GROUPS GENERATED BY UNITARY REFLECTIONS OF PERIOD TWO

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1. Introduction. In complex affine n-space with a unitary metric, a *reflection* is a congruent transformation leaving invariant all the points of a hyperplane. Thus the characteristic roots of a unitary reflection of period p consist of a primitive pth root of unity and $n-1$ unities. A group generated by n reflections is conveniently represented by a graph having a node for each generator and a branch for each pair of non-commutative generators. For a generator of period p, the node is generally marked p, but we find it convenient to omit the mark when $p = 2$, as in the case of real reflections (**8**, p. 619). Whenever two such involutory generators do not commute, the corresponding nodes are joined by a branch which is marked with the period of their product, except that for simplicity the mark is omitted when this period takes its most prevalent value 3.

Nodes representing commutative generators are not (directly) joined. This convention has the happy result that the graph for a (completely) reducible group consists of disconnected pieces representing the irreducible components.

For any finite irreducible group generated by n involutory reflections, Shephard (**29**) and Todd (**31**) showed, in effect, that the generators may be so chosen that the graph either is a tree or contains just one triangular circuit. Since the abstract group defined by

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$$R_1^2 = R_2^2 = R_3^2 = (R_2R_3)^{l_1} = (R_3R_1)^{l_2} = (R_1R_2)^{l_3} = E \qquad (l_1, l_2, l_3 > 2)$$

is infinite (**32**, pp. 26–29), the three generators represented by the nodes of such a circuit must satisfy some further relation. One of the purposes of the present paper is to indicate the advantages of using the relation

$$(R_1R_2R_3R_2)^m = E,$$

which is equivalent to $(R_2R_3R_1R_3)^m = E$ or $(R_3R_1R_2R_1)^m = E$ if, as can always be arranged, at least two of the l's take the value 3. Accordingly, we complete the graphical symbol by writing m inside the triangle. This agrees with the notation of Shephard (**29**, 370–374) when $m = 3$; for other values of m there is an essential difference, as we shall see.

The finite irreducible groups generated by three real reflections are the symmetry groups of the Platonic solids (**26a**, pp. 20, 24). We shall exhibit all the remaining finite three-dimensional irreducible groups generated by three reflections as instances of $[1\ 1\ 1^l]^m$ (see 3.8), which is finite when l and m satisfy the inequality 3.7. In particular, $[1\ 1\ 1^4]^4$, which is "No. 24" in the list of Shephard and Todd (**31**, p. 301), is interesting because of its connection with the new senary extreme form discovered by Barnes (see the preceding paper (**4**)).

Groups in more than three dimensions are derived by adding "tails" to the graph for $[1\ 1\ 1^l]^m$. The graph so obtained enables us to compute certain numbers $m_1, m_2, \ldots, m_n$ which quickly yield both the order of the group and the order of its centre.

Reflections of period $p > 2$ will be discussed in a subsequent paper, where it will be shown that the proper interpretation for a branch marked 4 or 5 is not

$$(R_1R_2)^4 = E \qquad \text{or} \qquad (R_1R_2)^5 = E$$

but

$$R_1R_2R_1R_2 = R_2R_1R_2R_1 \text{ or } R_1R_2R_1R_2R_1 = R_2R_1R_2R_1R_2.$$

2. Reflections in the coordinate hyperplanes. In complex affine n-space, with the "contravariant" notation $(x^1, \ldots, x^n)$ for coordinates, any finite group of affine collineations (i.e., linear transformations) leaves invariant a positive definite Hermitian form, say

2.1 $$\sum\sum a_{jk}\, x^j\, \bar{x}^k \qquad (a_{jk} = \bar{a}_{kj})$$

(**6**, p. 253). This form determines a unitary metric in which the point $(x^1, \ldots, x^n)$ may be usefully regarded as having also covariant coordinates $(x_1, \ldots, x_n)$, defined by

2.2 $$x_j = \sum a_{jk}\, \bar{x}^k \qquad (j = 1, \ldots, n).$$

In this notation, the form 2.1 may be expressed either as

2.3 $$\sum x_j\, x^j$$

or as

2.4 $$\sum \sum a^{jk} x_j \bar{x}_k,$$

where the new coefficients $a^{jk} = \bar{a}^{kj}$ are given by

$$\sum a_{hk} a^{jk} = \delta_h^j,$$

so that

2.5 $$x^j = \sum a^{jk} \bar{x}_k, \quad \bar{x}^k = \sum a^{jk} x_j.$$

Embedding the affine n-space in a projective n-space, we may define an *affine reflection* (**36**, pp. 109, 115) to be a homology whose centre is a point at infinity, and define a *unitary reflection* (**28**, p. 82) to be an affine reflection that leaves invariant the form 2.1 or 2.3 or 2.4.

For a group generated by n unitary reflections, we may choose such a frame of reference that the centres of the n homologies are the points at infinity on the contravariant axes. Then R_k, the kth generating reflection, leaves invariant all the contravariant coordinates x^j except x^k. Let us suppose that it transforms x^k into

$$\sum c_l x^l.$$

Since 2.1 is invariant, we have

$$\begin{aligned} a_{kk} x^k \bar{x}^k &+ \sum_{j \neq k} (a_{jk} x^j \bar{x}^k + a_{kj} x^k \bar{x}^j) \\ &= a_{kk} \sum c_l x^l \sum \bar{c}_l \bar{x}^l + \sum_{j \neq k} (a_{jk} x^j \sum \bar{c}_l \bar{x}^l + a_{kj} \bar{x}^j \sum c_l x^l), \end{aligned}$$

whence, by comparing coefficients of $x^j \bar{x}^k$ $(j \neq k)$,

2.6 $$a_{jk} = a_{kk} c_j \bar{c}_k + a_{jk} \bar{c}_k.$$

In the present paper we restrict consideration to cases where R_k is involutory (i.e., of period 2), so that

$$c_k = -1$$

and therefore $2a_{jk} + a_{kk} c_j = 0$ $(j \neq k)$. Since the vanishing of a_{kk} would imply $a_{jk} = 0$ for all j, we must have $a_{kk} \neq 0$ for each k. Thus

2.7 $$c_j = -2a_{jk}/a_{kk} \qquad (j \neq k),$$

and R_k is the transformation leaving invariant every x^j except x^k, which becomes

2.8 $$\sum c_j x^j = x^k - \frac{2 \sum a_{jk} x^j}{a_{kk}} = x^k - \frac{2\bar{x}_k}{a_{kk}}$$

(cf. **17**, p. 403). Thus R_k leaves invariant every point on the hyperplane

$$x_k = 0.$$

In other words, the reflecting hyperplanes are the covariant coordinate hyperplanes.

