

THE NULLITY OF A WILD KNOT IN A COMPACT 3-MANIFOLD

Dedicated to the memory of Hanna Neumann

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(Received 18 April 1972)

Communicated by M. F. Newman

The nullity of the Alexander module of the fundamental group of the complement of a knot in S^3 was one of the invariants of wild knot type defined and investigated by E. J. Brody in [1], in which he developed a generalised elementary divisor theory applicable to infinitely generated modules over a unique factorisation domain. Brody asked whether the nullity of a knot with one wild point was bounded above by its enclosure genus; for knots in S^3 , the present author showed in [6] that this was indeed the case. In [7], it was (prematurely) stated by the author that this was also the case for knots k embedded in a 3-manifold M so that $H_1(M - k)$ was torsion-free.

The aim of this paper is to set the record straight. Let k be an oriented knot in the interior of a connected compact triangulated 3-manifold M , and suppose that k has only one wild point. Let $P(k, p)$ be the penetration index of k at this point p . Let $\pi_1(M - k)$ be the fundamental group of $M - k$; the first Betti group $B_1(M - k)$, being the torsion-free part of $H_1(M - k)$, is a free abelian group of finite rank. Let JB be the integer group ring of $B_1(M - k)$; JB is therefore a unique factorisation domain, which means that we can apply the results of [1] and [6] to obtain invariants of $M - k$, and therefore of the embedding type of k in M . The Betti module of k is the unique JB -module determined by $\pi_1(M - k)$; the main theorem of this paper, theorem 1, states that the nullity ν of the Betti module of k is bounded above by the expression

$$\frac{1}{2}P(k, p) + n(M) - 1,$$

in which $n(M)$ denotes the maximum possible nullity of the Betti module of a tame knot in M , and is an invariant of M . The paper closes with some questions about $n(M)$, and the conjecture that

$$\nu \leq e(k) + n(M) - 1$$

for all compact 3-manifolds M , where $e(k)$ denotes the enclosure genus of k (cf. [6], p. 550).

The author is indebted to the referee for his criticism and suggestions.

Proof of theorem 1

It will be assumed throughout that readers are familiar with the terminology, notation, and results of [1] and [6]; in particular, if G is a group and $\tau: G \rightarrow G/G'$ the canonical map, $\beta: G/G' \rightarrow B$ the canonical map whose kernel is the torsion subgroup of G/G' , then the Jacobian JB -module of G evaluated at $\beta\tau$ (cf. [1], p. 146, and [6], p. 546 f.) will be called the *Betti module of G* .

PROPOSITION 1. *There exists a closed 3-cell neighbourhood U of p such that the map $H_1(M - k - U) \rightarrow H_1(M - k)$, induced by inclusion, is an epimorphism.*

PROOF. Let N be any tame closed 3-cell neighbourhood of p , whose boundary is in general position with respect to k and meets k in $2n$ ($\neq 0$) points. The reduced Mayer-Vietoris sequence

$$\dots \rightarrow H_1(N - k) \oplus H_1(M - k - N) \rightarrow H_1(M - k) \rightarrow \tilde{H}_0(\text{Bd}N - k) = 0$$

shows that $H_1(M - k)$ is a quotient of the direct sum of $H_1(N - k)$ (which is a free group on n generators) and $H_1(M - k - N)$. This last group is finitely generated, as it is isomorphic to the first homology group of the compact space obtained by removing an open regular neighbourhood of $k \cup N$ from M ; this means that $H_1(M - k)$ is finitely generated.

Let $\alpha_1, \dots, \alpha_m$ be a set of paths whose homology classes generate $H_1(M - k)$, and let the distance of k from $\alpha_1 \cup \alpha_2 \cup \dots \cup \alpha_m$, measured with the barycentric metric on M , be ε . Then any 3-cell neighbourhood U of p which lies in the interior of a ball of radius ε and centred at p will have the property we require. Q.E.D.

