}$,
- a sublattice $N^\circ \subset N$ of finite index such that $\{N^\circ, N\} \subset \mathbb{Z}$,
- positive integers $\delta_1, \ldots, \delta_r$ such that there is a basis $(e_1, \ldots, e_r)$ of N, where $(\delta_1 e_1, \ldots, \delta_r e_r)$ is a basis of N°,
- $M = \mathrm{Hom}(N, \mathbb{Z})$ and $M^\circ = \mathrm{Hom}(N^\circ, \mathbb{Z})$.

Let $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. Let $\langle n, m \rangle$ denote the canonical paring either for $N^\circ \times M^\circ$ or for $N \times M_\mathbb{R}$. For $n \in N$, $n \neq 0$, let $n^\perp := \{z \in M_\mathbb{R} \mid \langle n, z \rangle = 0\}$.

Let $\mathfrak{s} = (e_1, \ldots, e_r)$ be a *seed* for Γ, which are a basis of N such that $(\delta_1 e_1, \ldots, \delta_r e_r)$ are a basis of N°. Let

$$N^+ = \left\{ \sum_{i=1}^r a_i e_i \mid a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^r a_i > 0 \right\} \tag{2.1}$$

be the set of the positive vectors of N with respect to $\mathfrak{s}$. Let N_{pr}^+ denote the set of primitive elements in N^+. The degree function $\deg : N^+ \to \mathbb{Z}_{>0}$ is defined by $\deg(\sum_{i=1}^r a_i e_i) := \sum_{i=1}^r a_i$. Let $\mathfrak{g}$ be the N^+-graded Lie algebra defined by

$$\mathfrak{g} = \bigoplus_{n \in N^+} \mathbb{Q} X_n, \quad [X_n, X_{n'}] := \{n, n'\} X_{n+n'}. \tag{2.2}$$

Let $\widehat{\mathfrak{g}}$ be the completion of $\mathfrak{g}$ with respect to deg, and let $G = G_{\Gamma,\mathfrak{s}} = \exp(\widehat{\mathfrak{g}})$ be the exponential group of $\widehat{\mathfrak{g}}$. Namely, an element of G is given by a formal symbol $\exp(X)$ $(X \in \widehat{\mathfrak{g}})$ and the product is defined by the Baker–Campbell–Hausdorff formula. We call G the *structure group* for the forthcoming scattering diagrams. Let $G^{>\ell}$ be the normal subgroup of G generated by the elements with degrees greater than ℓ, and let $G^{\leq \ell} := G/G^{>\ell}$ be its quotient. For each $n \in N_{\mathrm{pr}}^+$, let $G_n^{\|}$ be the abelian subgroup of G generated by elements $\exp(c_{jn} X_{jn})$ $(j > 0, c_{jn} \in \mathbb{Q})$ admitting the infinite product.

2.2 Initial seed for cluster pattern

Let $(e_1^*, \ldots, e_r^*)$ be the basis of M which is dual to $\mathfrak{s} = (e_1, \ldots, e_r)$. Let $f_i = \delta_i^{-1} e_i^*$. Then, $(f_1, \ldots, f_r)$ is a basis of M°, which is dual to the basis $(\delta_1 e_1, \ldots, \delta_r e_r)$ of N°. Let $\mathbf{x} = (x_i)_{i=1}^r$, $\mathbf{y} = (y_i)_{i=1}^r$, and $B = B_{\Gamma,\mathfrak{s}} = (b_{ij})_{i,j=1}^r$ with

$$x_i = x^{f_i}, \quad y_i = y^{e_i}, \quad b_{ij} = \{\delta_i e_i, e_j\}, \tag{2.3}$$

where x and y are formal variables for formal exponentials. Then, B is a skew-symmetrizable integer matrix with a left skew-symmetrizer Δ^{-1}, where we set $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_r)$. Thus, $\Sigma = (\mathbf{x}, \mathbf{y}, B)$ is naturally identified with the *initial* seed for the cluster pattern $\Sigma_\mathfrak{s}$ corresponding to $\mathfrak{D}_\mathfrak{s}$. Conversely, for a given skew-symmetrizable integer matrix B, there is a (not unique) pair of Γ and $\mathfrak{s}$ such that $B = B_{\Gamma,\mathfrak{s}}$. Moreover, two groups $G_{\Gamma,\mathfrak{s}}$ and $G_{\Gamma',\mathfrak{s}'}$ are isomorphic if $B_{\Gamma,\mathfrak{s}} = B_{\Gamma',\mathfrak{s}'}$. In this paper, we concentrate on y-variables only.

2.3 y-representation

We introduce a monoid $Q := N^+ \sqcup \{0\} \subset N$, the monoid algebra $\mathbb{Q}[Q]$ of Q over $\mathbb{Q}$, and its completion $\mathbb{Q}[[Q]]$ with respect to deg. An element of $\mathbb{Q}[[Q]]$ is written as a formal infinite sum with a formal variable y as

$$\sum_{n \in Q} c_n y^n, \quad (c_n \in \mathbb{Q}). \tag{2.4}$$

It is also regarded as a formal power series of the *initial y-variables* $\mathbf{y} = (y_1, \ldots, y_r)$ in (2.3). We consider an action of X_n $(n \in N^+)$ on $\mathbb{Q}[[Q]]$

$$X_n(y^{n'}) = \{n, n'\} y^{n'+n}, \tag{2.5}$$

which is a derivation, and we linearly extend it to the action of any $X \in \widehat{\mathfrak{g}}$. Then, the resulting map $\widehat{\mathfrak{g}} \to \mathrm{Der}(\mathbb{Q}[[Q]])$ is a Lie algebra homomorphism. Moreover, its exponential action

$$(\mathrm{Exp}\, X)(y^{n'}) := \sum_{j=0}^{\infty} \frac{1}{j!} X^j(y^{n'}) \quad (X \in \widehat{\mathfrak{g}}) \tag{2.6}$$

is well defined on $\mathbb{Q}[[Q]]$. Then, $\mathrm{Exp}\, X$ is an algebra automorphism of $\mathbb{Q}[[Q]]$ (e.g., [17, §1.2]). Thus, we have a representation of G

$$\begin{aligned} \rho_y : \quad G \quad &\to \quad \mathrm{Aut}(\mathbb{Q}[[Q]]) \\ \exp X \quad &\mapsto \quad \mathrm{Exp}\, X. \end{aligned} \tag{2.7}$$

We call it the *y-representation* of G.

2.4 Dilogarithm elements and pentagon relation

Let us introduce some distinguished elements in G.

DEFINITION 2.1 (Dilogarithm element).   For any $n \in N^+$, we define

$$\Psi[n] := \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} X_{jn} \right) \in G_{n_0}^{\parallel}, \tag{2.8}$$

where $n_0 \in N_{\mathrm{pr}}^+$ is the one such that $n = j n_0$ for some $j \in \mathbb{Z}_{>0}$. We call it the *dilogarithm element* for n.

The element $\Psi[n]$ acts on $\mathbb{Q}[[Q]]$ under ρ_y as

$$\begin{aligned} \Psi[n](y^{n'}) &= \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} X_{jn} \right)(y^{n'}) \\ &= y^{n'} \exp\left(\{n, n'\} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} y^{jn} \right) \\ &= y^{n'} (1 + y^n)^{\{n, n'\}}. \end{aligned} \tag{2.9}$$

Observe that this is essentially the automorphism part of the Fock–Goncharov decomposition of mutations of y-variables [7, §2.1].

It is easy to see that the dilogarithm elements $\Psi[n]^c$ $(n \in N^+, c \in \mathbb{Q})$ are generators of G admitting the infinite product. Moreover, they satisfy the following remarkable relations.

PROPOSITION 2.2.   *Let $n_1, n_2 \in N^+$. The following relations hold in G.*
(a) *If $\{n_2, n_1\} = 0$, for any $c_1, c_2 \in \mathbb{Q}$,*

$$\Psi[n_2]^{c_2}\Psi[n_1]^{c_1} = \Psi[n_1]^{c_1}\Psi[n_2]^{c_2}. \tag{2.10}$$

(b) *(Pentagon relation [14, Exam. 1.14], [29, Prop. III.1.13].) If $\{n_2, n_1\} = c$ $(c \in \mathbb{Q}, c \neq 0)$,*

$$\Psi[n_2]^{1/c}\Psi[n_1]^{1/c} = \Psi[n_1]^{1/c}\Psi[n_1 + n_2]^{1/c}\Psi[n_2]^{1/c}. \tag{2.11}$$

Proof.   The equality (2.10) is clear by (2.9). One can prove (2.11) by comparing the actions of both sides on $y^{n'}$ using (2.9). □

2.5 Cluster scattering diagrams

A wall $\mathbf{w} = (\mathfrak{d}, g)_n$ for $\mathfrak{s}$ is a triplet such that $n \in N_{\mathrm{pr}}^+$, a cone $\mathfrak{d} \subset n^\perp$ of codimension 1, and $g \in G_n^\parallel$. We call n, $\mathfrak{d}$, and g the *normal vector*, the *support*, and the *wall element* of $\mathbf{w}$, respectively. Let $p^* : N \to M^\circ$ and $n \mapsto \{\cdot, n\}$. We say that a wall $\mathbf{w} = (\mathfrak{d}, g)_n$ is *incoming* if $p^*(n) \in \mathfrak{d}$ holds.

