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EXISTENCE RESULTS FOR NONLINEAR PERIODIC BOUNDARY-VALUE PROBLEMS

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Abstract We study a class of second-order nonlinear differential equations on a finite interval with periodic boundary conditions. The nonlinearity in the equations can take negative values and may be unbounded from below. Criteria are established for the existence of non-trivial solutions, positive solutions and negative solutions of the problems under consideration. Applications of our results to related eigenvalue problems are also discussed. Examples are included to illustrate some of the results. Our analysis relies mainly on topological degree theory.

Keywords: solutions; boundary-value problems; cone; spectral radius; topological degree

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Secondary 34B18

1. Introduction

Let T be a fixed positive number. In this paper, we are concerned with the existence of solutions of the boundary-value problem (BVP) consisting of the equation

$$-(p(t)u')' + q(t)u = w(t)f(t, u), \quad t \in (0, T), \tag{1.1}$$

and the periodic boundary condition (BC)

$$u(0) = u(T), \qquad u^{[1]}(0) = u^{[1]}(T), \tag{1.2}$$

where

$$u^{[1]}(t) = p(t)u'(t)$$

denotes the quasi-derivative of $u(t)$. As applications of our results, we also study the BVP consisting of the equation

$$-(p(t)u')' + q(t)u = \lambda w(t)f(t, u), \quad t \in (0, T), \tag{1.3}$$

and the BC (1.2), where λ is a positive parameter. We assume throughout, and without further mention, that the following conditions hold:

$$\left.\begin{aligned} \frac{1}{p}, q, w \in L(0,T),\\ p, w > 0, \quad q \geqslant 0 \quad \text{and} \quad q \not\equiv 0 \text{ a.e. on } (0,T),\\ f \in C([0,T] \times \mathbb{R}). \end{aligned}\right\} \tag{H}$$

By a *solution* of BVP (1.1), (1.2), we mean a function $u \in C^1[0,T]$ such that $p(t)u'$ is absolutely continuous on $(0,T)$, u satisfies equation (1.1) a.e. on $(0,T)$, and u satisfies BC (1.2). Moreover, if $u(t) > 0$ for $t \in [0,T]$, then $u(t)$ is said to be a *positive solution* of BVP (1.1), (1.2). Similar definitions also apply for BVP (1.3), (1.2) as well as for *negative solutions* of these problems.

BVPs with periodic BCs have been extensively studied in the literature. As examples of recent work, we mention the papers of Atici and Guseinov [**1**], Graef *et al*. [**3**], Jiang *et al*. [**6**], Lan [**10**], O'Regan and Wang [**11**], Torres [**15**], Yao [**19**] and Zhang and Wang [**21**]. In particular, Atici and Guseinov [**1**] used Krasnosel'skii's fixed-point theorem to obtain sufficient conditions for the existence of positive solutions of BVPs (1.1), (1.2) and (1.3), (1.2) when the nonlinearity f is non-negative over $[0,T] \times \mathbb{R}^+$ with $\mathbb{R}^+ = [0,\infty)$. In a very nice paper, Lan [**10**] considered the problem (1.1), (1.2) with $p(t) \equiv 1$, $q(t) \equiv a > 0$, $w(t) \equiv 1$ and $f(t,x)$ non-negative but with a possible singularity at $x = 0$. He obtained sufficient conditions for the existence of one or several positive solutions. Yao [**19**] used fixed-point theory in a cone to obtain the existence of one or more positive solutions of the problem (1.1), (1.2) with $p(t) \equiv 1$, $q(t) \equiv 0$ and $w(t) \equiv 1$. The function $f(t,x)$ could be singular at $x = 0$ here as well.

In this paper, by means of topological degree theory, we derive new criteria for the existence of non-trivial solutions, positive solutions and negative solutions of these problems when f is a sign-changing function and not necessarily bounded from below even over $[0,T] \times \mathbb{R}^+$. Our existence conditions are determined by the relationship between the behaviour of the quotient $f(t,x)/x$ for x near 0 and $\pm\infty$ and the smallest eigenvalue λ_0 (given in (2.2), below) of a related linear problem. Our work extends some results in [**1**] (see Remarks 2.5 and 2.10).

The proofs of our results are partly motivated by a recent paper by Han and Wu [**5**], where the BVP consisting of the equation

$$-u'' = g(t)h(u), \quad t \in (0,1), \tag{1.4}$$

and the BC

$$u(0) = u(1) = 0 \tag{1.5}$$

is studied for the case when h is, loosely speaking, superlinear. By a topological degree argument, conditions are given in terms of the relative behaviour of the quotient $h(x)/x$ for x near 0 and ∞ with respect to the smallest eigenvalue of a related linear problem. Existence criteria of the kind in [**5**] are obtained in [**2**, **13**, **14**, **16**, **17**] for other different types of BVPs.