Since theorem 1 is trivial if $P(k, p)$ is infinite, we may assume that $P = P(k, p)$ is finite. Let U be a 3-cell of the type given by proposition 1. An *admissible sequence of 3-cells for the knot k* is a sequence $U_1 \supset U_2 \supset U_3 \supset \dots$ such that, for each $i = 1, 2, \dots$,

(i) U_i is a tame closed 3-cell neighbourhood of p whose boundary is in general position with respect to k and meets k in precisely P points,

(ii) $U_{i+1} \subset \text{Int } U_i$ and $U_i \subset \text{Int } U$, and

(iii) $U_i \cap U_i = \{p\}$.

Let $U_1 \supset U_2 \supset U_3 \supset \dots$ be an admissible sequence for k , and let G_i and G be the fundamental groups of $M - k - U_i$ and $M - k$ respectively. Then the inclusion maps $M - k - U_i \subset M - k - U_{i+1}$ and $M - k - U_i \subset M - k$ induce maps $\phi_i: G_i \rightarrow G_{i+1}$ and $\psi_i: G_i \rightarrow G$ such that $\psi_{i+1}\phi_i = \psi_i$; note that G is the direct limit of the sequence

$$\dots \rightarrow G_i \xrightarrow{\phi_i} G_{i+1} \rightarrow \dots,$$

by [3].

Let $\mathbf{a}: \mathcal{G} \rightarrow \mathcal{A}$ be the abelianisation functor, given by $\mathbf{a}(G) = G/G'$ for all $G \in \mathcal{G}$. Note that $\mathbf{a}(G_i) \cong H_1(M - k - U_i)$ and $\mathbf{a}(G) \cong H_1(M - k)$, so we have a commuting diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\psi_i} & G \\ \tau_i \downarrow & & \downarrow \tau \\ H_1(M - k - U_i) & \xrightarrow{\eta \mathbf{a}(\psi_i) \eta_i^{-1}} & H_1(M - k) \xrightarrow{\beta} B_1(M - k) = B \end{array}$$

in which τ_i is the composition of the Hurewicz isomorphism $\eta_i: G_i/G_i' \rightarrow H_1(M - k - U_i)$ with the canonical map $G_i \rightarrow G_i/G_i'$ (and $\eta: G/G' \rightarrow H_1(M - k)$ is another Hurewicz isomorphism).

We shall use ζ_i to denote the map $\eta \mathbf{a}(\psi_i) \eta_i^{-1}$; note that ζ_i is really just the homology map induced by the inclusion $M - k - U_i \subset M - k$. Since $U_i \subset \text{Int } U$, proposition 1 shows that ζ_i is an epimorphism. We may therefore make G_i into a B -group $[G_i, \alpha_i]$ by setting $\alpha_i = \beta \zeta_i \tau_i$; then ϕ_i and ψ_i become morphisms in the category \mathcal{B} of B -groups, because the diagrams

$$\begin{array}{ccc} G_i & \xrightarrow{\phi_i} & G_{i+1} \\ \alpha_i \searrow & & \swarrow \alpha_{i+1} \\ & B & \end{array} \quad \begin{array}{ccc} G_i & \xrightarrow{\psi_i} & G \\ \alpha_i \searrow & & \swarrow \beta \tau \\ & B & \end{array}$$

commute (cf. [6], p. 545).

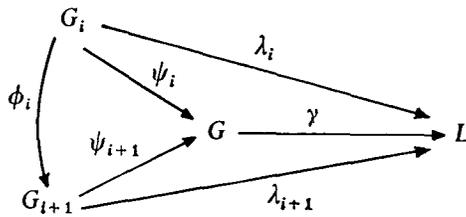
PROPOSITION 2. $[G, \beta\tau]$ is the direct limit in \mathcal{B} of the sequence

$$\dots \rightarrow [G_i, \alpha_i] \xrightarrow{\phi_i} [G_{i+1}, \alpha_{i+1}] \rightarrow \dots$$

PROOF. (Note that this proposition is analogous to (5.1) of [6]; a proof is included here because the proof of (5.1) contains a lacuna.)