DEFINITION 2.3 (Scattering diagram).   A *scattering diagram* $\mathfrak{D} = \{\mathbf{w}_\lambda = (\mathfrak{d}_\lambda, g_\lambda)_{n_\lambda}\}_{\lambda \in \Lambda}$ for $\mathfrak{s}$ is a collection of walls for $\mathfrak{s}$ satisfying the following finiteness condition: *For any degree ℓ, there are only finitely many walls such that $\pi_\ell(g_\lambda) \neq \mathrm{id}$, where $\pi_\ell : G \to G^{\leq \ell}$ is the canonical projection.*

For a scattering diagram $\mathfrak{D}$, we define

$$\mathrm{Supp}(\mathfrak{D}) = \bigcup_{\lambda \in \Lambda} \mathfrak{d}_\lambda, \quad \mathrm{Sing}(\mathfrak{D}) = \bigcup_{\lambda \in \Lambda} \partial \mathfrak{d}_\lambda \cup \bigcup_{\substack{\lambda, \lambda' \in \Lambda \\ \dim \mathfrak{d}_\lambda \cap \mathfrak{d}_{\lambda'} = r-2}} \mathfrak{d}_\lambda \cap \mathfrak{d}_{\lambda'}. \tag{2.12}$$

A curve $\gamma : [0, 1] \to M_\mathbb{R}$ is *admissible* for $\mathfrak{D}$ if it satisfies the following properties:

(i)   The end points of γ are in $M_\mathbb{R} \setminus \mathrm{Supp}(\mathfrak{D})$.
(ii)   It is a smooth curve, and it intersects $\mathrm{Supp}(\mathfrak{D})$ transversally.
(iii) γ does not intersect $\mathrm{Sing}(\mathfrak{D})$.

For any admissible curve γ, the path-ordered product $\mathfrak{p}_{\gamma, \mathfrak{D}} \in G$ is defined as the product of the wall elements $g_\lambda^{\epsilon_\lambda}$ of walls $\mathbf{w}_\lambda$ of $\mathfrak{D}$ intersected by γ in the order of intersection, where ϵ_λ is the *intersection sign* defined by

$$\epsilon_\lambda = \begin{cases} 1, & \langle n_\lambda, \gamma' \rangle < 0, \\ -1, & \langle n_\lambda, \gamma' \rangle > 0, \end{cases} \tag{2.13}$$

and γ' is the velocity vector of γ at the wall $\mathbf{w}_\lambda$. The product $\mathfrak{p}_{\gamma, \mathfrak{D}}$ is an infinite one in general, and it is well defined in G due to the finiteness condition. See [14, §1.1] for a more precise definition. We say that a pair of scattering diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ are *equivalent* if $\mathfrak{p}_{\gamma, \mathfrak{D}} = \mathfrak{p}_{\gamma, \mathfrak{D}'}$ for any admissible curve γ for both $\mathfrak{D}$ and $\mathfrak{D}'$. We say that a scattering diagram $\mathfrak{D}$ is *consistent* if $\mathfrak{p}_{\gamma, \mathfrak{D}} = \mathrm{id}$ for any admissible loop (i.e., closed curve) γ for $\mathfrak{D}$.

DEFINITION 2.4 (Cluster scattering diagram).   A *CSD* $\mathfrak{D}_\mathfrak{s}$ for $\mathfrak{s}$ is a consistent scattering diagram whose set of incoming walls are given by

$$\mathrm{In}_\mathfrak{s} := \{\mathbf{w}_{e_i} = (e_i^\perp, \Psi[e_i]^{\delta_i})_{e_i} \mid i = 1, \ldots, r\}. \tag{2.14}$$

For $n \in N^+$, let $\delta(n)$ be the smallest positive rational number such that $\delta(n)n \in N^\circ$, which is called the *normalization factor* of n. For example, $\delta(e_i) = \delta_i$. Note that $\delta(tn) = \delta(n)/t$ holds for any $n \in N_{\mathrm{pr}}^+$ and $t \in \mathbb{Z}_{>0}$. Also, $\delta(n)$ is an integer for any $n \in N_{\mathrm{pr}}^+$.

The following is the most fundamental theorem on CSDs.

THEOREM 2.5 [14, Ths. 1.12 and 1.13].

(a) *There exists a CSD $\mathfrak{D}_{\mathfrak{s}}$ uniquely up to equivalence.*
(b) *There exists a (still not unique) CSD $\mathfrak{D}_{\mathfrak{s}}$ such that every wall element has the form*

$$\Psi[tn]^{s\delta(tn)} \quad (n \in N_{\mathrm{pr}}^+, \ t, s \in \mathbb{Z}_{>0}). \tag{2.15}$$

In this paper, we exclusively use $\mathfrak{D}_{\mathfrak{s}}$ given in Theorem 2.5(b), which we call a *positive realization*, a CSD $\mathfrak{D}_{\mathfrak{s}}$.

REMARK 2.6. The fact $s \in \mathbb{Z}_{>0}$ is the key to prove the *Laurent positivity* of cluster variables and theta functions in [14], although we do not use this connection in this paper.

§3. Dilogarithm identities in CSDs

3.1 *y*-variables for CSD

Let us extend the notion of y-variables (coefficients) for a cluster pattern $\mathbf{\Sigma}_{\mathfrak{s}}$ to a CSD $\mathfrak{D}_{\mathfrak{s}}$. We say that a curve is *weakly admissible* for $\mathfrak{D}$ if it satisfies the conditions (ii) and (iii) for an admissible curve. The definition of the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}}$ is extended to a weakly admissible curve γ by ignoring the contribution from the walls at the end points.

DEFINITION 3.1 (*y*-variable/*c*-vector for CSD). Let $\mathbf{w} = (\mathfrak{d}, g)_n$ be any wall of $\mathfrak{D}_{\mathfrak{s}}$ with $g = \Psi[tn]^{s\delta(tn)}$ as in (2.15). Let $z \in \mathfrak{d}$ with $z \notin \mathrm{Sing}(\mathfrak{D}_{\mathfrak{s}})$. Let

$$\mathcal{C}^+ := \{ z \in M_{\mathbb{R}} \mid \langle e_i, z \rangle \geq 0 \ (i = 1, \dots, r) \}. \tag{3.1}$$

Let γ_z be any weakly admissible curve in $\mathfrak{D}_{\mathfrak{s}}$ from z to any point in $\mathrm{Int}(\mathcal{C}^+)$. Then, we define a *y-variable $y_z[tn]$ at z with the c-vector tn* by

$$y_z[tn] := \mathfrak{p}_{\gamma_z, \mathfrak{D}_{\mathfrak{s}}}(y^{tn}) \in \mathbb{Q}[[Q]], \tag{3.2}$$

where the path-ordered product $\mathfrak{p}_{\gamma_z, \mathfrak{D}_{\mathfrak{s}}} \in G$ acts on y^{tn} under the y-representation ρ_y.

Since any $g \in G_n^{\|}$ acts trivially on y^{tn}, $y_z[tn]$ is independent of the choice of γ_z due to the consistency of $\mathfrak{D}_{\mathfrak{s}}$. Also, due to our assumption on $\mathfrak{D}_{\mathfrak{s}}$, any wall element of $\mathfrak{D}_{\mathfrak{s}}$ has the form (2.15). Therefore, $\mathfrak{p}_{\gamma_z, \mathfrak{D}_{\mathfrak{s}}}$ acts as a (possibly infinite) product of mutations in (2.9).

REMARK 3.2.

(a) If z belongs to a codimension 1 face of a cluster chamber (G-cone) of $\mathfrak{D}_{\mathfrak{s}}$, we have $t = s = 1$ by the mutation invariance of $\mathfrak{D}_{\mathfrak{s}}$ [14]. Moreover, $n = \varepsilon_{i;t} \mathbf{c}_{i;t}$ and $y_z[n] = y_{i;t}^{\varepsilon_{i;t}}$, where $y_{i;t}$ is an ordinary y-variable, and $\varepsilon_{i;t}$ and $\mathbf{c}_{i;t}$ are the tropical sign and the c-vector for $y_{i;t}$, respectively. The notion of a *seed* cannot be entirely extended for $\mathfrak{D}_{\mathfrak{s}}$, because there is no overall chamber structure therein. Thus, the composite mutation (3.2) directly connects the initial y-variables $\mathbf{y}$ and a given *single* y-variable $y_z[tn]$ for $\mathfrak{D}_{\mathfrak{s}}$.
(b) Unlike usual y-variables for a cluster pattern, all y-variables here have *positive* c-vectors.

(c) Let $\theta_{Q,m}$ be the *theta functions* in [14], where $Q \in \text{Int}(\mathcal{C}^+)$ and $m \in \mathbb{Z}^r$. When m belongs to a cluster chamber, the following formula holds [14, Th. 4.9]:

$$\theta_{Q,m} = \mathfrak{p}_{\gamma,\mathfrak{D}_\mathfrak{s}}(x^m), \tag{3.3}$$

where the action of $\mathfrak{p}_{\gamma,\mathfrak{D}_\mathfrak{s}}$ on x^m is given by the *(principal) x-representation* [29, §III.4]. Clearly, the definition (3.2) is parallel to the formula (3.3). On the other hand, the formula (3.3) is not valid for general m. Thus, the relation between our y-variables and the theta functions is not clear in general.

(d) The variable $y_z[tn]$ changes discontinuously when $z \in \mathfrak{d}$ crosses the codimension 2 intersection (a joint) $\mathfrak{d} \cap \mathfrak{d}'$ with the support $\mathfrak{d}'$ of another wall in $\mathfrak{D}_\mathfrak{s}$, because $\mathfrak{p}_{\gamma_z,\mathfrak{D}_\mathfrak{s}}$ changes.