We now introduce some notation that will be used throughout this paper. For $b \geqslant 0$, denote by ϕ_b and ψ_b the unique solutions of the initial-value problems

$$-(p(t)u')' + (q(t) + bw(t))u = 0, \quad u(0) = 1, \quad u^{[1]}(0) = 0,$$

and

$$-(p(t)u')' + (q(t) + bw(t))u = 0, \quad u(0) = 0, \quad u^{[1]}(0) = 1,$$

respectively. Let

$$D = \phi_b(T) + \psi_b^{[1]}(T) - 2. \tag{1.6}$$

Then, by [**1**, Lemma 2.3], $D > 0$. Define $H(t, s; b)$ by

$$\begin{aligned} H(t, s; b) = {} & \frac{\psi_b(T)}{D}\phi_b(t)\phi_b(s) - \frac{\phi_b(T)}{D}\psi_b(t)\psi_b(s) \\ & + \begin{cases} \dfrac{\psi_b^{[1]}(T) - 1}{D}\phi_b(t)\psi_b(s) - \dfrac{\phi_b(T) - 1}{D}\phi_b(s)\psi_b(t), & 0 \leqslant s \leqslant t \leqslant T, \\[2ex] \dfrac{\psi_b^{[1]}(T) - 1}{D}\phi_b(s)\psi_b(t) - \dfrac{\phi_b(T) - 1}{D}\phi_b(t)\psi_b(s), & 0 \leqslant t \leqslant s \leqslant T. \end{cases} \end{aligned} \tag{1.7}$$

From [**1**, Theorem 2.5],

$$H(t, s; b) > 0 \quad \text{for } t, s \in [0, T]. \tag{1.8}$$

When $b = 0$, we denote $H(t, s; 0)$ by $G(t, s)$, i.e.

$$G(t, s) = H(t, s; 0). \tag{1.9}$$

Let

$$m = \min_{t,s \in [0,T]} G(t, s) \quad \text{and} \quad M = \max_{t,s \in [0,T]} G(t, s). \tag{1.10}$$

Then $M > m > 0$.

Remark 1.1. For some special coefficients p and q, the constants m and M defined by (1.10) can be explicitly computed. For instance, when $p(t) > 0$ and $q(t) = \rho^2/p(t)$ on $[0, T]$ for some $\rho > 0$, m and M are given by (see, for example, [**1**, p. 354])

$$m = \frac{\exp\{(\rho/2)\int_0^T \mathrm{d}s/p(s)\}}{\rho(\exp\{\rho\int_0^T \mathrm{d}s/p(s)\} - 1)} \quad \text{and} \quad M = \frac{\exp\{\rho\int_0^T \mathrm{d}s/p(s)\} + 1}{2\rho(\exp\{\rho\int_0^T \mathrm{d}s/p(s)\} - 1)}.$$

We also refer the reader to [**15**, Proposition 2.0.1] for general formulae to compute m and M.

This paper is organized as follows. In § 2, we state the main results and give several examples to illustrate the significance of the results. Some preliminary lemmas are presented in § 3 and the proofs of the main results are given in § 4.

2. Main results

It is well known that the eigenvalue problem consisting of the equation

$$-(p(t)u')' + q(t)u = \lambda w(t)u, \quad t \in (0, T), \tag{2.1}$$

and BC (1.2) has a countable number of eigenvalues λ_i, $i = 0, 1, 2, \dots$, which are bounded below and unbounded above and can be ordered to satisfy

$$0 < \lambda_0 < \lambda_1 \leqslant \lambda_2 < \cdots < \lambda_{2n-3} \leqslant \lambda_{2n-2} < \lambda_{2n-1} \leqslant \lambda_{2n} < \lambda_{2n+1} \leqslant \cdots . \tag{2.2}$$

The reader is referred to [**7**, Theorem 2.1 and Corollary 2.1] and [**18**, Theorem 13.7] for the proofs of these results. In some of our results we will use the first eigenvalue λ_0.

We need the following assumptions.

(A1) There exist $b \geqslant 0$, $c > 0$, $\alpha > 1$ and $0 < r < 1$ such that

$$f(t, x) + bx + c|x|^\alpha \geqslant 0 \quad \text{for } (t, x) \in [0, T] \times [-r, 0]. \tag{2.3}$$

(A2) There exists $b \geqslant 0$ such that

$$x(f(t, x) + bx) \geqslant 0 \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}. \tag{2.4}$$

(A3) There exist $0 < r_1 < r_2 < \infty$ such that

$$f(t, x) \geqslant \frac{1}{m \int_0^T w(s)\,\mathrm{d}s} x \quad \text{for } (t, x) \in [0, T] \times [0, r_1] \tag{2.5}$$

and

$$f(t, x) \leqslant \frac{1}{M \int_0^T w(s)\,\mathrm{d}s} x \quad \text{for } (t, |x|) \in [0, T] \times [r_2, \infty], \tag{2.6}$$

where m and M are given in (1.10).

For convenience, we introduce the following notation:

$$f_0 = \liminf_{x \to 0^+} \min_{t \in [0,T]} \frac{f(t, x)}{x} \quad \text{and} \quad f^\infty = \limsup_{|x| \to \infty} \max_{t \in [0,T]} \left| \frac{f(t, x)}{x} \right|. \tag{2.7}$$

We first state the results for BVP (1.1), (1.2). The first two results establish the existence of at least one non-trivial solution.

Theorem 2.1. *Assume that (A1) holds. If*

$$f^\infty < \lambda_0 < f_0, \tag{2.8}$$

then BVP (1.1), (1.2) has at least one non-trivial solution.

Corollary 2.2. *Assume that (A1) and (A3) hold. Then BVP (1.1), (1.2) has at least one non-trivial solution.*

The next two results provide sufficient conditions for the existence of multiple solutions of BVP (1.1), (1.2).

Theorem 2.3. *Assume that (A2) and (2.8) hold. Then BVP (1.1), (1.2) has at least one positive solution and one negative solution.*

Corollary 2.4. *Assume that (A2) and (A3) hold. Then BVP (1.1), (1.2) has at least one positive solution and one negative solution.*

Remark 2.5. We make the following observations.

(a) In Theorem 2.1 and Corollary 2.2, to guarantee the existence of non-trivial solutions, all we need is the behaviour of f for x near 0 and $\pm\infty$.