Expressing R_k in terms of covariant coordinates, we find

2.9 $$x'_j = \sum a_{jl}\bar{x}^{l\prime} = \sum_{l\neq k} a_{jl}\bar{x}^l + a_{jk}\left(\bar{x}^k - \frac{2x_k}{a_{kk}}\right) = x_j - \frac{2a_{jk}x_k}{a_{kk}}$$

(**14**, p. 182). Thus the matrix for the covariant transformation is the transpose of the matrix for the contravariant transformation (as it clearly must be for any transformation of period 2).

3. Groups in unitary 3-space. In the case of the ternary Hermitian form

3.1 $$x^1\bar{x}^1 + x^2\bar{x}^2 + x^3\bar{x}^3 - \tfrac{1}{2}(x^2\bar{x}^3 + x^3\bar{x}^2 + x^3\bar{x}^1 + x^1\bar{x}^3 + cx^1\bar{x}^2 + \bar{c}x^2\bar{x}^1)$$

(which is semidefinite when $c = 1$), we have

$$a_{kk} = 1, \qquad a_{23} = a_{31} = -\tfrac{1}{2}, \qquad a_{12} = -\tfrac{1}{2}c.$$

The three reflections, expressed as matrices, are

3.2 $$R_1 = \begin{pmatrix} -1 & \bar{c} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 \\ c & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

We may easily verify that these three transformations R_k are of period 2, while their products R_2R_3 and R_3R_1 are of period 3. Certain abstract definitions obtained by Shephard and Todd (**31**, p. 299) suggest that we may have a sufficient set of defining relations for the abstract group $\{R_1, R_2, R_3\}$ as soon as we have specified also the periods of the products R_1R_2 and $R_1R_2R_3R_2$. Since

$$R_1R_2R_3R_2 = R_1R_3R_2R_3 \qquad \text{and} \qquad R_2R_3R_1R_3 = R_2R_1R_3R_1,$$

the relation $(R_1R_2R_3R_2)^m = E$ can be replaced by $(R_2R_3R_1R_3)^m = E$ or by $(R_3R_1R_2R_1)^m = E$.

To find how the periods of R_1R_2 and $R_1R_2R_3R_2$ depend on c, we observe that the characteristic equations for

$$R_1R_2 = \begin{pmatrix} c\bar{c}-1 & -\bar{c} & \bar{c}+1 \\ c & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_1R_2R_3R_2 = \begin{pmatrix} c\bar{c}+c+\bar{c} & -1 & -\bar{c} \\ c+1 & 0 & -1 \\ c+1 & -1 & 0 \end{pmatrix}$$

are respectively

$$(\lambda - 1)\{(\lambda+1)^2 - c\bar{c}\lambda\} = 0, \quad (\lambda-1)\{(\lambda+1)^2 - (c+1)(\bar{c}+1)\lambda\} = 0.$$

Comparing these with the characteristic equation

$$(\lambda+1)^2 - \left(2\cos\frac{\pi}{p}\right)^2\lambda = 0$$

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for an ordinary rotation through $2\pi/p$, we see that the reflections 3.2 satisfy the relations

3.3 $$\mathrm{R}_k^2 = (\mathrm{R}_2\mathrm{R}_3)^3 = (\mathrm{R}_3\mathrm{R}_1)^3 = (\mathrm{R}_1\mathrm{R}_2)^l = (\mathrm{R}_1\mathrm{R}_2\mathrm{R}_3\mathrm{R}_2)^m = \mathrm{E}$$

if

3.4 $$c\bar{c} = 4\cos^2\frac{\pi}{l}, \quad (c+1)(\bar{c}+1) = 4\cos^2\frac{\pi}{m},$$

i.e., if $c = 2\cos\pi/l\cdot e^{\pi i/s}$, where s is given by

3.5 $$2\cos\frac{\pi}{s} = \left(4\cos^2\frac{\pi}{m} - 4\cos^2\frac{\pi}{l} - 1\right)\Big/\left(2\cos\frac{\pi}{l}\right).$$

The most significant cases are worked out in Table I on page 269, where we use the abbreviations

$$\tau = \frac{\sqrt{5}+1}{2}, \quad \omega = \frac{-1+i\sqrt{3}}{2}$$

(so that $\tau^2 - \tau - 1 = 0$, $\omega^2 + \omega + 1 = 0$).

The form 3.1, having determinant

3.6 $$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \tfrac{1}{8}\begin{vmatrix} 2 & -c & -1 \\ -\bar{c} & 2 & -1 \\ -1 & -1 & 2 \end{vmatrix}$$
$$= \tfrac{1}{8}\{5 - c\bar{c} - (c+1)(\bar{c}+1)\}$$
$$= \tfrac{1}{8}\left(5 - 4\cos^2\frac{\pi}{l} - 4\cos^2\frac{\pi}{m}\right),$$

is positive definite if

$$4\cos^2\frac{\pi}{l} + 4\cos^2\frac{\pi}{m} < 5,$$

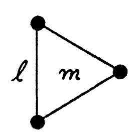
i.e., if

3.7 $$2\cos\frac{2\pi}{l} + 2\cos\frac{2\pi}{m} < 1.$$

When l (or m) has the value 3, m (or l) is unrestricted. The only other possibility is that l (or m) is 4 while m (or l) is 4 or 5. In each case, the results of Shephard and Todd will enable us to establish the sufficiency of 3.3 for an abstract definition of the group.

In the graphical notation of §1, the abstract group 3.3 is

3.8



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for which a convenient abbreviation is $[1\ 1\ 1^l]^m$, with the l omitted when $l = 3$. (We place the first node at the top, the second at the bottom, and the third on the right.)