3.2 Euler and Rogers dilogarithms

We define the Euler dilogarithm [23]

$$\text{Li}_2(x) = \sum_{j=1}^{\infty} \frac{1}{j^2} x^j \quad (|x| < 1) \tag{3.4}$$

$$= -\int_0^x \frac{\log(1-y)}{y} \, dy \quad (x \le 1) \tag{3.5}$$

and the Rogers dilogarithm

$$L(x) = \text{Li}_2(x) + \frac{1}{2} \log x \log(1-x) \quad (0 \le x \le 1). \tag{3.6}$$

The celebrated *pentagon identity* (Abel's identity, the five-term relation) for $\text{Li}_2(x)$ is neatly expressed in terms of $L(x)$ as $(0 \le x, y < 1)$

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right). \tag{3.7}$$

We introduce a variant of the Rogers dilogarithm, which we call the *modified Rogers dilogarithm*, as follows:

$$\tilde{L}(x) := L\left(\frac{x}{1+x}\right) = -\text{Li}_2(-x) - \frac{1}{2} \log x \log(1+x) \tag{3.8}$$

$$= \frac{1}{2} \int_0^x \left\{ \frac{\log(1+y)}{y} - \frac{\log y}{1+y} \right\} dy \quad (0 \le x). \tag{3.9}$$

The function $\tilde{L}(x)$ is smooth but not analytic at $x = 0$; however, it has the following Puiseux expansion around $x = 0$ *with* log *factor*:

$$\tilde{L}(x) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^2} x^j - \frac{1}{2} \log x \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} x^j. \tag{3.10}$$

The pentagon identity (3.7) is expressed in terms of $\tilde{L}(x)$ as follows (e.g., [28, §5.3]):

$$\begin{aligned}
\tilde{L}(y_2(1+y_1)) &+ \tilde{L}(y_1) \\
&= \tilde{L}(y_1(1+y_2+y_1y_2)^{-1}) + \tilde{L}(y_1y_2(1+y_2)^{-1}) + \tilde{L}(y_2).
\end{aligned} \tag{3.11}$$

The above arguments in $\tilde{L}(x)$ are identified with the y-variables with positive c-vectors in the Y-pattern of type A_2 with the initial exchange matrix

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.12}$$

as follows:

$$\begin{pmatrix} \boxed{y_1} \\ y_2 \end{pmatrix} \overset{\mu_1}{\rightarrow} \begin{pmatrix} y_1^{-1} \\ \boxed{y_2(1+y_1)} \end{pmatrix} \overset{\mu_2}{\rightarrow} \begin{pmatrix} y_1^{-1}(1+y_2+y_1y_2) \\ y_2^{-1}(1+y_1)^{-1} \end{pmatrix},$$

$$\begin{pmatrix} y_1 \\ \boxed{y_2} \end{pmatrix} \overset{\mu_2}{\rightarrow} \begin{pmatrix} \boxed{y_1y_2(1+y_2)^{-1}} \\ y_2^{-1} \end{pmatrix} \overset{\mu_1}{\rightarrow} \begin{pmatrix} y_1^{-1}y_2^{-1}(1+y_2) \\ \boxed{y_1(1+y_2+y_1y_2)^{-1}} \end{pmatrix} \tag{3.13}$$

$$\overset{\mu_2}{\rightarrow} \begin{pmatrix} y_2^{-1}(1+y_1)^{-1} \\ y_1^{-1}(1+y_2+y_1y_2) \end{pmatrix}.$$

Also, observe the *pentagon periodicity* [10, §2]. Namely, both results in (3.13) coincide up to the transposition τ_{12}.

REMARK 3.3.   A dilogarithm element $\Psi[n]$ in (2.8) is *formally* expressed as

$$\Psi[n] = \exp(-\mathrm{Li}_2(-X_n)) \tag{3.14}$$

if we *identify* the power X_n^j with X_{jn}. Be careful, however, that this is only formal, because X_n^j and X_{jn} act differently on $\mathbb{Q}[[Q]]$.

3.3 Dilogarithm identity associated with a loop in $\mathfrak{D}_{\mathfrak{s}}$

Let $\mathfrak{D}_{\mathfrak{s},\ell}$ be the scattering diagram consisting of walls of $\mathfrak{D}_{\mathfrak{s}}$ such that $\pi_\ell(g_\lambda) \neq \mathrm{id}$ in $G^{\leq\ell}$. We call $\mathfrak{D}_{\mathfrak{s},\ell}$ the *reduction* of $\mathfrak{D}_{\mathfrak{s}}$ at degree ℓ. Due to the finiteness condition, $\mathfrak{D}_{\mathfrak{s},\ell}$ has only finitely many walls.

Let γ be any admissible loop for $\mathfrak{D}_{\mathfrak{s}}$. Let us fix degree ℓ. Suppose that γ intersects walls of $\mathfrak{D}_{\mathfrak{s},\ell}$ at $z_1, \ldots, z_{P-1}$ in this order. There might be multiple walls with a common normal vector intersected by γ at a time. We distinguish them by allowing the multiplicity $z_a = z_{a+1}$. Then, the walls crossed at $z_0, \ldots, z_{P-1}$ are parametrized as $\mathbf{w}_0, \ldots, \mathbf{w}_{P-1}$. By Theorem 2.5(b), each wall $\mathbf{w}_a$ has the form

$$\mathbf{w}_a = \left(\mathfrak{d}_a, \Psi[t_a n_a]^{s_a \delta(t_a n_a)} \right)_{n_a}. \tag{3.15}$$

By the consistency of $\mathfrak{D}_{\mathfrak{s}}$, we have $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathfrak{s}}} = \mathrm{id}$. Thus, we have a consistency relation around γ,

$$\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathfrak{s},\ell}} = \Psi[t_{P-1}n_{P-1}]^{\epsilon_{P-1}s_{P-1}\delta(t_{P-1}n_{P-1})} \cdots \Psi[t_1 n_1]^{\epsilon_1 s_1 \delta(t_1 n_1)} \equiv \mathrm{id} \\ \mathrm{mod}\ G^{>\ell}, \tag{3.16}$$

where ϵ_a is the intersection sign defined by (2.13).

For the variable $\mathbf{y} = (y_1 \ldots, y_r)$, we call the following formal sum

$$f(\mathbf{y}) + \sum_{i=1}^{r}(\log y_i)g_i(\mathbf{y}) \quad (f(\mathbf{y}), g_i(\mathbf{y}) \in \mathbb{Q}[[Q]]) \tag{3.17}$$

a *formal (generalized) Puiseux series in $\mathbf{y}$ with* log *factor*. Let $\mathcal{F}^{>\ell}$ be the set of all formal power series $f(\mathbf{y}) \in \mathbb{Q}[[Q]]$ such that all coefficients of $f(\mathbf{y})$ vanish up to the total

order ℓ. Let

$$\mathcal{F}_{\log}^{>\ell} := \mathcal{F}^{>\ell} + \sum_{i=1}^{r} (\log y_i) \mathcal{F}^{>\ell}. \tag{3.18}$$

We first prove the reduced DI at level ℓ associated with γ.

THEOREM 3.4. *The following identity holds as a formal Puiseux series in* **y** *with* log *factor:*

$$\sum_{a=0}^{P-1} \epsilon_a s_a \delta(t_a n_a) \tilde{L}(y_{z_a}[t_a n_a]) \equiv 0 \mod \mathcal{F}_{\log}^{>\ell}. \tag{3.19}$$

Here, we present a proof based on the classical mechanical method employed in [13] with some modification. (An alternative proof will be given later by Theorem 4.6.) First, let us give an outline of the method. The basic observation therein and also here, which is originated in [8], is that the action of $\Psi[n]$ in (2.9) (i.e., a mutation) is described by a Hamiltonian system with a log-canonical Poisson bracket

$$\{y^n, y^{n'}\}_P := \{n, n'\} y^n y^{n'}, \quad \{e_i, e_j\} = \delta_i^{-1} b_{ij} \tag{3.20}$$

and a Hamiltonian

$$\mathscr{H}[n] := \mathrm{Li}_2(-y^n). \tag{3.21}$$

Here, we change the normalizations of (3.20) and (3.21) from the convention in [13]. Indeed,

$$\{\mathscr{H}[n], y^{n'}\}_P = -\{n, n'\} y^{n'} \log(1 + y^n), \tag{3.22}$$

which is the infinitesimal (or log) form of the action of $\Psi[n]^{-1}$ in (2.9). We also note that the calculation is essentially the same as (2.9). The main ingredient of the method in [13] is the *canonical coordinates* $\mathbf{u} = (u_1, \ldots, u_r)$ and $\mathbf{p} = (p_1, \ldots, p_r)$ satisfying

$$\{p_i, u_j\}_P = \delta_{ij}, \quad \{u_i, u_j\}_P = \{p_i, p_j\}_P = 0. \tag{3.23}$$

Then, the y-variables **y** are represented as

$$y_i = \exp((\delta_i^{-1} p_i + w_i)/\sqrt{2}), \quad w_i = \sum_{j=1}^{r} b_{ji} u_j. \tag{3.24}$$