(b) Corollary 2.4 extends [**1**, Theorem 3.5], where only the existence of positive solutions was established.

We now state the existence results for BVP (1.3), (1.2), which are immediate consequences of the above results.

Theorem 2.6. *Assume that (A1) holds. If*

$$\frac{\lambda_0}{f_0} < \lambda < \frac{\lambda_0}{f^\infty}, \tag{2.9}$$

then BVP (1.3), (1.2) has at least one non-trivial solution.

Corollary 2.7. *Assume that (A1) holds. If*

$$\frac{1}{mf_0\int_0^T w(s)\,\mathrm{d}s} \leqslant \lambda \leqslant \frac{1}{Mf^\infty\int_0^T w(s)\,\mathrm{d}s}, \tag{2.10}$$

then BVP (1.3), (1.2) has at least one non-trivial solution.

Theorem 2.8. *Assume that (A2) and (2.9) hold. Then BVP (1.3), (1.2) has at least one positive solution and one negative solution.*

Corollary 2.9. *Assume that (A2) and (2.10) hold. Then BVP (1.3), (1.2) has at least one positive solution and one negative solution.*

Remark 2.10. Corollary 2.9 extends [**1**, Theorem 4.3 and Corollary 4.4], where only the existence of positive solutions was established.

In the remainder of this section, we give three simple examples to illustrate some of our results. To the best of our knowledge, no previous criteria can be applied to these examples.

Example 2.11. Consider the BVP consisting of the equation

$$-u'' + u = f(t,u), \quad t \in (0,1), \tag{2.11}$$

and the BC

$$u(0) = u(1), \quad u'(0) = u'(1), \tag{2.12}$$

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where

$$f(t,x)=\begin{cases}-12t^2+13+(|x|^{1/2}-2)x^{1/3}, & x<-4,\\ -t^2(x^2+x)+3|x|+1, & -4\leqslant x\leqslant 0,\\ 1-tx^{1/2}, & x>0.\end{cases}\tag{2.13}$$

We claim that BVP (2.11), (2.12) has at least one non-trivial solution.

Here, $T=1$ and $p(t)=q(t)=w(t)=1$ on $(0,T)$ so (H) holds. Now with $b=c=1$ and $\alpha=2$, from (2.13), we see that (2.3) holds for any $r\in(0,1)$, and so (A1) holds. Moreover, in view of (2.7), we have that $f_0=\infty$ and $f^\infty=0$. It is well known that, for the problem consisting of the equation

$$-u''=\lambda u,\quad t\in(0,1),$$

and BC (2.12), the first eigenvalue is 0 (see, for example, [**12**, p. 428]). It follows that the first eigenvalue is $\lambda_0=1$ for the problem consisting of the equation

$$-u''+u=\lambda u,\quad t\in(0,1),$$

and BC (2.12). Hence, (2.8) holds. The conclusion then follows from Theorem 2.1.

Example 2.12. Consider the BVP consisting of the equation

$$-u''+8u=\lambda f(t,u),\quad t\in(0,1),\tag{2.14}$$

and BC (2.12), where λ is a positive parameter and

$$f(t,x)=x^{1/3}-2t^2x.\tag{2.15}$$

We claim that, for each $0<\lambda<4$, BVP (2.14), (2.12) has at least one positive solution and one negative solution.

Here, $T=1$, $p(t)=w(t)=1$ and $q(t)=8$ on $(0,T)$ so again (H) holds and, for $b=2$, from (2.15), it is clear that (A2) holds. Moreover, in view of (2.7), we have that $f_0=\infty$ and $f^\infty=2$. Reasoning as in Example 2.11, we see that the first eigenvalue for the problem consisting of the equation

$$-u''+8u=\lambda u,\quad t\in(0,1),$$

and BC (2.12) is $\lambda_0=8$. The conclusion then follows from Theorem 2.8.

Example 2.13. Consider the BVP consisting of the equation

$$-((t^2+1)u')'+\frac{64}{t^2+1}u=t^{-1/2}f(t,u),\quad t\in(0,1),\tag{2.16}$$

and the BC

$$u(0)=u(1),\quad u'(0)=2u'(1),\tag{2.17}$$

where

$$f(t,x)=\begin{cases}-(\mathrm{e}^{2\pi}+1)^2-2\mathrm{e}^{2\pi}t(x^{1/3}+1)x^{2/3}, & x<-1,\\ (\mathrm{e}^{2\pi}+1)^2x, & -1\leqslant x\leqslant 1,\\ (\mathrm{e}^{2\pi}+1)^2+2\mathrm{e}^{2\pi}t^3(1-x^{1/2})x^{1/2}, & x>1.\end{cases}\tag{2.18}$$

We claim that, for each λ with

$$\frac{4(e^{2\pi}-1)}{e^{\pi}(e^{2\pi}+1)^2} \leqslant \lambda \leqslant \frac{4(e^{2\pi}-1)}{e^{2\pi}(e^{2\pi}+1)},$$

BVP (2.16), (2.17) has at least one positive solution and one negative solution.