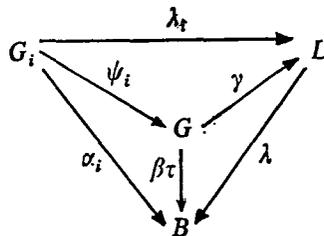
Notice that $[G, \beta\tau]$ and the maps $\psi_i: [G_i, \alpha_i] \rightarrow [G, \beta\tau]$ satisfy all the commuting requirements for a direct limit. Let $[L, \lambda]$ be a B -group and $\lambda_i: [G_i, \alpha_i] \rightarrow [L, \lambda]$ a family of B -homomorphisms such that $\lambda_{i+1} \phi_i = \lambda_i$ for each i ; we wish to show that there exists a unique homomorphism γ in \mathcal{B} such that $\lambda_i = \gamma \psi_i$ for each i , where $\gamma: [G, \beta\tau] \rightarrow [L, \lambda]$.

Now G is the direct limit of the sequence in \mathcal{G} , so there exists a unique homomorphism $\gamma: G \rightarrow L$ such that the diagram



commutes for each i ; we shall show that $\lambda\gamma = \beta\tau$.

Consider the diagram



in which every triangle commutes, with the possible exception of GLB . Some chasing shows that $\lambda\gamma\psi_i = \beta\tau\psi_i$, and this is true for all i . Since the group G is generated by the image groups $\psi_i(G_i)$, it follows that $\lambda\gamma = \beta\tau$; that is, γ is a \mathcal{B} -map. This shows that $[G, \beta\tau]$ is the direct limit of the sequence

$$\dots \rightarrow [G_i, \alpha_i] \xrightarrow{\phi_i} [G_{i+1}, \alpha_{i+1}] \rightarrow \dots$$

in \mathcal{B} .

Q.E.D.

From proposition 2, and theorem 1 of [6], it follows that the Betti module $M[G, \beta\tau]$ of G is the direct limit of the Jacobian modules of the G_i evaluated at α_i ; since G is generated by the image groups $\psi_i(G_i)$, $M[G, \beta\tau]$ is the union of its submodules $M\psi_i(M[G_i, \alpha_i])$, and each of these submodules is finitely generated because each G_i is. We therefore have a sequence

$$M\psi_1(M[G_1, \alpha_1]) \subset \dots \subset M\psi_i(M[G_i, \alpha_i]) \subset M\psi_{i+1}(M[G_{i+1}, \alpha_{i+1}]) \subset \dots$$

whose union is $M[G, \beta\tau]$. Then the nullity of $M[G, \beta\tau]$ as a JB -module is the supremum of the nullities of the $M\psi_i(M[G_i, \alpha_i])$; by (2.2) of [1], it follows that

$$v(M[G, \beta\tau]) \leq \sup v(M[G_i, \alpha_i]).$$

This brings us to the statement of the main theorem.

THEOREM 1. *Let k be an oriented knot with one wild point p , lying in the interior of the compact 3-manifold M : let $P(k, p)$ be the (3-cell) penetration index of k at p (see [7], p. 176). If v denotes the nullity of the Betti module of k , and if $n(M)$ is the largest integer n such that there exists a tame knot in M whose Betti module has nullity n (or $n(M) = \infty$ if no largest such n exists), then*

$$v \leq \frac{1}{2}P(k, p) + n(M) - 1.$$

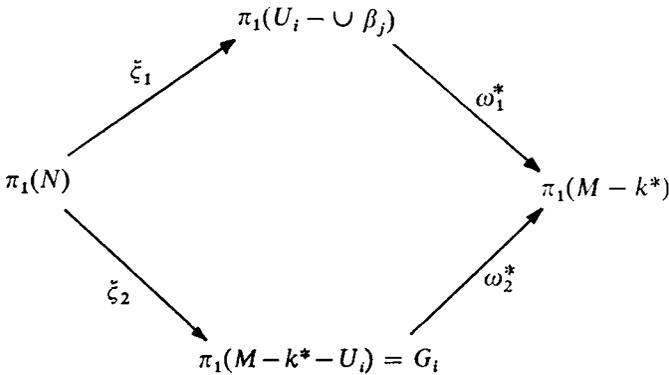
PROOF. There is nothing to prove if $P(k, p)$ is infinite, so we assume that $P(k, p) = P$ is finite and takes the value $2n$. Then to prove the theorem it is sufficient to prove that