In the phase space $\tilde{M} \simeq \mathbb{R}^{2r}$ with the canonical coordinates $(\mathbf{u}, \mathbf{p})$, the point $(\overline{\mathbf{u}}, \overline{\mathbf{p}})$ moves to $(\overline{\mathbf{u}}', \overline{\mathbf{p}}')$ under the time-one flow of $\mathscr{H}[n]$ as

$$\overline{u}_i' = \overline{u}_i - \frac{1}{\sqrt{2}} n_i \delta_i^{-1} \log(1 + \overline{y}^n), \tag{3.25}$$

$$\overline{p}_i' = \overline{p}_i + \frac{1}{\sqrt{2}} \left(\sum_{j=1}^{r} n_j b_{ij} \right) \log(1 + \overline{y}^n), \tag{3.26}$$

where $\overline{y}_i = \exp((\delta_i^{-1} \overline{p}_i + \overline{w}_i)/\sqrt{2})$. We concentrate on the subspace (called the *small phase space*) $\tilde{M}_0$ of $\tilde{M}$ defined by $\delta_i^{-1} p_i = w_i$ ($i = 1, \ldots, r$). Then, the modified Rogers dilogarithm

$$\mathscr{L}[n] := \tilde{L}(y^n) \tag{3.27}$$

appears as the *Lagrangian*. Moreover, it is constant under the above time-one flow because $\{\mathscr{H}[n], y^n\}_P = 0$. We regard $\mathscr{L}[n]$ as a function of $\overline{\mathbf{u}}$ and consider its infinitesimal variation $\delta\mathscr{L}[n]$ by $\overline{\mathbf{u}} + \delta\overline{\mathbf{u}}$. Then, the following formula holds.

LEMMA 3.5 (cf. [13, Lem. 6.4]).   *In the small phase space $\tilde{M}_0$, we have*

$$\delta\mathscr{L}[n] = \sum_{i=1}^{r} \overline{p}_i' \delta\overline{u}_i' - \sum_{i=1}^{r} \overline{p}_i \delta\overline{u}_i. \tag{3.28}$$

The formula is identical to the one in [13, Lem. 6.4]. However, the proof therein is applicable only when n is a c-vector of the cluster pattern for B. So, we give a different line of a proof.

Proof.   By (3.9), the left-hand side is written as

$$\delta\mathscr{L}[n] = \frac{1}{\sqrt{2}}\left(\sum_{i,j=1}^{r} n_j b_{ij} \delta\overline{u}_i\right)\left\{\log(1+\overline{y}^n) - \frac{\overline{y}^n \log\overline{y}^n}{1+\overline{y}^n}\right\}. \tag{3.29}$$

By (3.25), we have

$$\delta\overline{u}_i' = \delta\overline{u}_i - \frac{1}{\sqrt{2}}n_i\delta_i^{-1}\left(\sum_{j,k=1}^{r} n_j b_{kj} \delta\overline{u}_k\right)\frac{\overline{y}^n}{1+\overline{y}^n}. \tag{3.30}$$

Let us write (3.26) and (3.30) as $\overline{p}_i' = \overline{p}_i + A_i$ and $\delta\overline{u}_i' = \delta\overline{u}_i - B_i$. Then, the right-hand side of (3.28) is given by

$$\sum_{i=1}^{r} A_i \delta\overline{u}_i - \sum_{i=1}^{r} \overline{p}_i B_i - \sum_{i=1}^{r} A_i B_i. \tag{3.31}$$

They are calculated as

$$\sum_{i=1}^{r} A_i \delta\overline{u}_i = \frac{1}{\sqrt{2}}\left(\sum_{i,j=1}^{r} n_j b_{ij} \delta\overline{u}_i\right)\log(1+\overline{y}^n), \tag{3.32}$$

$$\sum_{i=1}^{r} \overline{p}_i B_i = \frac{1}{\sqrt{2}}\left(\sum_{i,j=1}^{r} n_j b_{ij} \delta\overline{u}_i\right)\frac{\overline{y}^n \log\overline{y}^n}{1+\overline{y}^n}, \tag{3.33}$$

$$\sum_{i=1}^{r} A_i B_i = \frac{1}{2}\left(\sum_{i,j=1}^{r} n_j b_{ij} n_i \delta_i^{-1}\right)\left(\sum_{i,j=1}^{r} n_j b_{ij} \delta\overline{u}_i\right)\frac{\overline{y}^n \log(1+\overline{y}^n)}{1+\overline{y}^n} = 0, \tag{3.34}$$

where the last equality is due to the skew-symmetry of $\delta_i^{-1} b_{ij}$. Thus, the equality (3.28) holds. $\square$

Now, we are ready to prove Theorem 3.4.

Proof of Theorem 3.4.   We prove (3.19) based on Lemma 3.5.

(a) Assume that B is nonsingular. Then, the y-variables $\mathbf{y}$ represented in (3.24) are algebraically independent on the small phase space $\tilde{M}_0$.

We consider the Hamiltonian system in the small phase space $\tilde{M}_0$ for the time span $[0, P]$, where P is the one in (3.19). The Hamiltonian for the time span $[a, a+1]$ $(a = 0, \ldots, P)$ is given by

$$\mathscr{H}_a = \epsilon_a s_a \delta(t_a n_a) \mathrm{Li}_2(-y^{t_a n_a}). \tag{3.35}$$

Accordingly, for a trajectory $\alpha = (\overline{\mathbf{u}}(t), \overline{\mathbf{p}}(t))_{t \in [0,j]}$ in the small phase space $\tilde{M}_0$, we consider the quantity

$$S[\alpha] = \sum_{a=0}^{P-1} \mathscr{L}_a, \quad \mathscr{L}_a = \epsilon_a s_a \delta(t_a n_a) \tilde{L}(\overline{y}(a)^{t_a n_a}), \tag{3.36}$$

where

$$\overline{y}_i(a) = \exp(\sqrt{2} \delta_i^{-1} \overline{p}_i(a)) = \exp\left(\sqrt{2} \sum_{j=1}^r b_{ji} \overline{u}_j(a) \right). \tag{3.37}$$

For the ith coordinate function y_i at time 0, let y_i' be the one at time P after the above time development. Then, by (3.22), it is described as a function of $\mathbf{y}$ by

$$y_i' = \mathfrak{p}_{\gamma, \mathfrak{D}_{s,\ell}}^{-1}(y_i), \tag{3.38}$$

where

$$\mathfrak{p}_{\gamma, \mathfrak{D}_{s,\ell}}^{-1} = \Psi[t_0 n_0]^{-\epsilon_0 s_0 \delta(t_0 n_0)} \cdots \Psi[t_{P-1} n_{P-1}]^{-\epsilon_{P-1} s_{P-1} \delta(t_{P-1} n_{P-1})}. \tag{3.39}$$

Note that the dilogarithm elements act in the opposite order along the time development on the coordinate functions.

For simplicity, suppose that equality for $\mathfrak{p}_{\gamma, \mathfrak{D}_{s,\ell}}$ in (3.16) is the exact one *without* modulo $G^{>\ell}$. Then, by (3.38), we have

$$y_i' = y_i. \tag{3.40}$$

It follows form Lemma 3.5 that $\delta S[\alpha] = 0$ under any infinitesimal deformation $\alpha + \delta\alpha$. This implies that $S[\alpha]$ is constant with respect to α. Moreover, it is easy to show that the constant is zero by taking the limit $\mathbf{y} \to \mathbf{0}$. On the other hand, $S[\alpha]$ can be viewed as a formal Puiseux series in $\mathbf{y} = \overline{\mathbf{y}}(0)$ with log factor. More explicitly, $\overline{y}(a)^{t_a n_a}$ in (3.36) is replaced with

$$\begin{aligned} &\Psi[t_0 n_0]^{-\epsilon_0 s_0 \delta(t_0 n_0)} \cdots \Psi[t_{a-1} n_{a-1}]^{-\epsilon_{a-1} s_{a-1} \delta(t_{a-1} n_{a-1})}(y^{t_a n_a}) \\ &= \mathfrak{p}_{\gamma_a, \mathfrak{D}_{s,\ell}}^{-1}(y^{t_a n_a}), \end{aligned} \tag{3.41}$$

where γ_a is the subpath of γ from the base point of γ to z_a. Take any admissible path γ_0 from the base point of γ to a point in $\mathrm{Int}(\mathcal{C}^+)$ and apply $\mathfrak{p}_{\gamma_0, \mathfrak{D}_{s,\ell}}$ to $S[\alpha]$. Then, we obtain the desired formula (3.19).