Here, $T=1$, $p(t)=t^2+1$, $q(t)=64/(t^2+1)$ and $w(t)=t^{-1/2}$ on $(0,T)$. Then, (H) holds and, with $b=2e^{2\pi}$, from (2.18), we see that (A2) holds. Moreover, in view of (2.7), we have that $f_0=(e^{2\pi}+1)^2$ and $f^{\infty}=2e^{2\pi}$. For m and M defined by (1.10), and noting Remark 1.1, it follows that

$$m=\frac{e^{\pi}}{8(e^{2\pi}-1)} \quad \text{and} \quad M=\frac{e^{2\pi}+1}{16(e^{2\pi}-1)}.$$

Thus,

$$\frac{1}{mf_0\int_0^T w(s)\,\mathrm{d}s}=\frac{4(e^{2\pi}-1)}{e^{\pi}(e^{2\pi}+1)^2} \quad \text{and} \quad \frac{1}{Mf^{\infty}\int_0^T w(s)\,\mathrm{d}s}=\frac{4(e^{2\pi}-1)}{e^{2\pi}(e^{2\pi}+1)}.$$

The conclusion then follows from Corollary 2.9.

Examples may also be readily given to illustrate the other results. We leave this to the interested reader.

3. Preliminary lemmas

Our first lemma follows from [**1**, Theorem 2.4]; it can also be verified directly.

Lemma 3.1. *For any $b \geqslant 0$ and $h \in L(0,T)$, the BVP consisting of the equation*

$$-(p(t)u')' + (q(t)+bw(t))u = h(t), \quad t \in (0,T),$$

and BC (1.2) has a solution $u(t)$ if and only if

$$u(t)=\int_0^T H(t,s;b)h(s)\,\mathrm{d}s,$$

where $H(t,s;b)$ is defined by (1.7).

We refer the reader to Theorem A.3.3 (ix) and Lemma 2.5.1 in [**4**], respectively, for the proofs of the following two well-known lemmas. In the rest of this paper, the bold $\mathbf{0}$ denotes the zero element in any given Banach space.

Lemma 3.2. *Let Ω be a bounded open set in a real Banach space X and let $T:\bar{\Omega}\to X$ be compact. If there exists $u_0 \in X$, $u_0 \neq \mathbf{0}$, such that*

$$u - Tu \neq \tau u_0 \quad \text{for all } u \in \partial\Omega \text{ and } \tau \geqslant 0,$$

then the Leray–Schauder degree

$$\deg(I-T,\Omega,\mathbf{0})=0.$$

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Lemma 3.3. *Let Ω be a bounded open set in a real Banach space X with $\mathbf{0} \in \Omega$ and $T : \bar{\Omega} \to X$ be compact. If*

$$Tu \neq \tau u \quad \text{for all } u \in \partial\Omega \text{ and } \tau \geqslant 1,$$

then the Leray–Schauder degree

$$\deg(I - T, \Omega, \mathbf{0}) = 1.$$

Now assume that X is a real Banach space with the norm $\|\cdot\|$, X^* is the dual space of X, P is a total cone in X, i.e. $X = \overline{P - P}$, and P^* is the dual cone of P, i.e.

$$P^* = \{g \in X^* : g(u) \geqslant 0 \text{ for all } u \in P\}.$$

The following Krein–Rutman theorem can be found in [**20**, Proposition 7.26].

Lemma 3.4. *Let $L : X \to X$ be a compact linear positive operator, let L^* be the dual operator of L and let r_L be the spectral radius of L. If $r_L > 0$, then r_L is an eigenvalue of L and L^* with eigenfunctions in $P \setminus \{\mathbf{0}\}$ and $P^* \setminus \{\mathbf{0}\}$, respectively.*

Let L, L^* and r_L be given as in Lemma 3.4. If $r_L > 0$, then, from Lemma 3.4, there exist $\varphi \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that

$$L\varphi = r_L\varphi \quad \text{and} \quad L^*h = r_Lh. \tag{3.1}$$

Choose $\delta > 0$ and define

$$P(h, \delta) = \{u \in P : h(u) \geqslant \delta\|u\|\}. \tag{3.2}$$

Then $P(h, \delta)$ is a cone in X.

Lemma 3.5. *Assume that the following conditions hold.*

(C1) *There exist $\varphi \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that (3.1) holds and $L(P) \subseteq P(h, \delta)$.*

(C2) *$A : X \to P$ is a continuous operator and there exist $\alpha > 1$ and $K > 0$ such that $\|Au\| \leqslant K\|u\|^\alpha$ for all $u \in X$.*

(C3) *$\boldsymbol{f} : X \to X$ is a bounded continuous operator and there exists $r^* > 0$ such that*

$$\boldsymbol{f}u + Au \in P \quad \text{for all } u \in X \text{ with } \|u\| < r^*.$$

(C4) *There exist $\eta > 0$ and $r^{**} > 0$ such that*

$$L\boldsymbol{f}u \geqslant r_L^{-1}(1 + \eta)Lu \quad \text{for all } u \in X \text{ with } \|u\| < r^{**}.$$

Let $T = L\boldsymbol{f}$. There then exists $0 < R < \min\{r^*, r^{**}\}$ such that the Leray–Schauder degree

$$\deg(I - T, B(\mathbf{0}, R), \mathbf{0}) = 0,$$

where $B(\mathbf{0}, R) = \{u \in X : \|u\| < R\}$.