$$v(M[G_i, \alpha_i]) \leq n + n(M) - 1$$

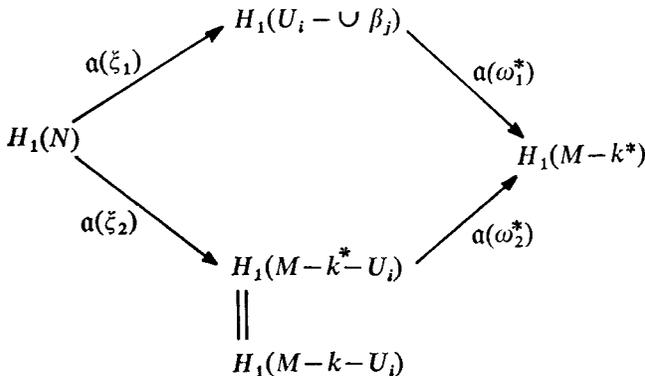
for each i .

We join the P points of $k \cap \text{Bd}U_i$ by n oriented arcs β_1, \dots, β_n whose interiors lie in $\text{Int } U_i$, so that the components of k in $M - U_i$ form a consistently oriented knot k^* when joined together by the β_j 's. We choose the β_j 's so that they are unknotted and do not link each other; that is, so that $\pi_1(U_i - \cup \beta_j)$ is a free group on n generators. The fact that we may choose the β_j 's in this way can be proven by induction on n .

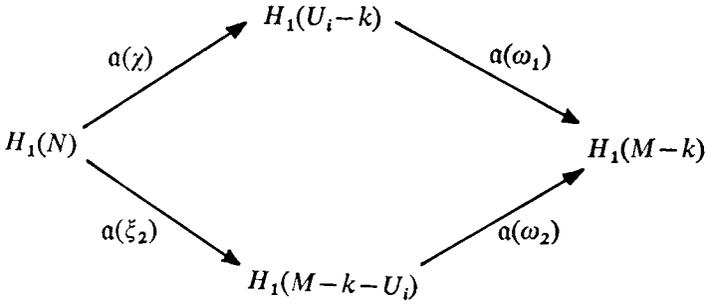
Let N be an open regular neighbourhood of $\text{Bd}U_i - k^* = \text{Bd}U_i - k$ in $M - k^*$; the open sets $N \cup (M - U_i - k^*)$ and $N \cup (U_i - \cup \beta_j)$ have the same homotopy types as $M - U_i - k^*$ and $U_i - \cup \beta_j$ respectively, so we can apply the van Kampen theorem to show that the diagram



is a pushout in \mathcal{G} , where all the maps are induced by inclusion maps. Since the functor \mathbf{a} preserves pushouts, the diagram



is a pushout in \mathcal{A} (where we have translated every group L/L' by the Hurewicz isomorphism to the first homology group of the appropriate space; this means that in the diagram above, the maps between homology groups have all been induced by inclusion maps). Similarly, the diagram

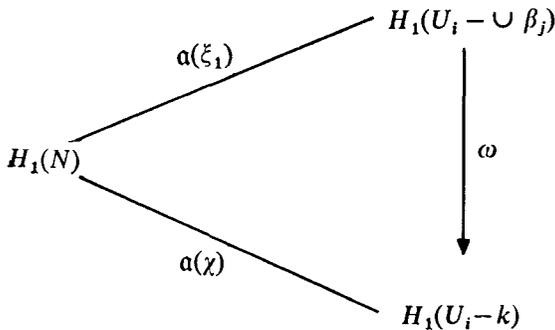


in which the maps are induced by inclusion maps, is a pushout in \mathcal{A} .