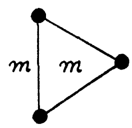
The substitution $R_2 \leftrightarrow R_2R_3R_2 = R_3R_2R_3$ shows that l and m are interchangeable:

3.9 $$[1\ 1\ 1^l]^m \simeq [1\ 1\ 1^m]^l.$$

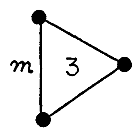
In the case of $[1\ 1\ 1^m]^3$, we can replace $(R_1R_2R_3R_2)^3 = E$ or $(R_2R_3R_1R_3)^3 = E$ by the equivalent relation

$$(R_3R_1R_2)^2 = (R_1R_2R_3)^2,$$

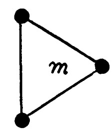
which enables us to identify this group with Shephard's $[1\ 1;\ 1]^m$ or Todd's $G(m, m, 3)$ (**29**, p. 374; **31**, p. 277). Thus Shephard's



is our



or



$[1,\ 1;\ 1]^m$ $\qquad$ $[1\ 1\ 1^m]^3 \simeq [1\ 1\ 1]^m$,

which is 3.3 with $l = 3$. It follows that the order of $[1\ 1\ 1]^m$ is $6m^2$. The first two values of m yield symmetric groups:

$$[1\ 1\ 1]^1 \simeq \mathfrak{S}_3, \qquad \text{generated by } (2\ 3),\ (3\ 1),\ (1\ 2);$$

$$[1\ 1\ 1]^2 \simeq \mathfrak{S}_4, \qquad \text{generated by } (1\ 4),\ (2\ 4),\ (3\ 4).$$

Since the relations 3.3 with $l = 1$ or 2 imply $m = 3$, there are no other groups with l or $m = 1$ or 2.

The substitution $R_3 \leftrightarrow R_1R_3R_1$ enables us to express $[1\ 1\ 1^4]^4$ in the form

$$R_k^2 = (R_2R_1R_3R_1)^3 = (R_3R_1)^3 = (R_1R_2)^4 = (R_2R_3)^4 = E,$$

which is No. 24, of order 336, in the list of Shephard and Todd (**31**, p. 299). Similarly $[1\ 1\ 1^5]^4$ is No. 27, of order 2160. Thus the relations 3.3 suffice for an abstract definition in every case.

4. Extension to higher spaces. In accordance with the graphical symbolism described in §1, we derive from $[1\ 1\ 1^l]^m$ a group

$$[p\ q\ r^l]^m \simeq [q\ p\ r^l]^m$$

in unitary $(p + q + r)$-space, by adding "tails" of $p - 1$, $q - 1$, $r - 1$ branches to the three nodes in 3.8.

The isomorphism 3.9 is easily seen to be maintained when we add a tail to either or both of the first two nodes (which are joined by the marked branch):

4.1 $$[p\ q\ 1^l]^m \simeq [p\ q\ 1^m]^l.$$

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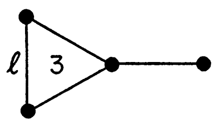
When $l = 3$, we have the group

$$[p\ q\ r]^m,$$

which involves the three numbers p, q, r symmetrically.

As this notation scarcely differs from Shephard's (29, p. 371), we can make use of his results in order to interpret the graphical symbol both abstractly and geometrically. For instance, the group



$$[1\ 1\ 2^l]^3,$$

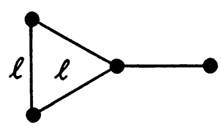
defined by

$$\begin{aligned} \text{4.2} \qquad R_k^2 &= (R_2R_3)^3 = (R_3R_1)^3 = (R_1R_2)^l = (R_1R_2R_3R_2)^3 \\ &= (R_1R_4)^2 = (R_2R_4)^2 = (R_3R_4)^3 = E, \end{aligned}$$

is generated by reflections in the covariant coordinate hyperplanes when the metric is determined by the form 3.1 with the extra terms

$$x^4\bar{x}^4 - \tfrac{1}{2}(x^3\bar{x}^4 + x^4\bar{x}^3).$$

In Shephard's notation (29, p. 379), this group of order $24l^3$ is



$$[1\ 1;\ 2]^l.$$

Extending the tail, we have $[1\ 1\ (n-2)^l]^3$, which is his $[1\ 1;\ n-2]^l$, of order $l^{n-1}\,n!$. In fact, all his graphical symbols (29, pp. 371, 374, 379, 382, 383) can be amended by the following simple modification. Whenever he draws a triangle with a number (m or 4) inside, this inner number should be changed to 3.

Let $2^{-p-q-r} f(p, q, r)$ denote the determinant of the form corresponding to $[p\ q\ r^l]^m$; for instance,

$$f(1, 1, 2) = \begin{vmatrix} 2 & -c & -1 & 0 \\ -\bar{c} & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 6 - 4\cos^2\frac{\pi}{l} - 8\cos^2\frac{\pi}{m}.$$

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We easily find (cf. **17**, p. 426) the recursion formulae

$$\begin{aligned} f(p, q, r) &= 2f(p-1, q, r) - f(p-2, q, r) \\ &= f(q, p, r) \\ &= 2f(p, q, r-1) - f(p, q, r-2), \end{aligned}$$

with the initial values

$$f(1, 1, 1) = 5 - 4\cos^2\frac{\pi}{l} - 4\cos^2\frac{\pi}{m},$$

$$f(1, 1, 0) = 4\sin^2\frac{\pi}{l}, \quad f(p, 0, r) = p + r + 1,$$

whence

$$f(p, q, r) = (p+1)(q+1) + r - 4pq\left(\cos^2\frac{\pi}{l} + r\cos^2\frac{\pi}{m}\right)$$

(cf. **29**, p. 372). The necessary condition

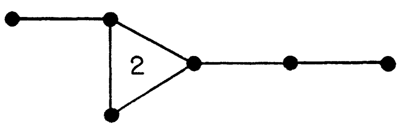
$$f(p, q, r) > 0$$

yields the finite groups listed in Table II. The identification with Shephard's list can be completed as follows.

Setting $l = 3$, we see that $[p\ q\ r]^m$ occurs whenever

4.3 $$p + q + r + 1 - 4pqr\cos^2\frac{\pi}{m} > 0.$$

In particular, we have $[p\ q\ r]^2$ for all values of p, q, r; but this is merely an unusual way of generating $\mathfrak{S}_{p+q+r+1}$, the symmetry group of the regular simplex α_{p+q+r} in real $(p+q+r)$-space. For instance,



$[2\ 1\ 3]^2$

is $\mathfrak{S}_7$, generated by the transpositions

$$\begin{matrix} (1\ 5), & (1\ 4), & & & \\ & & (3\ 4), & (3\ 6), & (6\ 7). \\ & (2\ 4), & & & \end{matrix}$$

The criterion 4.3 shows that, when $m > 2$, the numbers p, q, r cannot all be greater than 1. Thus every such finite group (with $l = 3$) is expressible as

4.4 $$[p\ q\ 1]^m \simeq [p\ q\ 1^m]^3,$$

which we shall usually write with the 1 in the middle: $[p\ 1\ q]^m$. It is evidently Shephard's $[p\ 1; q]^m$ (**29**, p. 373).