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Proof. We first claim that there exists $0 < R < \min\{r^*, r^{**}\}$ such that

$$u - Tu \neq \tau\varphi \quad \text{for all } u \in \partial B(\mathbf{0}, R) \text{ and } \tau \geqslant 0. \tag{3.3}$$

If this is not the case, then, for all $0 < R < \min\{r^*, r^{**}\}$, there exist $u_1 \in \partial B(\mathbf{0}, R)$ and $\tau_1 \geqslant 0$ such that

$$u_1 - L\boldsymbol{f}u_1 = \tau_1\varphi. \tag{3.4}$$

Then, from (3.1) and (C4), we have

$$\begin{aligned} h(u_1) &= h(L\boldsymbol{f}u_1) + \tau_1 h(\varphi) \\ &\geqslant h(L\boldsymbol{f}u_1) \\ &\geqslant r_L^{-1}(1+\eta)h(Lu_1) \\ &= r_L^{-1}(1+\eta)(L^*h)(u_1) \\ &= r_L^{-1}(1+\eta)r_L h(u_1) \\ &= (1+\eta)h(u_1). \end{aligned}$$

Hence, $h(u_1) \leqslant 0$. This, together with (3.1) and (C2), implies that

$$\begin{aligned} h(u_1 + LAu_1) &= h(u_1) + h(LAu_1) \\ &= h(u_1) + (L^*h)(Au_1) \\ &\leqslant (L^*h)(Au_1) \\ &\leqslant r_L h(Au_1) \\ &\leqslant r_L K\|h\|\,\|u_1\|^\alpha \\ &= D_1\|u_1\|^\alpha, \end{aligned} \tag{3.5}$$

where $D_1 = r_L K\|h\|$. From (3.1) and (3.4), we see that

$$\begin{aligned} u_1 + LAu_1 &= L\boldsymbol{f}u_1 + LAu_1 + \tau_1\varphi \\ &= L(\boldsymbol{f}u_1 + Au_1) + \tau r_L^{-1}L\varphi. \end{aligned}$$

In view of (C1) and (C3), we see that $u_1 + LAu_1 \in P(h, \delta)$. Thus, from (3.2),

$$h(u_1 + LAu_1) \geqslant \delta\|u_1 + LAu_1\| \geqslant \delta\|u_1\| - \delta\|LAu_1\|,$$

and so

$$\|u_1\| \leqslant \delta^{-1}h(u_1 + LAu_1) + \|LAu_1\|.$$

Hence, from (C2) and (3.5),

$$\begin{aligned} R = \|u_1\| &\leqslant \delta^{-1}D_1\|u_1\|^\alpha + K\|L\|\,\|u_1\|^\alpha \\ &= D_2\|u_1\|^\alpha && (3.6) \\ &= D_2R^\alpha, && (3.7) \end{aligned}$$

where $D_2 = \delta^{-1}D_1 + K\|L\|$. Since $\alpha > 1$, (3.6) cannot hold if R is sufficiently small. Therefore, there exists $0 < R < \min\{r^*, r^{**}\}$ such that (3.3) holds. Note that the operator T is compact. The conclusion now readily follows from Lemma 3.2, and this completes the proof of the lemma. $\square$

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4. Proofs of the main results

In what follows, let $X = C[0,T]$ be the Banach space equipped with the norm $\|u\| = \max_{t\in[0,T]} |u(t)|$. Define a total cone P in X by

$$P = \{u \in X : u(t) \geqslant 0 \text{ for } t \in [0,T]\}$$

and operators L, $\boldsymbol{f}$ $T : X \to X$ by

$$Lu(t) = \int_0^T H(t,s;b)w(s)u(s)\,\mathrm{d}s, \tag{4.1}$$

$$\boldsymbol{f}u(t) = f(t,u(t)) + bu(t), \tag{4.2}$$

and

$$Tu(t) = L\boldsymbol{f}u(t) = \int_0^T H(t,s;b)w(s)\boldsymbol{f}u(s)\,\mathrm{d}s, \tag{4.3}$$

where b is given in (A1) or (A2) and $H(t,s;b)$ is defined by (1.7) with this b. Then $L : X \to X$ is compact, linear and positive, $\boldsymbol{f} : X \to X$ is bounded and continuous and $T : X \to X$ is compact. Moreover, by Lemma 3.1, $u(t)$ is a solution of BVP (1.1), (1.2) if and only if it is a fixed point of T.

Proof of Theorem 2.1. We first verify that conditions (C1)–(C4) of Lemma 3.5 are satisfied.

From (1.8), we see that $(Lv)(t) > 0$ for $t \in [0,T]$ for any $v \in P$ with $v(t) \not\equiv 0$ on $[0,T]$. Thus, there exists $d > 0$ such that $d(Lv)(t) \geqslant v(t)$ for $t \in [0,T]$. Then, from [**8**, Chapter 5, Theorem 2.1], it follows that the spectral radius, r_L, of L satisfies $r_L > 0$. Hence, by Lemma 3.4, there exist $\varphi \in P \setminus \{\mathbf{0}\}$ and $h \in P^* \setminus \{\mathbf{0}\}$ such that (3.1) holds. Moreover, it is easy to see that $r_L^{-1} = \lambda_0 + b$, where λ_0 is given in (2.2). We now show that h can be explicitly given by

$$h(u) = \int_0^T w(t)\varphi(t)u(t)\,\mathrm{d}t, \quad u \in X. \tag{4.4}$$

In fact, from (1.7), it is clear that $H(t,s;b) = H(s,t;b)$ for $t,s \in [0,T]$. Then, for $u \in X$, (4.4) implies that

$$\begin{aligned}
r_L h(u) &= \int_0^T w(t)(r_L\varphi(t))u(t)\,\mathrm{d}t \\
&= \int_0^T w(t)u(t)\left(\int_0^T H(t,s;b)w(s)\varphi(s)\,\mathrm{d}s\right)\mathrm{d}t \\
&= \int_0^T w(s)\varphi(s)\left(\int_0^T H(t,s;b)w(t)u(t)\,\mathrm{d}t\right)\mathrm{d}s \\
&= \int_0^T w(s)\varphi(s)\left(\int_0^T H(s,t;b)w(t)u(t)\,\mathrm{d}t\right)\mathrm{d}s \\
&= \int_0^T w(s)\varphi(s)(Lu)(s)\,\mathrm{d}s \\
&= h(Lu) = (L^*h)(u),
\end{aligned}$$

i.e. h satisfies the second equality in (3.1). Thus, h can be explicitly given as in (4.4). Note that $\varphi(t) > 0$ on $[0, T]$, so there exists $\delta_1 > 0$ such that

$$\varphi(s) > \delta_1 H(t, s; b) \quad \text{for } t, s \in [0, T]. \tag{4.5}$$