Now, we consider the general case in (3.16). The equality (3.40) is replaced with

$$y_i' = y_i(1 + h_i(\mathbf{y})) \tag{3.42}$$

for some $h_i(\mathbf{y}) \in \mathcal{F}^{>\ell}$. Then, again by Lemma 3.5, under any infinitesimal variation $\alpha + \delta\alpha$,

$$\delta S[\alpha] = \sum_{i=1}^r (\overline{p}_i' \delta \overline{u}_i' - \overline{p}_i \delta \overline{u}_i), \tag{3.43}$$

$$\delta_i^{-1} \overline{p}_i = \sum_{j=1}^r b_{ji} \overline{u}_j = \frac{1}{\sqrt{2}} \log \overline{y}_i, \quad \delta_i^{-1} \overline{p}_i' = \sum_{j=1}^r b_{ji} \overline{u}_j' = \frac{1}{\sqrt{2}} \log \overline{y}_i', \tag{3.44}$$

where we set $\overline{p}_i = \overline{p}_i(0)$ and $\overline{p}_i' = \overline{p}_i'(P)$, and so forth. From now on, we view $\overline{p}_i, \overline{p}_i', \overline{u}_i, \overline{u}_i'$ as formal Puiseux series in $\mathbf{y} = \overline{\mathbf{y}}(0)$ with log factor as before, and also we omit the bars, for simplicity. By (3.42) and (3.44), we can write them as

$$p'_i - p_i = f_i, \quad u'_j - u_j = g_i \quad (f_i, g_i \in \mathcal{F}^{>\ell}). \tag{3.45}$$

Then, we have

$$\delta S[\alpha] = \sum_{i=1}^{r}\left((p_i + f_i)\left(\delta u_i + \sum_{j=1}^{r}\frac{\partial g_i}{\partial y_j}\frac{\partial y_j}{\partial u_i}\delta u_i\right) - p_i\delta u_i\right)$$

$$= \sum_{i=1}^{r}((\log y_i)\tilde{f}_i + \tilde{g}_i)\delta u_i \quad (\tilde{f}_i, \tilde{g}_i \in \mathcal{F}^{>\ell}) \tag{3.46}$$

$$= \sum_{i=1}^{r}((\log y_i)\hat{f}_i + \hat{g}_i)\delta y_i \quad (\hat{f}_i, \hat{g}_i \in \mathcal{F}^{>\ell-1}).$$

It follows that

$$\frac{\partial S[\alpha]}{\partial y_i} \in \mathcal{F}^{>\ell-1}_{\log}. \tag{3.47}$$

Therefore, we have

$$S[\alpha] \in \mathcal{F}^{>\ell}_{\log}, \tag{3.48}$$

where the constant term is shown to be zero as before. Then, applying $\mathfrak{p}_{\gamma_0, \mathfrak{D}_{\mathfrak{s},\ell}}$ to $S[\alpha]$, we obtain the desired formula (3.19).

(b) When B is singular, we have the problem that $\mathbf{y}$ represented in (3.24) are not algebraically independent on the small phase space $\tilde{M}_0$. To remedy it, we apply the standard principal extension technique [12], [14]. Namely, let $\tilde{N} = N \oplus M^{\circ}$. We extend the skew-symmetric bilinear form $\{\cdot, \cdot\}$ on N to the one on $\tilde{N}$ as

$$\{(n, m), (n', m')\} := \{n, n'\} + \langle n', m \rangle - \langle n, m' \rangle. \tag{3.49}$$

Then, we extend the y-representation (2.5) of G to the one (the *principal y-representation*) on $\mathbb{Q}[[\tilde{\mathbf{y}}]]$ ($\tilde{\mathbf{y}} = (y_1, \ldots, y_{2r})$) by

$$X_n(\tilde{y}^{(n', m')}) = \{(n, 0), (n', m')\}\tilde{y}^{(n'+n, m')}. \tag{3.50}$$

Accordingly, we extend the canonical coordinates $\mathbf{u}$ and $\mathbf{p}$ to $\tilde{\mathbf{u}} = (u_1, \ldots, u_{2r})$ and $\tilde{\mathbf{p}} = (p_1, \ldots, p_{2r})$ to express $\tilde{y}_i$ in the same way as (3.24), where the matrix B is replaced with the principally extended matrix of B

$$\tilde{B} = \begin{pmatrix} B & -I \\ I & O \end{pmatrix}. \tag{3.51}$$

Then, in particular, the subvariables $\mathbf{y} = (y_1, \ldots, y_r)$ are now algebraically independent on the small phase space for $(\tilde{\mathbf{u}}, \tilde{\mathbf{p}})$. After this, the proof of (a) is applicable. □

Now, we take the limit $\ell \to \infty$. We parametrize the intersection of γ and the walls in $\mathfrak{D}_{\mathfrak{s}}$ as z_a ($a \in J$) by a countable and totally ordered set J so that γ crosses z_a earlier than $z_{a'}$ only if $a < a'$. Then, the consistent relation is presented by the infinite product in the increasing order in J from right to left,

$$\mathfrak{p}_{\gamma, \mathfrak{D}_{\mathfrak{s}}} = \prod_{a \in J}^{\leftarrow} \Psi[t_a n_a]^{\epsilon_a s_a \delta(t_a n_a)} = \mathrm{id}. \tag{3.52}$$

Thus, we obtain the first main theorem of the paper.

THEOREM 3.6. *For any admissible loop γ with the consistent relation (3.52), the following identity holds as a formal Puiseux series in $\mathbf{y}$ with $\log$ factor:*

$$\sum_{a \in J} \epsilon_a s_a \delta(t_a n_a) \tilde{L}(y_{z_a}[t_a n_a]) = 0. \tag{3.53}$$

Proof. This is immediately obtained by taking the limit $\ell \to \infty$ of (3.19). $\square$

We call the identity (3.53), together with its reduction (3.19) at degree ℓ, the *DI* associated with a loop γ in $\mathfrak{D}_\mathfrak{s}$.

§4. Construction and reduction of DIs by pentagon identity

4.1 Construction and reduction of CSD by pentagon relation

Let us briefly review the construction of a CSD $\mathfrak{D}_\mathfrak{s}$ satisfying the property in Theorem 2.5(b) by [14], [29].

Let Γ be fixed data of rank 2. We say that a product of dilogarithm elements $\Psi[n]^c$ ($n \in N^+$, $c \in \mathbb{Q}_{>0}$) is *ordered* (resp. *anti-ordered*) if, for any adjacent pair $\Psi[n']^{c'}\Psi[n]^c$, $\{n', n\} \leq 0$ (resp. $\{n', n\} \geq 0$) holds.

The following is a key lemma which we use in the construction of the abovementioned CSD.

PROPOSITION 4.1 (Ordering lemma [29, Prop. III.5.4]). *Let $\mathfrak{s}$ be a seed for a fixed data Γ of rank 2. Let*

$$C^{\mathrm{in}} = \Psi[t'_j n'_j]^{s'_j \delta(t'_j n'_j)} \cdots \Psi[t'_1 n'_1]^{s'_1 \delta(t'_1 n'_1)} \quad (n'_a \in N^+_{\mathrm{pr}}, \ s'_a, t'_a \in \mathbb{Z}_{>0}) \tag{4.1}$$

be any finite anti-ordered product. Then, C^{in} equals to a (possibly infinite) ordered product C^{out} of factors of the same form

$$\Psi[t_a n_a]^{s_a \delta(t_a n_a)} \quad (n_a \in N^+_{\mathrm{pr}}, \ s_a, t_a \in \mathbb{Z}_{>0}). \tag{4.2}$$

Moreover, the above relation $C^{\mathrm{in}} = C^{\mathrm{out}}$ is obtained from a trivial relation $C^{\mathrm{in}} = C^{\mathrm{in}}$ by applying the relations in Proposition 2.2 possibly infinitely many times.

The explicit algorithm of obtaining C^{out} from C^{in} is given in [29, Algorithm III.5.7]. We recall important notions for a scattering diagram [14].

DEFINITION 4.2 (Parallel/perpendicular joint).

(a) Let $\mathfrak{D}$ be a scattering diagram. For any pair of walls $\mathbf{w}_i = (\mathfrak{d}_i, g_i)_{n_i}$ ($i = 1, 2$), the intersection of their supports $\mathfrak{j} = \mathfrak{d}_1 \cap \mathfrak{d}_2$ is called a *joint* of $\mathfrak{D}$ if $\mathfrak{j}$ is a cone of codimension 2.

(b) For any joint $\mathfrak{j}$, let

$$N_{\mathfrak{j}} := \{n \in N \mid \langle n, z \rangle = 0 \ (z \in \mathfrak{j})\}, \tag{4.3}$$

which is the rank 2 sublattice of N. Then, a joint $\mathfrak{j}$ is *parallel* (resp. *perpendicular*) if the skew-symmetric form $\{\cdot, \cdot\}$ restricted on $N_{\mathfrak{j}}$ vanishes (otherwise).

Based on Proposition 4.1, we present the construction of a CSD $\mathfrak{D}_\mathfrak{s}$, which was given by [14] and modified with Proposition 4.1 by [29]. The resulting CSD satisfies the property in Theorem 2.5(b).
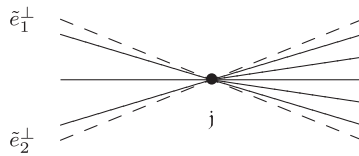
Figure 1.
Walls containing a perpendicular joint j.