The isomorphism 4.4, which is a special case of 4.1, has an interesting counterpart for $[1\ 1\ r^l]^3$. Consider, for instance, $[1\ 1\ 2^l]^3$, defined by 4.2. Instead of R_1, we may introduce a new generator

$$\overline{R}_1 = R_4R_3R_1R_3R_4$$

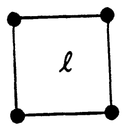
by writing

$$R_1 = R_3R_4\overline{R}_1R_4R_3.$$

The consequent relations

$$\begin{aligned}\overline{R}_1^{\,2} = R_2^2 = R_3^2 = R_4^2 &= (R_2R_3)^3 = (R_4\overline{R}_1)^3 \\ &= (R_2R_3R_4\overline{R}_1R_4R_3)^l = (\overline{R}_1R_2)^3 \\ &= (\overline{R}_1R_3)^2 = (R_2R_4)^2 = (R_3R_4)^3 = E\end{aligned}$$

are naturally represented by the graphical symbol



$[1\ 1\ 1\ 1^l]$.

More generally, an alternative symbol for $[1\ 1\ r^l]^3$ $(r = n - 2)$ consists of an n-gon with the mark l inside to indicate any one of n equivalent relations such as

$$(\overline{R}_1R_2R_3 \ldots R_nR_{n-1} \ldots R_2)^l = E.$$

When $n = 3$, we simply have $[1\ 1\ 1]^l$ as an alternative symbol for $[1\ 1\ 1^l]^3$.

5. Commutator subgroups. If the words in the defining relations for a given abstract group $\mathfrak{G}$ are such that each involves an even number of letters, then every element is either "even" or "odd" according to the parity of the number of letters occurring in any expression for it. The "even" elements form a subgroup $\mathfrak{G}^+$ of index 2.

When $\mathfrak{G}$ is $[p\ q\ r^l]^m$, one possible set of $p + q + r - 1$ generators for $\mathfrak{G}^+$ is provided by the products R_jR_k of period 3, represented in the graph by the unmarked branches. (We may ignore the branch marked l, even when $l = 3$.) Any such product is a commutator:

$$R_jR_k = R_kR_jR_kR_j = R_k^{-1}\,R_j^{-1}\,R_kR_j.$$

Moreover, any commutator

$$(R_i \ldots R_h)(R_k \ldots R_j)(R_h \ldots R_i)(R_j \ldots R_k)$$

is the product of an even number of R's. Hence

The commutator subgroup of $[p\ q\ r^l]^m$ *is* $[p\ q\ r^l]^{m+}$.

In particular, the commutator subgroup of $[1\ 1\ 1^l]^m$ is $[1\ 1\ 1^l]^{m+}$, whose generators

$$S_1 = R_1R_3, \quad S_2 = R_3R_2$$

satisfy the relations

5.1 $$S_1^{\ 3} = S_2^{\ 3} = (S_1S_2)^{\ l} = (S_1^{-1}S_2)^{\ m} = E.$$

These relations suffice for an abstract definition, since we can reconstruct 3.3 by adjoining to 5.1 a new element T, of period 2, which transforms both the S's into their inverses (and then defining $R_1 = S_1T$, $R_2 = TS_2$, $R_3 = T$). Thus

5.2 $$[1\ 1\ 1^l]^{m+} \simeq (3, 3 \mid l, m)$$

in the notation of **(12**, pp. 77–85**)**. (In order to interchange l and m, we merely have to replace one of the S's by its inverse.)

Setting $l = 3$, we recognize $[1\ 1\ 1]^{m+}$ as a group of order $3m^3$ considered by Edington **(21**, p. 208**)**. This is generated by the permutations

$$\begin{aligned} S_1 &= (3m\ 1\ 2)(3\ 4\ 5)\ldots(3m-3\ \ 3m-2\ \ 3m-1),\\ S_2 &= (2\ 3\ 4)(5\ 6\ 7)\ldots(3m-1\ \ 3m\ \ 1)\end{aligned}$$

which yields, for $[1\ 1\ 1]^m$,

$$\begin{aligned} R_1 &= (1\ 2)(4\ 5)\ldots(3m-2\ \ 3m-1),\\ R_2 &= (2\ 3)(5\ 6)\ldots(3m-1\ \ 3m),\\ R_3 &= (3\ 4)(6\ 7)\ldots(3m\ \ \ 1).\end{aligned}$$

A more interesting case is

5.3 $$[1\ 1\ 1^4]^{4+} \simeq (3, 3 \mid 4, 4) \simeq LF(2, 7)$$

(12, pp. 83–84**)**. The generators

$$S_1 = (0\ 1\ \infty)(2\ 6\ 4) = \left(\frac{1}{1-x}\right) \pmod 7$$

$$S_2 = (0\ 2\ 3)(1\ 6\ 5) = (4x+2) \pmod 7$$

are transformed into their inverses by

$$(0\ 6)(1\ 2)(3\ 5)(4\ \infty) = \left(\frac{x+1}{2x-1}\right) \pmod 7.$$

Hence in this case we have an *inner* automorphism, and

5.4 $$[1\ 1\ 1^4]^4 \simeq \mathfrak{C}_2 \times LF(2, 7).$$

6. Exponents and invariants. By 2.9 with $a_{kk} = 1$, the reflection R_k (in covariant coordinates) is

$$x'_j = x_j - 2a_{jk}x_k \qquad (j = 1, \ldots, n)$$

or

$$x_j = x'_j - 2a_{jk}x'_k.$$

Since this is the same expression as for real reflections (**16**, pp. 766–767), the characteristic equation for the product

$$R_1 R_2 \ldots R_n$$

is again

$$\text{6.1} \qquad \begin{vmatrix} \frac{1}{2}(\lambda+1) & a_{12}\lambda & a_{13}\lambda & \ldots & a_{1n}\lambda \\ a_{21} & \frac{1}{2}(\lambda+1) & a_{23} & \ldots & a_{2n}\lambda \\ & \ldots & & \ldots & \\ a_{n1} & a_{n2} & a_{n3} & & \frac{1}{2}(\lambda+1) \end{vmatrix} = 0$$

(**25**, p. 20). In the case of $[1\ 1\ 1^l]^m$, given by 3.1, we take $a_{12} = -\frac{1}{2}c$, $a_{21} = -\frac{1}{2}\bar{c}$, while the remaining a_{jk} $(j \neq k)$ are all $-\frac{1}{2}$, so that the equation is

$$\text{6.2} \qquad \begin{vmatrix} \lambda+1 & -c\lambda & -\lambda \\ -\bar{c} & \lambda+1 & -\lambda \\ -1 & -1 & \lambda+1 \end{vmatrix} = 0$$

or

$$\lambda^3 - (c\bar{c} + c - 1)\lambda^2 - (c\bar{c} + \bar{c} - 1)\lambda + 1 = 0.$$

This is unchanged when c is replaced by $-\bar{c} - 1$; therefore, by 3.4, it involves l and m symmetrically.