Let $\delta = r_L \delta_1$. We will show that $L(P) \subseteq P(h, \delta)$. For any $u \in P$, from (4.4) and (4.5), we have that

$$\begin{aligned} h(Lu) &= r_L \int_0^T w(t)\varphi(s)u(s)\,\mathrm{d}s \\ &\geqslant r_L \delta_1 \int_0^T H(t, s; b)w(s)u(s)\,\mathrm{d}s \\ &= \delta(Lu)(t) \end{aligned}$$

for $t \in [0, T]$. Hence, $h(Lu) \geqslant \delta\|Lu\|$, i.e. $L(P) \subseteq P(h, \delta)$. Therefore, condition (C1) of Lemma 3.5 holds.

Let $Au(t) = c|u(t)|^\alpha$ for $u \in X$, where c and α are given in (A1). Then, with $K = c$, (C2) of Lemma 3.5 holds.

Let r be given in (A1). Since $f_0 > \lambda_0$, there exist $\eta > 0$ and $0 < \varepsilon_1 < 1$ such that

$$\begin{aligned} f(t, x) + bx &\geqslant (\lambda_0 + b)(1 + \eta)x \\ &= r_L^{-1}(1 + \eta)x \\ &\geqslant 0 \quad \text{for } (t, x) \in [0, T] \times [0, \varepsilon_1]. \end{aligned} \tag{4.6}$$

Let $\boldsymbol{f}$ be defined by (4.2). Now, in view of (2.3) and (4.6), we see that condition (C3) of Lemma 3.5 holds with $r^* = \min\{\varepsilon_1, r\}$.

From (2.3), it follows that

$$f(t, x) + bx \geqslant -c|x|^\alpha \quad \text{for } (t, x) \in [0, T] \times [-r, 0].$$

Choose $0 < \varepsilon_2 < \min\{\varepsilon_1, r\}$ to be sufficiently small that $-c|x|^\alpha \geqslant r_L^{-1}(1 + \eta)x$ for $x \in [-\varepsilon_2, 0]$. Then

$$f(t, x) + bx \geqslant r_L^{-1}(1 + \eta)x \quad \text{for } (t, x) \in [0, T] \times [-\varepsilon_2, 0]. \tag{4.7}$$

From (4.6) and (4.7), we have

$$f(t, x) + bx \geqslant r_L^{-1}(1 + \eta)x \quad \text{for } (t, x) \in [0, T] \times [-\varepsilon_2, \varepsilon_2],$$

which clearly implies that

$$L\boldsymbol{f}u \geqslant r_L^{-1}(1 + \eta)Lu \quad \text{for all } u \in X \text{ with } \|u\| < \varepsilon_2.$$

Hence, (C4) of Lemma 3.5 holds with $r^{**} = \varepsilon_2$.

We have verified that all the conditions of Lemma 3.5 hold, so there exists $R_1 > 0$ such that

$$\deg(I - T, B(\mathbf{0}, R_1), \mathbf{0}) = 0, \tag{4.8}$$

where $B(\mathbf{0}, R_1) = \{u \in X : \|u\| < R_1\}$.

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Next, since $f^{\infty} < \lambda_0$, there exist $0 < \nu < 1$ and $R_3 > R_1$ such that

$$\begin{aligned} |f(t,x) + bx| &\leqslant (\lambda_0 + b)(1-\nu)|x| \\ &= r_L^{-1}(1-\nu)|x| \quad \text{for } (t,|x|) \in [0,T] \times (R_3, \infty). \end{aligned} \tag{4.9}$$

Let

$$L = \sup_{u \in X,\, \|u\| \leqslant R_3} \max_{t \in [0,T]} \int_0^T H(t,s;b) w(s) |\boldsymbol{f} u(s)| \, \mathrm{d}s. \tag{4.10}$$

Then $0 < L < \infty$. Choose R_2 sufficiently large that

$$R_2 > \max\{R_3, \nu^{-1} L\}. \tag{4.11}$$

For any $u \in X$, let

$$\begin{aligned} I_1^u &= \{t \in [0,T] : |u(t)| > R_3\}, \\ I_2^u &= [0,T] \setminus I_1^u, \end{aligned}$$

and

$$\tilde{u}(t) = \min\{|u(t)|, R_3\}.$$

Define

$$B(\mathbf{0}, R_2) = \{u \in X : \|u\| < R_2\}.$$

We claim that

$$Tu \neq \tau u \quad \text{for all } u \in \partial B(\mathbf{0}, R_2) \text{ and } \tau \geqslant 1. \tag{4.12}$$