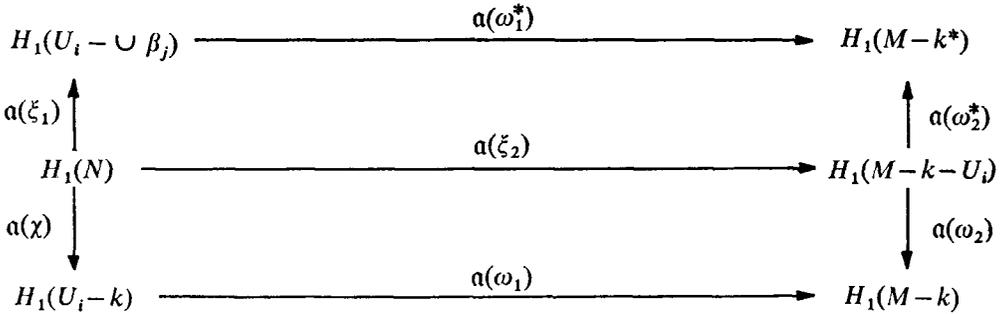
Now we choose a starting point on k lying in $M - U_i$, and order the points of $k \cap \text{Bd}U_i$ with the order in which we meet them as we travel along k in its positive direction. If the points of $k \cap \text{Bd}U_i = k^* \cap \text{Bd}U_i$ are x_1, x_2, \dots, x_{2n} , then x_{2m-1} and x_{2m} are joined by an arc of k ; we may also assume that they are joined by the subarc β_m of k^* . Let $t_1, t_2, \dots, t_{2n-1}$ be $2n - 1$ disjoint simple loops on $\text{Bd}U_i - k$, chosen so that each t_j bounds a disc D_j on $\text{Bd}U_i$ which contains no other t_m in its interior, and whose interior contains x_j but no other points of $k \cap \text{Bd}U_i$.

Let the n components of $k \cap U_i$ be ordered so that k_j starts at x_{2j-1} and finishes at x_{2j} . Then $H_1(U_i - k)$ is generated by the homology classes of the loops t_{2j-1} in $U_i - k$; in fact, $H_1(U_i - k)$ is the free abelian group generated by the elements $\mathbf{a}(\chi)(t_1), \mathbf{a}(\chi)(t_3), \mathbf{a}(\chi)(t_5), \dots, \mathbf{a}(\chi)(t_{2n-1})$. Similarly, $H_1(U_i - \cup \beta_j)$ is the free abelian group on the n generators $\mathbf{a}(\xi_1)(t_1), \mathbf{a}(\xi_1)(t_3), \dots, \mathbf{a}(\xi_1)(t_{2n-1})$. Notice also that $\mathbf{a}(\chi)(t_{2m}) = \mathbf{a}(\chi)(t_{2m-1})$ and $\mathbf{a}(\xi_1)(t_{2m}) = \mathbf{a}(\xi_1)(t_{2m-1})$ for each m .

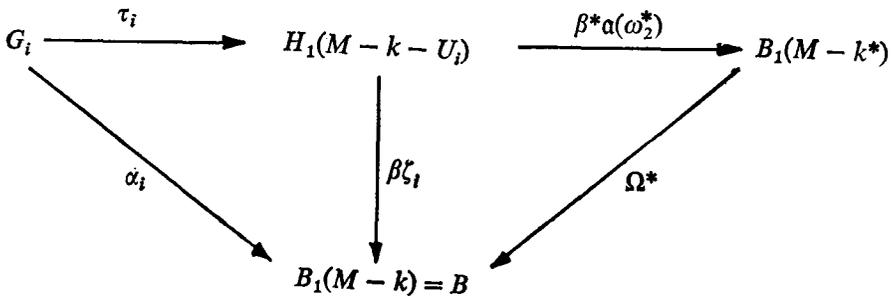
There is therefore an isomorphism $\omega: H_1(U_i - \cup \beta_j) \rightarrow H_1(U_i - k)$ which makes the diagram



commute. Some chasing around the diagram



(in which the small rectangles are both pushouts in \mathcal{A}) shows that this isomorphism ω induces an isomorphism $\Omega: H_1(M - k^*) \rightarrow H_1(M - k)$ such that $\Omega\alpha(\omega_1^*) = \alpha(\omega_1)\omega$ and $\Omega\alpha(\omega_2^*) = \alpha(\omega_2)$. If $\beta^*: H_1(M - k^*) \rightarrow B_1(M - k^*)$ is the canonical map to the Betti group of k^* , then Ω induces an isomorphism $\Omega^*: B_1(M - k^*) \rightarrow B_1(M - k)$ such that $\Omega^*\beta^* = \beta\Omega$. We therefore have a commutative diagram



(because $\zeta_i = \alpha(\omega_2)$ is the map induced by the inclusion of $M - k - U_i$ in $M - k$).