In the case of $[1\ 1\ 1]^m$, we have $c = e^{2\pi i/m}$, the equation reduces to

$$\lambda^3 - c\lambda^2 - \bar{c}\lambda + 1 = 0$$

(**31**, p. 295, with c instead of θ^{-1}), and the roots are

$$c, \quad \pm\bar{c}^{\frac{1}{2}}.$$

In the case of $[1\ 1\ 1^l]^4$, we have $(c+1)(\bar{c}+1) = 2$, and the equation reduces to

$$\lambda^3 + \bar{c}\lambda^2 + c\lambda + 1 = 0$$

with $c = \frac{1}{2}(-1 + i\sqrt{7})$ when $l = 4$, and $c = \omega\tau$ when $l = 5$ (**31**, p. 296).

Similarly, we can reconstruct the characteristic equation for $R_1R_2 \ldots R_n$ when $n > 3$:

for $[1\ 1\ 2]^4$, $\qquad \lambda^4 - i\lambda^3 - \lambda^2 + i\lambda + 1 = 0;$
for $[2\ 1\ 2]^3$, $\qquad \lambda^5 - \omega\lambda^4 + \bar{\omega}\lambda^3 + \omega\lambda^2 - \bar{\omega}\lambda + 1 = 0;$
for $[2\ 1\ 3]^6$, $\qquad \lambda^6 - \omega\lambda^5 + \bar{\omega}\lambda^4 - \lambda^3 + \omega\lambda^2 - \bar{\omega}\lambda + 1 = 0$

(**31**, p. 298). The last equation arises in the form

$$\begin{vmatrix} \lambda+1 & -\lambda & 0 & 0 & 0 & 0 \\ -1 & \lambda+1 & -\omega\lambda & -\lambda & 0 & 0 \\ 0 & -\bar{\omega} & \lambda+1 & -\lambda & 0 & 0 \\ 0 & -1 & -1 & \lambda+1 & -\lambda & 0 \\ 0 & 0 & 0 & -1 & \lambda+1 & -\lambda \\ 0 & 0 & 0 & 0 & -1 & \lambda+1 \end{vmatrix} = 0,$$

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which is the natural extension of 6.2 (with $c = \omega$) from $[1\ 1\ 1]^3$ to $[2\ 1\ 3]^3$. If the equation for $[p\ q\ r^l]^m$ is $f_r(\lambda) = 0$, we can use the recursion formula

$$f_r(\lambda) = (\lambda+1)f_{r-1}(\lambda) - \lambda f_{r-2}(\lambda).$$

By taking p, q, r in such an order that $q = 1$, we may name the n generators (as in the above example) in the order of a Hamiltonian path (**14**, p. 8). We start at the tip of the first tail, pass to the second node (where there is no second tail), and proceed along the third tail.

In the case of $[1\ 1\ (n-2)^l]^3$, we have $c = -\bar{b} - 1$ where $b = e^{2\pi i/l}$, the equation is

$$(\lambda - b)\ (\lambda^{n-1} - \bar{b}) = 0$$

(**31**, p. 295, with b instead of θ^{-1}), and the roots are

$$\epsilon^{n-1},\ \epsilon^{l-1},\ \epsilon^{2l-1},\ \ldots,\ \epsilon^{(n-1)l-1},$$

where $\epsilon = e^{2\pi i/(n-1)l}$.

In each case, to compute h, the period of $R_1R_2 \ldots R_n$, we observe that the characteristic roots are powers

$$\epsilon^{m_j} \qquad (j = 1, 2, \ldots, n; \quad m_1 \leqslant m_2 \leqslant \ldots \leqslant m_n)$$

of a primitive hth root of unity, $\epsilon = e^{2\pi i/h}$. Our choice of the positive value of s in 3.5, combined with the "natural" order of $R_1, R_2, \ldots, R_n$ in the chosen product, has the effect that

6.3 $$m_n = h - 1.$$

(By reversing either of these conventions we would have had $m_1 = 1$ instead of 6.3. In the case of real reflections, this important distinction disappears, as we have both $m_1 = 1$ and $m_n = h - 1$.) The actual values are given in Table II on page 270 (cf. **16**, p. 771). It was proved by Shephard and Todd (**31**, p. 289) that any finite group of unitary transformations which possesses a basic set of n invariant polynomial forms is a group generated by unitary reflections. Moreover, when the n invariant forms are chosen so as to have the smallest possible degrees, the product of the degrees is equal to the order of the group. They observed (**31**, pp. 283, 294) that these degrees are just the numbers

$$m_j + 1 \qquad (j = 1, 2, \ldots, n).$$

Therefore the Jacobian J of the n forms has degree $\sum m_j$, which is equal to the number of reflections in the group (**31**, p. 290). Since J changes sign when operated on by a reflection of period 2, it follows that J factorizes into $\sum m_j$ linear forms which, when equated to zero, give the $\sum m_j$ reflecting hyperplanes (**30**, p. 47).

For instance, the group $[1\ 1\ 1^4]^4$ has exponents $m_1 = 3$, $m_2 = 5$, $m_3 = 13$. Klein (**26**, pp. 446–448) described invariants f, ∇, C, of degrees 4, 6, 14, and observed that their Jacobian factorizes into 21 linear forms which, when equated to zero, give the axes of 21 harmonic homologies transforming the plane quartic curve $f = 0$ into itself.

Incidentally, we have the simple expression

6.4 $$(m_1 + 1)\,(m_2 + 1)\ldots(m_n + 1)$$

for the order of the group.