CONSTRUCTION 4.3 [14, Appendix C.3], [29, Construction III.5.14]. We construct scattering diagrams $\mathfrak{D}_1 \subset \mathfrak{D}_2 \subset \cdots$ as below. Then, a CSD $\mathfrak{D}_\mathfrak{s}$ in Theorem 2.5(b) is given by

$$\mathfrak{D}_\mathfrak{s} = \bigcup_{\ell=1}^{\infty} \mathfrak{D}_\ell. \tag{4.4}$$

(1) We start with $\mathfrak{D}_1 = \mathrm{In}_\mathfrak{s}$.
(2) We construct $\mathfrak{D}_{\ell+1}$ from $\mathfrak{D}_\ell$ as follows: For any *perpendicular* joint j of $\mathfrak{D}_\ell$, let N_j be the one in (4.3). Let $N_j^+ := N^+ \cap N_j$ and $N_{j,\mathrm{pr}}^+ := N_{\mathrm{pr}}^+ \cap N_j$. Then, there exists a unique pair $\tilde{e}_1, \tilde{e}_2 \in N_{j,\mathrm{pr}}^+$ such that $N_j^+ \subset \mathbb{Q}_{\geq 0}\tilde{e}_1 + \mathbb{Q}_{\geq 0}\tilde{e}_2$ and $\{\tilde{e}_2, \tilde{e}_1\} > 0$. See Figure 1. Thus, $p^*(n)$ is in the *second quadrant* in $M_\mathbb{R}$ with respect to $\tilde{e}_1^\perp$ and $\tilde{e}_2^\perp$. Then, by the construction of walls as described below, all wall elements in $\mathfrak{D}_\ell$ have the form in (2.15). We say that a product of dilogarithm elements $\Psi[n]^c$ ($n \in N_j^+$, $c \in \mathbb{Q}_{>0}$) is *ordered* (resp. *anti-ordered*) if, for any adjacent pair $\Psi[n']^{c'}\Psi[n]^c$, $\{n', n\} \leq 0$ (resp. $\{n', n\} \geq 0$) holds. We take the wall elements of all walls which contain j and lie in the second quadrant with respect to $\tilde{e}_1^\perp$ and $\tilde{e}_2^\perp$, and we consider the anti-ordered product of them

$$C^{\mathrm{in}} = \Psi[t'_{j'} n'_{j'}]^{s'_{j'}\delta(t'_{j'}n'_{j'})} \cdots \Psi[t'_1 n'_1]^{s'_1\delta(t'_1 n'_1)} \quad (\deg(t'_a n'_a) \leq \ell), \tag{4.5}$$

where $n'_a \in N_{j,\mathrm{pr}}^+$ and $s'_a, t'_a \in \mathbb{Z}_{>0}$. Then, we apply Proposition 4.1 to C^{in} modulo $G^{>\ell+1}$, and obtain a finite-ordered product

$$C^{\mathrm{out}}(\ell) = \Psi[t_j n_j]^{s_j\delta(t_j n_j)} \cdots \Psi[t_1 n_1]^{s_1\delta(t_1 n_1)} \quad (\deg(t_a n_a) \leq \ell+1) \tag{4.6}$$

such that

$$C^{\mathrm{in}} \equiv C^{\mathrm{out}}(\ell) \mod G^{>\ell+1}. \tag{4.7}$$

By the construction of walls which we describe below, all factors in (4.6) with $\deg(t_a n_a) \leq \ell$ already appear as wall elements in $\mathfrak{D}_\ell$. Now, we add new walls

$$(\sigma(\mathrm{j}, -p^*(n_a)), \Psi[t_a n_a]^{s_a\delta(t_a n_a)})_{n_a} \tag{4.8}$$

to $\mathfrak{D}_\ell$ for all factors in (4.6) with $\deg(t_a n_a) = \ell+1$. The cone $\sigma(\mathrm{j}, -p^*(n_a))$, which is generated by j and $-p^*(n_a)$, is of codimension 1 due to the perpendicular condition $p^*(n_a) \notin \mathbb{R}\mathrm{j}$. Also, it is outgoing because it is in the *fourth quadrant* in $M_\mathbb{R}$ with respect to $\tilde{e}_1^\perp$ and $\tilde{e}_2^\perp$. We do the procedure for all perpendicular joints of $\mathfrak{D}_\ell$ to obtain $\mathfrak{D}_{\ell+1}$.

The consistency around parallel joints of the above constructed $\mathfrak{D}_\mathfrak{s}$ is not obvious, but it was proved in [14, Appendix C.3].

The above construction of a CSD immediately implies the following important result, which is implicit in [14, Appendix C.3], and described explicitly in [29].

THEOREM 4.4 [29, Th. III.5.17]. *For any admissible loop γ for $\mathfrak{D}_{\mathfrak{s}}$, the consistency relation $\mathfrak{p}_{\gamma,\mathfrak{D}_{\mathfrak{s}}} = \mathrm{id}$ is reduced to a trivial one by applying the relations in Proposition 2.2 possibly infinitely many times.*

Proof. For the reader's convenience, we present the proof of [29, Th. III.5.17]. Fix $\ell > 0$, and consider the reduction $\mathfrak{D}_{\mathfrak{s},\ell}$ at ℓ. By the topological reason, any consistency relation is reduced to the consistency relations for admissible loops around joints in $\mathfrak{D}_{\mathfrak{s},\ell}$. For a parallel joint $\mathfrak{j}$, which is a trivial case, the consistency relation has the form

$$
\begin{aligned}
&\Psi[t'_{j'}n'_{j'}]^{s'_{j'}\delta(t'_{j'}n'_{j'})}\cdots\Psi[t'_1n'_1]^{s'_1\delta(t'_1n'_1)} \\
&= \Psi[t'_1n'_1]^{s'_1\delta(t'_1n'_1)}\cdots\Psi[t'_{j'}n'_{j'}]^{s'_{j'}\delta(t'_{j'}n'_{j'})},
\end{aligned}
\tag{4.9}
$$

where the left-hand side is an anti-ordered product. By applying (2.10), it is reduced to a trivial relation. For a perpendicular joint $\mathfrak{j}$, which is a nontrivial case, the consistency relation has the form

$$
\begin{aligned}
&\Psi[t'_{j'}n'_{j'}]^{s'_{j'}\delta(t'_{j'}n'_{j'})}\cdots\Psi[t'_1n'_1]^{s'_1\delta(t'_1n'_1)} \\
&\equiv \Psi[t_jn_j]^{s_j\delta(t_jn_j)}\cdots\Psi[t_1n_1]^{s_1\delta(t_1n_1)} \quad \mathrm{mod}\, G^{>\ell},
\end{aligned}
\tag{4.10}
$$

where the left- and right-hand sides are the ones in (4.5) and (4.6), respectively. The right-hand side is obtained from the left-hand side by Proposition 4.1 (more precisely, by Algorithm III.5.7 of [29]), which depends only on the relations (2.10) and (2.11). By applying the relations in the reverse way, the relation (4.10) is reduced to a trivial one. $\qquad\square$

4.2 Construction and reduction of DIs by pentagon identity

Based on Construction 4.3 and Theorem 4.4, we will give parallel results for DIs, where the role of the pentagon relation (2.11) is played by the pentagon identity (3.11) of the Rogers dilogarithm.

For this purpose, we reformulate the pentagon identity in (3.11) into a form which is closer to the pentagon relation (2.11).

LEMMA 4.5. *Let $n_1, n_2 \in N^+$. If $\{n_2, n_1\} = c$ $(c \in \mathbb{Q}, c \neq 0)$, the following pentagon identity holds:*

$$
\begin{aligned}
\frac{1}{c}\Psi[n_1]^{-1/c}(\tilde{L}(y^{n_2})) + \frac{1}{c}\tilde{L}(y^{n_1}) &= \frac{1}{c}\Psi[n_2]^{-1/c}\Psi[n_1+n_2]^{-1/c}(\tilde{L}(y^{n_1})) \\
&\quad + \frac{1}{c}\Psi[n_2]^{-1/c}(\tilde{L}(y^{n_1+n_2})) + \frac{1}{c}\tilde{L}(y^{n_2}).
\end{aligned}
\tag{4.11}
$$

Proof. we have

$$
\Psi[n_1]^{-1/c}(y^{n_2}) = y^{n_2}(1+y^{n_1}),
\tag{4.12}
$$

$$
\Psi[n_2]^{-1/c}(y^{n_1+n_2}) = y^{n_1+n_2}(1+y^{n_2})^{-1},
\tag{4.13}
$$

$$
\begin{aligned}
\Psi[n_2]^{-1/c}\Psi[n_1+n_2]^{-1/c}(y^{n_1}) &= \Psi[n_2]^{-1/c}(y^{n_1}(1+y^{n_1+n_2})^{-1}) \\
&= y^{n_1}(1+y^{n_2})^{-1}(1+y^{n_1+n_2}(1+y^{n_2})^{-1})^{-1} \\
&= y^{n_1}(1+y^{n_2}+y^{n_1+n_2})^{-1}.
\end{aligned}
\tag{4.14}
$$

Then, the identity (4.11) is obtained from (3.11) by setting $y_1 = y^{n_1}$ and $y_2 = y^{n_2}$. $\qquad\square$
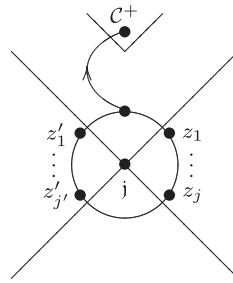
Figure 2.
Configuration for the consistency relation in (4.10).

The overall factor $1/c$ was put so that the identity (4.11) perfectly matches (the log form of) the corresponding pentagon relation (2.11). Also, observe that action of dilogarithm elements in (4.11) is parallel to the one in (3.41).

Thanks to the correspondence between (4.11) and (2.11), we obtain a parallel result to Construction 4.3 and Theorem 4.4 for the DIs. This is the second main result of the paper.