If this is not the case, then there exist $u^* \in \partial B(\mathbf{0}, R_2)$ and $\tau^* \geqslant 1$ such that $Tu^* = \tau^* u^*$. It follows that $u^* = s^* T u^*$, where $s^* = 1/\tau^*$. Clearly, $s^* \in (0,1)$. Assume that $\|u^*\| = |u^*(t^*)|$ for some $t^* \in [0,T]$. Then $R_2 = |u^*(t^*)| = s^* |Tu^*(t^*)|$. For h defined by (4.4), we have

$$\begin{aligned} h(R_2) &= h(|u^*(t^*)|) \\ &= s^* h(|Tu^*(t^*)|) \\ &\leqslant h(|Tu^*(t^*)|) \\ &= h\biggl(\biggl|\int_0^T H(t^*,s;b) w(s) \boldsymbol{f} u^*(s) \, \mathrm{d}s\biggr|\biggr) \\ &\leqslant h\biggl(\int_0^T H(t^*,s;b) w(s) |\boldsymbol{f} u^*(s)| \, \mathrm{d}s\biggr) \\ &= h\biggl(\int_{I_1^{u^*}} H(t^*,s;b) w(s) |\boldsymbol{f} u^*(s)| \, \mathrm{d}s\biggr) + h\biggl(\int_{I_2^{u^*}} H(t^*,s;b) w(s) |\boldsymbol{f} u^*(s)| \, \mathrm{d}s\biggr). \end{aligned}$$

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Now, from (4.9), (4.10) and the second equality in (3.1), we obtain

$$\begin{aligned}
&h\biggl(\int_{I_1^{u^*}} H(t^*,s;b)w(s)|\boldsymbol{f}u^*(s)|\,\mathrm{d}s\biggr)+h\biggl(\int_{I_2^{u^*}} H(t^*,s;b)w(s)|\boldsymbol{f}u^*(s)|\,\mathrm{d}s\biggr)\\
&\quad\leqslant r_L^{-1}(1-\nu)h\biggl(\int_{I_1^{u^*}} H(t^*,s;b)w(s)|u^*(s)|\,\mathrm{d}s\biggr)+h\biggl(\int_{I_2^{u^*}} H(t^*,s;b)w(s)|\boldsymbol{f}u^*(s)|\,\mathrm{d}s\biggr)\\
&\quad\leqslant r_L^{-1}(1-\nu)h\biggl(\int_0^T H(t^*,s;b)w(s)|u^*(s)|\,\mathrm{d}s\biggr)+h\biggl(\int_0^T H(t^*,s;b)w(s)|\boldsymbol{f}\tilde{u}^*(s)|\,\mathrm{d}s\biggr)\\
&\quad\leqslant r_L^{-1}(1-\nu)h(L|u^*(t^*)|)+h(L)\\
&\quad= r_L^{-1}(1-\nu)(L^*h)(|u^*(t^*)|)+h(L)\\
&\quad= r_L^{-1}(1-\nu)r_L h(|u^*(t^*)|)+h(L)\\
&\quad= (1-\nu)h(R_2)+h(L).
\end{aligned}$$

Thus,

$$h(R_2)\leqslant(1-\nu)h(R_2)+h(L),$$

which implies that

$$(\nu R_2-L)h(1)\leqslant 0.$$

In view of the fact that $h(1)>0$, it follows that $R_2\leqslant\nu^{-1}L$. This contradicts (4.11) and so (4.12) holds. By Lemma 3.3, we have

$$\deg(I-T,B(\mathbf{0},R_2),\mathbf{0})=1. \tag{4.13}$$

By the additivity property of the Leray–Schauder degree, (4.8) and (4.13), we obtain

$$\deg(I-T,B(\mathbf{0},R_2)\setminus\overline{B(\mathbf{0},R_1)})=1.$$

Thus, from the solution property of the Leray–Schauder degree, T has at least one fixed point u in $B(\mathbf{0},R_2)\setminus\overline{B(\mathbf{0},R_1)}$. Clearly, $u(t)$ is a non-trivial solution of BVP (1.1), (1.2), and this completes the proof of the theorem. □

The following lemma will be used in some of our remaining proofs.

Lemma 4.1. *Let λ_0 be given in (2.2). Then*

$$\frac{1}{M\int_0^T w(s)\,\mathrm{d}s}<\lambda_0<\frac{1}{m\int_0^T w(s)\,\mathrm{d}s},$$

where m and M are defined by (1.10).

Proof. Let the operator L_0 be defined by (4.1) with $b=0$, i.e. let

$$L_0u(t)=\int_0^T G(t,s)w(s)u(s)\,\mathrm{d}s. \tag{4.14}$$

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Now L_0 is u_0-positive with $u_0 \equiv 1$ (see [**9**] or [**20**]), so there exist $k_i(u) > 0$, $i = 1, 2$, such that

$$k_1(u) \cdot 1 \leqslant L_0 u \leqslant k_2(u) \cdot 1.$$

Note that

$$L_1 u(t) = \int_0^T m w(s) \, \mathrm{d}s \leqslant L_0 u(t),$$

so $r(L_1) \leqslant r(L_0)$, and hence

$$\int_0^T m w(s) \, \mathrm{d}s \leqslant r(L_0) = \frac{1}{\lambda_0}.$$

This proves the right-hand inequality in the lemma. The proof of the other half is similar. □

Proof of Corollary 2.2. Let f_0 and f^∞ be defined by (2.7). From (A3), we have that

$$f_0 \geqslant \frac{1}{m \int_0^T w(s) \, \mathrm{d}s} \quad \text{and} \quad f^\infty \leqslant \frac{1}{M \int_0^T w(s) \, \mathrm{d}s}. \tag{4.15}$$