7. Central quotient groups. Another interesting property of the degrees $m_j + 1$ is that their greatest common divisor

$$k = (m_1 + 1, \quad \ldots \quad, m_n + 1)$$

is equal to the order of the centre of the group. In fact, since

$$m_j \equiv -1 \pmod{k},$$

the characteristic roots of the element

$$(\mathrm{R}_1 \ldots \mathrm{R}_n)^{h/k}$$

are

$$\epsilon^{m_j h/k} = e^{2\pi i m_j/k} = e^{-2\pi i/k},$$

i.e., they are all equal, which means that this is the central transformation $e^{-2\pi i/k}\,\mathrm{I}$, of period k (**31**, p. 280).

By regarding the coordinates x_j or x^j as being homogeneous, we pass from complex affine n-space to complex projective $(n - 1)$-space. Instead of a group $\mathfrak{G}$ generated by involutory reflections, we now have the central quotient group

$$\mathfrak{G}/\mathfrak{C}_k,$$

generated by harmonic homologies (**31**, p. 275). Its abstract definition is given by the graphical symbol along with the single extra relation

7.1 $$(\mathrm{R}_1 \ldots \mathrm{R}_n)^{h/k} = \mathrm{E}.$$

In the complex projective plane, we have two groups:

$$[1\ 1\ 1^4]^4/\mathfrak{C}_2 \simeq LF(2, 7),$$

defined by 3.3 with $l = m = 4$ and $(\mathrm{R}_1\mathrm{R}_2\mathrm{R}_3)^7 = \mathrm{E}$ (cf. 5.3, 5.4), which is Klein's collineation group of order 168, containing 21 harmonic homologies (**26**, p. 440); and

$$[1\ 1\ 1^5]^4/\mathfrak{C}_6 \simeq \mathfrak{A}_6,$$

defined by 3.3, with $l = 5$, $m = 4$ and $(\mathrm{R}_1\mathrm{R}_2\mathrm{R}_3)^5 = \mathrm{E}$, which is Valentiner's collineation group (**35**, p. 227), of order 360, containing 45 harmonic homologies. When the latter is represented as the alternating group of degree six (**37**), the homologies appear as the double transpositions; e.g., the generators may be taken to be

$$\mathrm{R}_1 = (1\ 3)\ (4\ 6),\quad \mathrm{R}_2 = (3\ 4)\ (5\ 6),\quad \mathrm{R}_3 = (2\ 3)\ (5\ 6).$$

In complex projective 3-space, we have

$$[1\ 1\ 2]^4/\mathfrak{C}_4,$$

defined by

$$R_j^2 = (R_2R_3)^3 = (R_3R_1)^3 = (R_1R_2)^3 = (R_1R_2R_3R_2)^4$$
$$= (R_1R_4)^2 = (R_2R_4)^2 = (R_3R_4)^3 = (R_1R_2R_3R_4)^5 = E,$$

which is Bagnera's collineation group (**1**, p. 33) of order 1920, containing 40 harmonic homologies. This leaves invariant the Maschke quartic surface

$$x^4 + y^4 + z^4 + t^4 - 12xyzt = 0$$

(**27**, p. 504; **20a**), and a configuration consisting of five tetrahedra (**29**, p. 382), namely the tetrahedra (ab), (ac), (ad), (ae), (af) of Hudson (**24**, p. 43).

In complex projective 4-space, we have

$$[2\ 1\ 2]^3/\mathfrak{C}_2,$$

which is Mitchell's collineation group of order $36\cdot 6!$, containing 45 harmonic homologies whose centres are the nodes of the Burkhardt primal (**2**; **33**; **34**). Since the word

$$(R_1R_2R_3R_4R_5)^9$$

involves an odd number of letters, this element, which occurs in $[2\ 1\ 2]^3$ as the "central inversion", does not belong to the commutator subgroup $[2\ 1\ 2]^{3+}$; hence the central quotient group

$$[2\ 1\ 2]^3/\mathfrak{C}_2 \simeq [2\ 1\ 2]^{3+}$$

is the simple group of order 25920, and $[2\ 1\ 2]^3$ is its direct product with $\mathfrak{C}_2$.

In complex projective 5-space, we have

$$[2\ 1\ 3]^3/\mathfrak{C}_6$$

(**29**, p. 375), which is Mitchell's collineation group of order $18\cdot 9!$, containing 126 harmonic homologies whose centres form the Mitchell-Hamill configuration (**23**, p. 402). Its commutator subgroup is the simple group of order $9\cdot 9!$:

$$[2\ 1\ 3]^{3+}/\mathfrak{C}_6 \simeq HO(4, 3^2) \simeq PU_4^+(\mathbf{F}_9)$$

(**23**, p. 451; **19**, p. 310; **20**, p. 48).

8. Infinite groups. When the form 2.1 is not definite but only semi-definite, we have an infinite group generated by reflections 2.9 in n hyperplanes $x_k = 0$ forming an $(n-1)$-dimensional simplex. In fact, since $\det(a_{jk}) = 0$, there exist constants $z^1, \ldots, z^n$ such that

$$\sum z^j a_{jk} = 0$$

and therefore, by 2.9,

$$\sum z^j x'_j = \sum z^j x_j.$$

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Thus we may regard the group as operating in the $(n-1)$-space

$$\sum z^j x_j = 1,$$

in which the semidefinite form determines a unitary metric, and the R_k's appear as reflections in the simplex formed by the sections of the coordinate hyperplanes.

Adapting 3.7, we see that $[1\ 1\ 1^l]^m$ is infinite when

$$2\cos\frac{2\pi}{l} + 2\cos\frac{2\pi}{m} = 1.$$

When $l = 3$ and $m = \infty$, this is the group

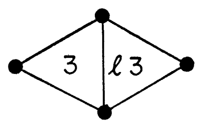
$$R_k^2 = (R_2R_3)^3 = (R_3R_1)^3 = (R_1R_2)^3 = E$$

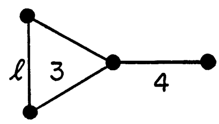
generated by real reflections in the sides of an ordinary equilateral triangle (**14**, p. 78). The only other two-dimensional instance is

$$[1\ 1\ 1^4]^6 \simeq [1\ 1\ 1^6]^4,$$

whose commutator subgroup $(3, 3 \mid 4, 6)$ is already known to be infinite (**12**, p. 95; **13**, p. 250).