This means that any Jacobian matrix of G_i , evaluated at α_i , is the image of a Jacobian matrix of G_i evaluated at $\beta^*\alpha(\omega_2^*)\tau_i$ under the ring isomorphism $J\Omega^*: J(B_1(M - k^*)) \rightarrow JB$; this is the result we will use to find an upper bound for $v(M[G_i, \alpha_i])$.

Now we direct our attention to finding a Betti matrix for $\pi_1(M - k^*)$. Since $\pi_1(M - k^*)$ is the quotient of the free product $G_i^*\pi_1(U_i - \cup \beta_j)$ by the normal subgroup generated by the $2n - 1$ elements of the form $\omega_1^*\xi_1(t_m)\omega_2^*\xi_2(t_m^{-1})$, it follows that $\pi_1(M - k^*)$ has a Betti matrix of the form

$$\left(\begin{array}{c} A \ 0 \\ C \end{array} \right),$$

where A is a Jacobian matrix of G_i evaluated at $\beta^* \mathbf{a}(\omega_2) \tau_i$, the matrix 0 has n columns of zeros, and the matrix C has $2n - 1$ rows, one for each of the relators $\omega_1^* \xi_1(t_m) \omega_2^* \xi_2(t_m^{-1})$, $m = 1, 2, \dots, 2n - 1$. Then by (4.1) of [6], the nullity v^* of the Betti module of $\pi_1(M - k^*)$ is at least as great as the nullity of the matrix $(A \ 0)$ reduced by the rank $r(C)$ of C . That is,

$$v^* \geq v(A) + n - r(C) \geq v(A) + n - (2n - 1);$$

hence

$$v(A) \leq v^* + n - 1.$$

Since $J\Omega^*$ is an isomorphism and $\Omega^* \beta^* \mathbf{a}(\omega_2^*) \tau_i = \alpha_i$, the nullity of a Jacobian matrix of G_i evaluated at $\beta^* \mathbf{a}(\omega^*) \tau_i$ is the nullity of any Betti matrix of G_i . Hence

$$\begin{aligned} v(M[G_i, \alpha_i]) &= v(A) \leq v^* + n - 1 \\ &\leq n(M) + n - 1. \end{aligned}$$

As mentioned earlier, this upper bound on $v(M[G_i, \alpha_i])$ yields an upper bound for $v(M[G, \beta\tau])$, and the theorem is proved. Q.E.D.

Some open questions

QUESTION 1. *How can one compute $n(M)$ for a given compact 3-manifold M ? Is there an algorithm for computing $n(M)$ from a triangulation of M ? Does there exist an M such that $n(M) = \infty$?*

We can at least give a lower bound for $n(M)$, as follows.

THEOREM 2: *For every compact 3-manifold M ,*

$$n(M) \geq 1 + v(M),$$

where $v(M)$ is the nullity of the Betti module of M .

PROOF. Let C be a tame closed 3-cell in the interior of M , and k a tame knot in the interior of C . Then

$$\begin{aligned} \pi_1(M - k) &\cong \pi_1(M) * \pi_1(C - k), \text{ and} \\ B_1(M - k) &\cong B_1(M) \oplus Z. \end{aligned}$$

Let β be the canonical map from $H_1(M - k)$ to $B_1(M - k)$; the symbol β will also be used to denote the induced map of integer group rings. If A is a matrix $\|a_{ij}\|$, $\beta(A)$ is the matrix $\|\beta(a_{ij})\|$. Now by [4], p. 206, $\pi_1(M - k)$ has an Alexander matrix of the form