Then the conclusion follows from Theorem 2.1 and Lemma 4.1. □

Proof of Theorem 2.3. For $u \in X$, let

$$\boldsymbol{f}_1 u(t) = \begin{cases} f(t, u(t)) + b u(t), & u(t) \geqslant 0, \\ -(f(t, u(t)) + b u(t)), & u(t) < 0. \end{cases} \tag{4.16}$$

In virtue of (2.4), we see that $\boldsymbol{f}_1 : X \to \mathbb{R}$ is continuous and non-negative. Define a compact operator $T_1 : X \to X$ by

$$T_1 u(t) = \int_0^T H(t, s; b) w(s) \boldsymbol{f}_1 u(s) \, \mathrm{d}s. \tag{4.17}$$

Note that $\boldsymbol{f}_1 u(t) + c|u(t)|^\alpha \geqslant 0$ for $u \in X$, where c and α are given in (A1). Then, as in the proof of Theorem 2.1, we see that conditions (C1)–(C4) of Lemma 3.5 hold, where A is defined as before, $\boldsymbol{f} = \boldsymbol{f}_1$ and $T = T_1$. Hence, by Lemma 3.5, there exists $R_1 > 0$ such that

$$\deg(I - T_1, B(\mathbf{0}, R_1), \mathbf{0}) = 0, \tag{4.18}$$

where $B(\mathbf{0}, R_1) = \{u \in X : \|u\| < R_1\}$.

Since $f^\infty < \lambda_0$, there exist $0 < \nu < 1$ and $R_3 > R_1$ such that

$$\begin{aligned} |f(t, x) + bx| &\leqslant (\lambda_1 + b)(1 - \nu)|x| \\ &= r_L^{-1}(1 - \nu)|x| \quad \text{for } (t, |x|) \in [0, T] \times (R_3, \infty). \end{aligned}$$

Let

$$L_1 = \sup_{u \in X, \, \|u\| \leqslant R_3} \max_{t \in [0, T]} \int_0^T H(t, s; b) w(s) |\boldsymbol{f}_1 u(s)| \, \mathrm{d}s.$$

Then $0 < L < \infty$. Choose R_2 large enough that

$$R_2 > \max\{R_3, \nu^{-1}L_1\}.$$

Define

$$B(\mathbf{0}, R_2) = \{u \in X : \|u\| < R_2\}.$$

An argument similar to the one used in deriving (4.12) yields

$$T_1 u \neq \tau u \quad \text{for all } u \in \partial B(\mathbf{0}, R_2) \text{ and } \tau \geqslant 1.$$

Thus, by Lemma 3.3, we have

$$\deg(I - T_1, B(\mathbf{0}, R_2), \mathbf{0}) = 1. \tag{4.19}$$

By the additivity property of the Leray–Schauder degree, (4.18), and (4.19), we obtain

$$\deg(I - T_1, B(\mathbf{0}, R_2) \setminus \overline{B(\mathbf{0}, R_1)}) = 1.$$

Thus, from the solution property of the Leray–Schauder degree, T_1 has at least one fixed point u in $B(\mathbf{0}, R_2) \setminus \overline{B(\mathbf{0}, R_1)}$. Then

$$u_1(t) = \int_0^T H(t, s; b) w(s) \boldsymbol{f}_1 u(s) \,\mathrm{d}s \quad \text{for } t \in [0, T],$$

which, together with (1.8), implies that $u_1(t) > 0$ on $[0, T]$. Therefore, from (4.16), $\boldsymbol{f}_1 u_1(t) = f(t, u_1(t)) + bu_1(t)$, and so $u_1(t)$ is a positive solution of BVP (1.1), (1.2).

For $u \in X$, let

$$\boldsymbol{f}_2 u(t) = \begin{cases} -(f(t, -u(t)) + b(-u(t))), & x \geqslant 0, \\ f(t, -u(t)) + b(-u(t)), & x < 0. \end{cases} \tag{4.20}$$

Then, from (2.4), we see that $\boldsymbol{f}_2 : X \to \mathbb{R}$ is continuous and non-negative. Define a compact operator $T_2 : X \to X$ by

$$(T_2 u)(t) = \int_0^T H(t, s; b) w(s) \boldsymbol{f}_2 u(s) \,\mathrm{d}s.$$

By an argument similar to that above, we see that T_2 has a fixed point v satisfying $v(t) > 0$ on $[0, T]$. From (4.20) and the fact that

$$v(t) = \int_0^T H(t, s; b) w(s) \boldsymbol{f}_2 v(s) \,\mathrm{d}s,$$

we obtain

$$-v(t) = \int_0^T H(t, s; b) w(s) (f(s, -v(s)) + b(-v(s))) \,\mathrm{d}s.$$

Therefore, $u_2(t) := -v(t)$ is a negative solution of BVP (1.1), (1.2), and the theorem is proved. □

Proof of Corollary 2.4. Since (A3) implies (4.15), the conclusion then follows from Theorem 2.3 and Lemma 4.1. □

Finally, by virtue of Lemma 4.1, Theorems 2.6 and 2.8 and Corollaries 2.7 and 2.9 are direct applications of Theorems 2.1 and 2.3 and Corollaries 2.2 and 2.4 with f in equation (1.1) replaced by λf. We omit the proofs here.

In conclusion, we note that it is reasonable to ask if results analogous to Theorems 2.1 and 2.3 hold if

$$f^0 < \lambda_0 < f_\infty,$$

where f^0 and f_∞ are defined analogously to f_0 and f^∞. The answer is 'yes' provided certain changes are made in conditions (A1)–(A3) as well as in the proofs of Theorems 2.1 and 2.3. These results will appear elsewhere.

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