The group $[1\ 1\ 1^l]^3$, of order $6l^2$, defined by 3.3 with $m = 3$, has two infinite extensions





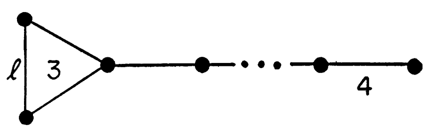
given by the extra relations

$$R_0^2 = (R_0R_1)^3 = (R_0R_2)^3 = (R_0R_1R_2R_1)^3 = E,$$
$$R_4^2 = (R_1R_4)^2 = (R_2R_4)^2 = (R_3R_4)^4 = E,$$

respectively. When $l = 2$, they reduce to the real groups

$$\square \quad \text{and} \quad \left[3, \begin{matrix}3\\4\end{matrix}\right]$$

(**14**, pp. 84, 85). More generally, $[1\ 1\ r^l]^3$ is a subgroup of the infinite group



(but on p. 265 we shall see that, as a geometrical group, this is discrete only if $l = 2, 3, 4$ or 6).

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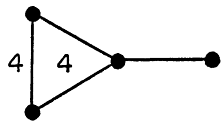
Similarly, the finite groups

$$[1\ 1\ 1^4]^4, \quad [2\ 1\ 1^4]^3, \quad [1\ 1\ 2]^4, \quad [2\ 1\ 3]^3$$

have infinite extensions

$$[1\ 1\ 2^4]^4, \quad [2\ 1\ 2^4]^3, \quad [1\ 1\ 3]^4, \quad [2\ 1\ 4]^3.$$

The last three were discovered by Shephard (**29**, pp. 382, 383, 381). But we must remember to change the mark inside his triangles from 4 to 3. His scheme does not include our



$[1\ 1\ 2^4]^4$,

whose determinant is

$$\frac{1}{16}\begin{vmatrix} 2 & -c & -1 & 0 \\ -\bar{c} & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix} = \tfrac{1}{16}\{6 - c\bar{c} - 2(c+1)(\bar{c}+1)\} = 0$$

(since $c\bar{c} = (c+1)\ (\bar{c}+1) = 2$).

9. Orthogonal coordinates. By working out the cofactors in the determinant 3.6 for $[1\ 1\ 1^l]^m$, we find the adjoint form of 3.1 to be a numerical multiple of

9.1
$$\begin{gathered} 3(x_1\bar{x}_1 + x_2\bar{x}_2) + (4 - c\bar{c})x_3\bar{x}_3 \\ + (c+2)\ (x_2\bar{x}_3 + x_3\bar{x}_1) + (\bar{c}+2)\ (x_3\bar{x}_2 + x_1\bar{x}_3) \\ + (2\bar{c}+1)x_1\bar{x}_2 + (2c+1)x_2\bar{x}_2. \end{gathered}$$

(See Table I, on page 269, for the values of c.)

In the case of $[1\ 1\ 1]^m$, where $l = 3$ and

$$c = e^{2\pi i/m},$$

9.1 may be expressed as

$$(x_1 + cx_2 + x_3)\ (\bar{x}_1 + \bar{c}\bar{x}_2 + \bar{x}_3) + (\bar{c}x_1 + x_2 + x_3)\ (c\bar{x}_1 + \bar{x}_2 + \bar{x}_3) \\ + (x_1 + x_2 + x_3)\ (\bar{x}_1 + \bar{x}_2 + \bar{x}_3).$$

In terms of orthogonal coordinates

$$\xi_1 = x_1 + cx_2 + x_3, \quad \xi_2 = \bar{c}x_1 + x_2 + x_3, \quad \xi_3 = x_1 + x_2 + x_3,$$

the form is $\xi_1\bar{\xi}_1 + \xi_2\bar{\xi}_2 + \xi_3\bar{\xi}_3$, and the reflecting planes are $x_k = 0$ where

9.2 $$(1-c)x_1 = c(\xi_2 - \xi_3),\ (1-c)x_2 = \xi_3 - \xi_1,\ (1-c)x_3 = \xi_1 - c\xi_2,$$

so that the reflections themselves are:

$$\begin{array}{lccl} R_1, & (\xi_1, \xi_2, \xi_3) & \to & (\xi_1, \xi_3, \xi_2); \\ R_2, & (\xi_1, \xi_2, \xi_3) & \to & (\xi_3, \xi_2, \xi_1); \\ R_3, & (\xi_1, \xi_2, \xi_3) & \to & (c\xi_2, \bar{c}\xi_1, \xi_3) \end{array}$$

(**31**, pp. 276, 295).

In the case of $[1\ 1\ 1^l]^3$, we have

$$c = -\bar{b} - 1, \qquad b = e^{2\pi i/l},$$

and the form 9.1 is

$$\begin{aligned} &3(x_1\bar{x}_1 + x_2\bar{x}_2) + (b-1)(\bar{b}-1)x_3\bar{x}_3 \\ &- (\bar{b}-1)(x_2\bar{x}_3 + x_3\bar{x}_1) - (b-1)(x_3\bar{x}_2 + x_1\bar{x}_3) \\ &\quad - (2b+1)x_1\bar{x}_2 - (2\bar{b}+1)x_2\bar{x}_1 \\ &= (x_1 - x_2)(\bar{x}_1 - \bar{x}_2) + (bx_1 - x_2)(\bar{b}\bar{x}_1 - \bar{x}_2) \\ &\quad + \{bx_1 - x_2 + (b-1)x_3\}\{\bar{b}\bar{x}_1 - \bar{x}_2 + (\bar{b}-1)\bar{x}_3\}. \end{aligned}$$

In terms of orthogonal coordinates

$$\xi_1 = b(x_1 - x_2), \quad \xi_2 = bx_1 - x_2, \quad \xi_3 = bx_1 - x_2 + (b-1)x_3,$$

the reflecting planes are $x_k = 0$ where

9.3 $$(1-b)x_1 = \bar{b}\xi_1 - \xi_2, \quad (1-b)x_2 = \xi_1 - \xi_2, \quad (1-b)x_3 = \xi_2 - \xi_3,$$

so that the reflections themselves are:

$$\begin{array}{lccl} R_1, & (\xi_1, \xi_2, \xi_3) & \to & (b\xi_2, \bar{b}\xi_1, \xi_3); \\ R_2, & (\xi_1, \xi_2, \xi_3) & \to & (\xi_2, \xi_1, \xi_3); \\ R_3, & (\xi_1, \xi_2, \xi_3) & \to & (\xi_1, \xi_3, \xi_2). \end{array}$$

The resemblance of 9.2 and 9.3 illustrates the fact that $[1\ 1\ 1^l]^m$ and $[1\ 1\ 1^m]^l$ are different ways of generating the same group.