THEOREM 4.6. *For any admissible loop γ for $\mathfrak{D}_{\mathfrak{s}}$, the associated DI in (3.53) is constructed from and reduced to a trivial one by applying the pentagon identity in (3.11) possibly infinitely many times.*

Proof. Fix $\ell > 0$, and consider the reduction $\mathfrak{D}_{\mathfrak{s},\ell}$ at ℓ. We may concentrate on a sufficiently small loop around a perpendicular joint $\mathfrak{j}$. The constancy relation is given in (4.10), where the underlying configuration in $\mathfrak{D}_{\mathfrak{s},\ell}$ is depicted in Figure 2. The corresponding DI in (3.19) has the form

$$\sum_{a=1}^{j'} s'_a \delta(t'_a n'_a) \tilde{L}(y_{z'_a}[t'_a n'_a]) \equiv \sum_{a=1}^{j} s_a \delta(t_a n_a) \tilde{L}(y_{z_a}[t_a n_a]) \mod \mathcal{F}_{\log}^{>\ell}. \tag{4.15}$$

In view of Figure 2, this is rewritten in the form

$$\sum_{a=1}^{j'} \Psi[t'_1 n'_1]^{-s'_1 \delta(t'_1 n'_1)} \cdots \Psi[t'_{a-1} n'_{a-1}]^{-s'_{a-1}\delta(t'_{a-1}n'_{a-1})} (s'_a \delta(t'_a n'_a)\tilde{L}(y^{t'_a n'_a}))$$
$$\tag{4.16}$$
$$\equiv \sum_{a=1}^{j} \Psi[t_1 n_1]^{-s_1 \delta(t_1 n_1)} \cdots \Psi[t_{a-1} n_{a-1}]^{-s_{a-1}\delta(t_{a-1}n_{a-1})} (s_a \delta(t_a n_a)\tilde{L}(y^{t_a n_a}))$$

modulo $\mathcal{F}_{\log}^{>\ell}$. Along the procedure, say, $\mathcal{P}$ of obtaining (4.6) from (4.5) by Proposition 4.1 (more precisely, by Algorithm III.5.7 of [29]), we do the following procedure:

- As the initial input, we set $F(y)$ to be the left-hand side of (4.16).
- If the pentagon relation (2.11) is applied to the adjacent pair $\Psi[t'n']^{c'}$ and $\Psi[tn]^c$ in the procedure $\mathcal{P}$, we apply the pentagon identity (4.11) to the corresponding terms in $F(y)$. We also apply the pentagon relation (2.11) itself to the corresponding pair $\Psi[t'n']^{c'}$ and $\Psi[tn]^c$ in $F(y)$. (This does not change $F(y)$ as a formal Puiseux series in **y** with log factor.)
- If the commutative relation modulo $G^{>\ell}$ is applied to the adjacent pair $\Psi[t'n']$ and $\Psi[tn]$ in the procedure $\mathcal{P}$, we also apply it to the corresponding pair $\Psi[t'n']^{c'}$ and $\Psi[tn]^c$ in $F(y)$. (This occurs for the relation (2.10) or the truncation of the relation (2.11) with

$\deg(n_1 + n_2) > \ell$. In the former case, this does not change $F(y)$, whereas in the latter case, the result equals to $F(y)$ modulo $\mathcal{F}_{\log}^{>\ell}$.)

Then, thanks to the correspondence between (4.11) and (2.11), the final result is the right-hand side of (4.16). Thus, the equality (4.16) is reduced to the trivial one. Also, by reversing the procedure, the identity (4.16) is obtained from a trivial one. $\qquad\square$

Note that this also provides an alternative proof of Theorem 3.4 without the classical mechanical method.

EXAMPLE 4.7 (Type B_2). Let us demonstrate how the procedure in the proof of Theorem 4.6 works in practice. Consider the consistency relation for a CSD of type B_2 obtained by the successive application of the pentagon relation [29, §III.2.2],

$$
\begin{aligned}
\begin{bmatrix}0\\1\end{bmatrix}\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right) &= \left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} \\
&= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)\begin{bmatrix}0\\1\end{bmatrix} \\
&= \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix},
\end{aligned}
\tag{4.17}
$$

where the pentagon relation (2.11) is applied to the pairs in the parentheses. The associated DI in the form (4.16) is obtained as follows:

$$
\begin{aligned}
&\Psi[\mathbf{e}_1]^{-1}\Psi[\mathbf{e}_2]^{-1}\tilde{L}(y^{\mathbf{e}_2}) + (\Psi[\mathbf{e}_1]^{-1}\tilde{L}(y^{\mathbf{e}_2}) + \tilde{L}(y^{\mathbf{e}_1})) \\
&= (\Psi[\mathbf{e}_2]^{-1}\Psi[(1,1)]^{-1}\Psi[\mathbf{e}_1]^{-1}\tilde{L}(y^{\mathbf{e}_2}) + \Psi[\mathbf{e}_2]^{-1}\Psi[(1,1)]^{-1}\tilde{L}(y^{\mathbf{e}_1})) \\
&\qquad + \Psi[\mathbf{e}_2]^{-1}\tilde{L}(y^{(1,1)}) + \tilde{L}(y^{\mathbf{e}_2}) \\
&= \Psi[\mathbf{e}_2]^{-1}\Psi[(1,1)]^{-1}\Psi[\mathbf{e}_2]^{-1}\Psi[(1,1)]^{-1}\tilde{L}(y^{\mathbf{e}_1}) \\
&\qquad + \Psi[\mathbf{e}_2]^{-1}\Psi[(1,1)]^{-1}\Psi[\mathbf{e}_2]^{-1}\tilde{L}(y^{(1,1)}) \\
&\qquad + (\Psi[\mathbf{e}_2]^{-1}\Psi[(1,1)]^{-1}\tilde{L}(y^{\mathbf{e}_2}) + \Psi[\mathbf{e}_2]^{-1}\tilde{L}(y^{(1,1)})) + \tilde{L}(y^{\mathbf{e}_2}) \\
&= \Psi[\mathbf{e}_2]^{-1}\Psi[\mathbf{e}_2]^{-1}\Psi[(1,2)]^{-1}\Psi[(1,1)]^{-1}\Psi[(1,1)]^{-1}\tilde{L}(y^{\mathbf{e}_1}) \\
&\qquad + \Psi[\mathbf{e}_2]^{-1}\Psi[\mathbf{e}_2]^{-1}\Psi[(1,2)]^{-1}\Psi[(1,1)]^{-1}\tilde{L}(y^{(1,1)}) \\
&\qquad + \Psi[\mathbf{e}_2]^{-1}\Psi[\mathbf{e}_2]^{-1}\Psi[(1,2)]^{-1}\tilde{L}(y^{(1,1)}) \\
&\qquad + \Psi[\mathbf{e}_2]^{-1}\Psi[\mathbf{e}_2]^{-1}\tilde{L}(y^{(1,2)}) + \Psi[\mathbf{e}_2]^{-1}\tilde{L}(y^{\mathbf{e}_2}) + \tilde{L}(y^{\mathbf{e}_2}).
\end{aligned}
\tag{4.18}
$$

For example, in the first equality, the pentagon identity (4.11) is applied to the second and third terms, whereas the corresponding pentagon relation (2.11) is applied to the first term.

Recall that a sequence of mutations for a seed of a cluster pattern is called a σ-*period* if it acts as a permutation σ of the indices of the seed [27]. Any σ-period of a cluster pattern corresponds to a loop γ contained in the support of the G-fan in the corresponding CSD. Thus, we have the following corollary.

COROLLARY 4.8.

(a) *For any σ-period of a cluster pattern, the corresponding consistency relation $\mathfrak{p}_{\gamma,\mathfrak{D}_s} =$ id is reduced to a trivial one by applying the commutative and pentagon relations in Proposition 2.2 possibly infinitely many times.*

(b) *Any DI associated with a σ-period of a cluster pattern in [27] is reduced to a trivial one by applying the pentagon identity in (3.11) possibly infinitely many times.*

Thus, our DIs have the *infinite reducibility* in view of the reducibility problem of the DIs to the pentagon relation [32, §2.A].

REMARK 4.9.

(a) The above corollary does *not* imply that any σ-period of a cluster pattern is reduced to a trivial one by applying the square and pentagon periodicities *in the cluster pattern itself*. In fact, Example 4.7 is such a case, where there are no square and pentagon periodicities at all in the cluster pattern. Some other examples are also known [9], [20].

(b) Even if the corresponding DI for a cluster pattern has only finitely many terms, it might be necessary to apply the pentagon relation *infinitely many times* to reduce it to a trivial one, in general.

§5. Examples: rank 2 CSDs of affine type

Let us present examples of DIs for rank 2 CSDs of affine type.

5.1 Type $A_1^{(1)}$

Without loss of generality, we may assume that $\{e_2, e_1\} = 1$ [29, §III.1.5]. We consider the case $\delta_1 = \delta_2 = 2$ so that the initial exchange matrix B in (2.3) is given by

$$B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}. \tag{5.1}$$

Let $\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$ denote the dilogarithm element $\Psi[n]$ with $n = n_1 e_1 + n_2 e_2$. The structure of the CSD is well known (see [14, Exam. 1.15], [31, Th. 6.1], [30, Th. 3.4]), and it is represented by the following relation:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}^2 \cdots \prod_{j=0}^{\infty} \begin{bmatrix} 2^j \\ 2^j \end{bmatrix}^{2^{2-j}} \cdots \begin{bmatrix} 2 \\ 3 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2. \tag{5.2}$$

See also [29, §III.5.7] and [25] for an alternative derivation of the relation by the pentagon identity. See Figure 3a. Here, the left-hand side is the path-ordered product $\mathfrak{p}_{\gamma_1, \mathfrak{D}_s}$ along the curve γ_1, whereas the right-hand side is the path-ordered product $\mathfrak{p}_{\gamma_2, \mathfrak{D}_s}$ along the curve γ_2. There are infinitely many walls with support $\mathfrak{d} = \mathbb{R}_{\geq 0}(1, -1) \subset (1, 1)^{\perp}$. We have $\delta(2^j(1, 1)) = 2^{1-j}$. Thus, the factor s_a is 2 for these walls, and 1 for the rest.