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where A_1 is an Alexander matrix for $\pi_1(M)$, and A_2 an Alexander matrix for $\pi_1(C - k)$; since $B_1(C - k) = H_1(C - k)$, it follows that $\beta(A_2) = A_2$ and therefore that $\pi_1(M - k)$ has a Betti matrix of the form

$$\begin{pmatrix} \beta(A_1) & 0 \\ 0 & A_2 \end{pmatrix}.$$

Then

$$v(k) = v(\beta(A_1)) + v(A_2) = v(\beta(A_1)) + 1 = 1 + v(M),$$

for every tame knot in a 3-cell has (Alexander) nullity equal to 1. Hence

$$n(M) \geq v(k) = 1 + v(M). \qquad \text{Q.E.D.}$$

Let $\chi(\text{Bd}M)$ denote the Euler characteristic of the boundary $\text{Bd} M$ of the 3-manifold M . Theorem 2 above then yields the corollary:

COROLLARY: *If M is compact, then*

(i) *if $\text{Bd} M$ is empty, or if $\text{Bd} M$ is not empty but $\chi(\text{Bd}M) \geq 3$, then $n(M) \geq 1$;*

(ii) *if $\text{Bd} M$ is not empty but $\chi(\text{Bd}M) \leq 2$, then*

$$n(M) \geq 2 - \frac{1}{2}\chi(\text{Bd}M).$$

PROOF. The nullity of any Jacobian module of a group G of deficiency d is bounded below by 0 if d is negative, or by d if $d \geq 0$. A lower bound for the deficiency of $\pi_1(M)$ will therefore yield a lower bound for $n(M)$, by theorem 2; we have used the lower bounds given in (6.2) of [5] (cf. theorem 2.2 of [2]) to obtain the lower bounds for $n(M)$. Q.E.D.

Note that if M is orientable and $\chi(\text{Bd}M) \geq 3$, the lower bound for $n(M)$ given above can be improved by using theorem 2.2 of [2] rather than (6.2) of [5].

QUESTION 2. *If M is a closed oriented 3-manifold different from S^3 then, by theorem 1 of [8], M can be written as the connected sum of prime 3-manifolds M_1, M_2, \dots, M_m . If M is not closed, the pair $(M, \text{Bd}M)$ can, by theorem 4.1 of [9], be written as the ‘‘connected sum along the boundary’’ of boundary-prime 3-manifold pairs $(M_1, \text{Bd}M_1), (M_2, \text{Bd}M_2), \dots, (M_m, \text{Bd}M_m)$. How is the number $n(M)$ related to the numbers $n(M_1), n(M_2), \dots, n(M_m)$?*

QUESTION 3. *Can the upper bound of theorem 1 be replaced by $e(k) + n(M) - 1$, where $e(k)$ is the enclosure genus of k ?*

In S^3 , or in a 3-cell, the answer is yes, by theorem 2 of [6]; I conjecture that the answer is ‘‘yes’’ for all compact 3-manifolds.

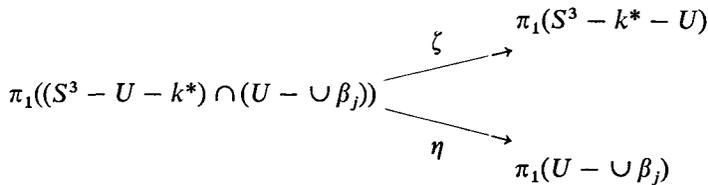
If k is a knot with a finite or empty set of wild points, a *meridian* of k is a simple loop in $M - k$ which bounds a disc in M which meets k in precisely one

point. Then the proof of theorem 2 of [6] can be modified to yield the following partial answer to question 3.

THEOREM 3. *Let k be an oriented knot with one wild point, lying in the interior of the compact 3-manifold M , and let $e(k)$ be the enclosure genus of k . Let v be the nullity of the Betti module of k . Then if $B_1(M - k)$ is the infinite cyclic group generated by a meridian of k ,*

$$v \leq e(k) + n(M) - 1.$$

For we can still use the results of p. 549 of [6], since in the diagram



which occurs on page 551, all the groups are \mathbf{Z} -groups and all the maps are \mathbf{Z} -homomorphisms. Q.E.D.

We conclude with another partial answer to question 3, which may be obtained by modifying the proof of theorem 2 above, and using theorem 2 of [6]:

If k lies in the interior of a 3-cell in M , then

$$v \leq e(k) + v(M).$$

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