We only need to consider the loop $\gamma_2^{-1} \circ \gamma_1$, where the consistency relation is given by the relation (5.2). Then, the associated DI in (3.19) is written as follows:

$$\tilde{L}(y_2(1 + y_1)^2) + \tilde{L}(y_1) = \tilde{L}(y_1(1 + y_2(1 + y_1)^2)^{-2}) + \Lambda + \tilde{L}(y_2), \tag{5.3}$$

$$\Lambda = \tilde{L}(y[(2, 1)]) + \tilde{L}(y[(3, 2)]) + \cdots$$
$$+ \sum_{j=0}^{\infty} 2^{1-j} \tilde{L}(y[2^j(1, 1)]) + \cdots + \tilde{L}(y[(2, 3)]) + \tilde{L}(y[(1, 2)]). \tag{5.4}$$

In the sum Λ, $y_{z_a}[t_a n_a]$ in (3.19) is unambiguously parametrized by $t_a n_a \in N^+$, so that z_a is omitted. Also, the common factor 2 is omitted. Since the left-hand side is a finite sum, the right-hand side converges as a function of $\mathbf{y} \in \mathbb{R}_{\geq 0}^2$.
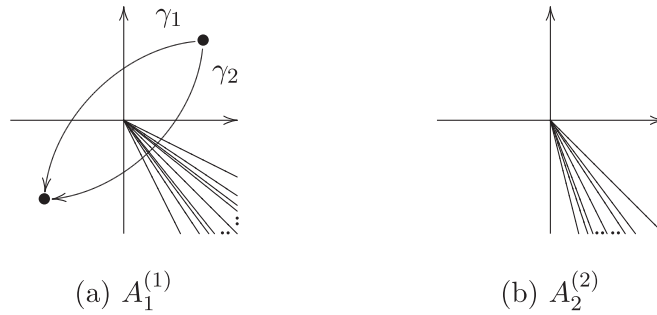
(a) $A_1^{(1)}$ (b) $A_2^{(2)}$

Figure 3.
Rank 2 CSDs of affine type, where types $A_1^{(1)}$ and $A_2^{(2)}$ are depicted in (a) and (b), respectively.

It seems difficult to obtain the explicit expression for $y[t_a n_a]$ in Λ, and we do not seek it here. Instead, we demonstrate the validity of the reduced identity (3.19) for small degrees ℓ.

(i) $\ell = 1$. The relation (5.2) is reduced at $\ell = 1$ to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \mod G^{>1}. \tag{5.5}$$

We have

$$y_2(1+y_1)^2 \equiv y_2, \quad y_1(1+y_2(1+y_1)^2)^{-2} \equiv y_1 \mod \mathcal{F}^{>1}. \tag{5.6}$$

The reduced DI at $\ell = 1$ is

$$\tilde{L}(y_2) + \tilde{L}(y_1) \equiv \tilde{L}(y_1) + \tilde{L}(y_2) \mod \mathcal{F}^{>1}_{\log}. \tag{5.7}$$

This holds trivially.

(ii) $\ell = 2$. The relation (5.2) is reduced at $\ell = 2$ to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \mod G^{>2}. \tag{5.8}$$

We have, modulo $\mathcal{F}^{>2}$,

$$\begin{aligned} y_2(1+y_1)^2 &\equiv y_2 + 2y_1 y_2, \\ y_1(1+y_2(1+y_1)^2)^{-2} &\equiv y_1 - 2y_1 y_2, \\ y[(1,1)] &\equiv y_1 y_2. \end{aligned} \tag{5.9}$$

Thus, the reduced DI at $\ell = 2$ is

$$\tilde{L}(y_2 + 2y_1 y_2) + \tilde{L}(y_1) \equiv \tilde{L}(y_1 - 2y_1 y_2) + 2\tilde{L}(y_1 y_2) + \tilde{L}(y_2) \mod \mathcal{F}^{>2}_{\log}. \tag{5.10}$$

This can be also verified directly by (3.10).

(iii) $\ell = 3$. The relation (5.2) is reduced at $\ell = 3$ to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}^2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 \mod G^{>3}. \tag{5.11}$$

We have, modulo $\mathcal{F}^{>3}$,

$$y_2(1+y_1)^2 \equiv y_2 + 2y_1y_2 + y_1^2y_2,$$
$$y_1(1+y_2(1+y_1)^2)^{-2} \equiv y_1 - 2y_1y_2 - 4y_1^2y_2 + 3y_1y_2^2,$$
$$y[(1,2)] \equiv y_1y_2^2. \tag{5.12}$$
$$y[(1,1)] \equiv y_1y_2(1+y_2)^{-2} \equiv y_1y_2 - 2y_1y_2^2.$$
$$y[(2,1)] \equiv y_1^2y_2.$$

Thus, the reduced DI at $\ell = 3$ is

$$\begin{aligned}
&\tilde{L}(y_2 + 2y_1y_2 + y_1^2y_2) + \tilde{L}(y_1) \\
&\equiv \tilde{L}(y_1 - 2y_1y_2 - 4y_1^2y_2 + 3y_1y_2^2) + \tilde{L}(y_1^2y_2) + 2\tilde{L}(y_1y_2 - 2y_1y_2^2) \\
&\quad + \tilde{L}(y_1y_2^2) + \tilde{L}(y_2) \mod \mathcal{F}_{\log}^{>3}.
\end{aligned} \tag{5.13}$$

Again, this can be also verified directly by (3.10).

5.2 Type $A_2^{(2)}$

We consider the case $\delta_1 = 1$, $\delta_2 = 4$ so that the initial exchange matrix B in (2.3) is given by

$$B = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}. \tag{5.14}$$

The structure of the CSD is well known [14], [30], and it is represented by the following relation:

$$\begin{aligned}
\begin{bmatrix} 0 \\ 1 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = &\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}^4 \begin{bmatrix} 5 \\ 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix}^4 \cdots \\
&\times \begin{bmatrix} 1 \\ 2 \end{bmatrix}^6 \prod_{j=1}^{\infty} \begin{bmatrix} 2^j \\ 2^{j+1} \end{bmatrix}^{2^{2-j}} \cdots \begin{bmatrix} 5 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}^4 \begin{bmatrix} 3 \\ 8 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}^4.
\end{aligned} \tag{5.15}$$

See also [29, §III.5.7] and [25] for an alternative derivation by the pentagon identity. Here, the canonical paring $\langle n, z \rangle$ for $N \times M_{\mathbb{R}}$ is given by

$$\langle n, z \rangle = (n_1, n_2) \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{5.16}$$

under the identification $N \simeq \mathbb{Z}^2$, $e_i \mapsto \mathbf{e}_i$ and $M_{\mathbb{R}} \simeq \mathbb{R}^2$, $f_i \mapsto \mathbf{e}_i$. There are infinitely many walls with support $\mathfrak{d} = \mathbb{R}_{\geq 0}(1, -2) \subset (1,2)^{\perp}$. We have $\delta(2^j(1,2)) = 2^{1-j}$. Thus, the factor s_a is 2 or 3 for these walls, and 1 for the rest. The associated DI (3.19) is written as follows:

$$4\tilde{L}(y_2(1+y_1)) + \tilde{L}(y_1) = \tilde{L}(y_1(1+y_2(1+y_1))^{-4}) + \Lambda + 4\tilde{L}(y_2), \tag{5.17}$$

$$\begin{aligned}
\Lambda = &4\tilde{L}(y[(1,1)]) + \tilde{L}(y[(3,4)]) + \cdots + 6\tilde{L}(y[(1,2)]) \\
&+ \sum_{j=1}^{\infty} 2^{2-j}\tilde{L}(y[2^j(1,2)]) + \cdots + 4\tilde{L}(y[(1,3)]) + \tilde{L}(y[(1,4)]).
\end{aligned} \tag{5.18}$$

Since the left-hand side is a finite sum, the right-hand side converges as a function of $\mathbf{y} \in \mathbb{R}_{\geq 0}^2$.

Let us concentrate on the reduction at $\ell = 3$. The relation (5.15) is reduced at $\ell = 3$ to

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 2 \end{bmatrix}^6 \begin{bmatrix} 0 \\ 1 \end{bmatrix}^4 \quad \mod G^{>3}. \tag{5.19}$$

We have, modulo $\mathcal{F}^{>3}$,

$$\begin{aligned}
y_1(1 + y_2(1 + y_1))^{-4} &\equiv y_1 - 4y_1 y_2 - 4y_1^2 y_2 + 10 y_1 y_2^2, \\
y[(1,2)] &\equiv y_1 y_2^2. \\
y[(1,1)] &\equiv y_1 y_2 (1 + y_2)^{-4} \equiv y_1 y_2 - 4 y_1 y_2^2.
\end{aligned} \tag{5.20}$$

Thus, the reduced DI at $\ell = 3$ is

$$\begin{aligned}
&4\tilde{L}(y_2 + y_1 y_2) + \tilde{L}(y_1) \\
&\equiv \tilde{L}(y_1 - 4y_1 y_2 - 4y_1^2 y_2 + 10 y_1 y_2^2) + 4\tilde{L}(y_1 y_2 - 4 y_1 y_2^2) \\
&\quad + 6\tilde{L}(y_1 y_2^2) + 4\tilde{L}(y_2) \quad \mod \mathcal{F}_{\log}^{>3}.
\end{aligned} \tag{5.21}$$

Again, this can be also verified directly by (3.10).

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