In the case of $[1\ 1\ 1^4]^4$, we have

9.4 $$c = \tfrac{1}{2}(-1 + i\sqrt{7}) = \beta + \beta^2 + \beta^4, \qquad \beta = e^{2\pi i/7},$$

so that c and $\bar{c}$ are the roots of the equation

9.5 $$x^2 + x + 2 = 0,$$

and the form 9.1 is

$$\begin{aligned} &3(x_1\bar{x}_1 + x_2\bar{x}_2) + 2x_3\bar{x}_3 - c^2(x_2\bar{x}_3 + x_3\bar{x}_1) \\ &\qquad - \bar{c}^2(x_3\bar{x}_2 + x_1\bar{x}_3) - (c - \bar{c})(x_1\bar{x}_2 - x_2\bar{x}_1) \\ &= \tfrac{1}{2}\{x_1\bar{x}_1 + (x_1 + cx_2)(\bar{x}_1 + \bar{c}\bar{x}_2) \\ &\qquad + (\bar{c}^2x_1 + c^2x_2 - 2x_3)(c^2\bar{x}_1 + \bar{c}^2\bar{x}_2 - 2\bar{x}_3)\}. \end{aligned}$$

In terms of orthogonal coordinates

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$$\xi_1 = x_1, \quad \xi_2 = x_1 + cx_2, \quad \xi_3 = \bar{c}^2 x_1 + c^2 x_2 - 2x_3,$$

the reflecting planes are $x_k = 0$, where

9.6 $$x_1 = \xi_1, \quad x_2 = \tfrac{1}{2}\bar{c}(-\xi_1 + \xi_2), \quad x_3 = \tfrac{1}{2}(-\xi_1 + c\xi_2 - \xi_3),$$

in agreement with Shephard and Todd (**31**, p. 295), whose α and x_j are our $-c$ and ξ_j. The reflections themselves are:

R_1, reversing the sign of ξ_1;

R_2, interchanging ξ_1 and ξ_2;

$$R_3, \begin{cases} \xi_1' = \tfrac{1}{2}(\xi_1 + c\xi_2 - \xi_3), \\ \xi_2' = \tfrac{1}{2}\,\bar{c}(\xi_1 + \xi_3), \\ \xi_3' = \tfrac{1}{2}(-\xi_1 + c\xi_2 + \xi_3). \end{cases}$$

In terms of the ξ's, the quartic invariant (Klein's f, mentioned near the end of p. 254) is

9.7 $$\xi_1^4 + \xi_2^4 + \xi_3^4 + 3\bar{c}(\xi_2^2\,\xi_3^2 + \xi_3^2\,\xi_1^2 + \xi_1^2\,\xi_2^2).$$

One can soon verify that this is transformed into itself not only by R_1 and R_2 but also by R_3 (cf. **22**, p. 338 with $b/a = 3\bar{c}$).

The remaining groups may be treated similarly, but the details are omitted because the results have all been obtained another way by Shephard (**29**, p. 373) and Todd (**31**, pp. 296, 298).

10. Polytopes and honeycombs. Let O denote the origin, i.e., the point of intersection of the hyperplanes of the generating reflections $R_1, \ldots, R_n$. Clearly, O is invariant, and the images of any other point are all equidistant from O. In particular, let P be a point on the line of intersection of $n-1$ of the n hyperplanes, say all except the kth. Then P is invariant under the subgroup generated by all the R's except R_k; but R_k transforms P into another point Q, and the whole group transforms P and Q into the vertices of a configuration called a *polytope* (**28**, p. 83) (or, when $n = 2$ or 3, a *polygon* or a *polyhedron*). Any subset of the R's (including R_k) generates a subgroup which transforms P into the vertices of a sub-configuration called an *element* of the polytope (e.g., a *side* of the polygon, or an *edge* or *face* of the polyhedron).

Similarly, when we have an infinite discrete group in $(n-1)$-space, generated by reflections $R_1, \ldots, R_n$ in n hyperplanes forming a simplex, as in §8, the images of the kth vertex P of the simplex are said to form a *honeycomb* (**29**, pp. 364, 375). The kth reflection R_k transforms P into another vertex Q of the honeycomb, and the subgroup generated by the remaining $n-1$ R's transforms Q into the *vertex figure* at P.

When the group is symbolized by a graph, the polytope or honeycomb is indicated by drawing a ring round the kth node (**14**, p. 87; **29**, p. 375), and an element can be found by deleting one or more nodes (along with any branches that occur there). Adapting the notation of Coxeter (**9**, p. 331) and Shephard (**29**, p. 378), we let

$$(p_p\ q\ r^l)^m, \quad (p\ q_q\ r^l)^m, \quad (p\ q\ r_r{}^l)^m$$

denote the polytopes (or honeycombs) indicated by ringing the tip of the first, second, or third tail. Of course,

$$(p_p\ q\ r^l)^m = (q\ p_p\ r^l)^m,$$

and there is complete symmetry when $l = 3$ (in which case the l is omitted).

The same reasoning that led to 3.9 and 4.1 shows that the two polytopes

$$(p_p\ q\ 1^l)^m, \quad (p_p\ q\ 1^m)^l$$

have the same vertices; in fact, they have the same j-dimensional elements for $j = 0, 1, \ldots, p$. Similarly, $(1_1\ 1\ r^l)^3$ has the same vertices and edges as

$$(1_1\ 1 \ldots 1)^l,$$

whose graphical symbol consists of an n-gon ($n = r + 2$) with one vertex ringed (see p. 251).

To compute the number of vertices of $(p_p\ q\ r^l)^m$ or of $(p\ q\ r_r{}^l)^m$, as in Table III, we divide the order of the group $[p\ q\ r^l]^m$ by the order of the subgroup that leaves one vertex invariant. This subgroup is given by reducing p to $p - 1$, or r to $r - 1$, respectively. To make this rule apply to $(p\ 1_1\ r^l)^m$ or to $(p\ q\ 1_1{}^l)^m$, we have to interpret the symbols

$$[p\ 0\ r^l]^m, \quad [p\ q\ 0^l]^m.$$

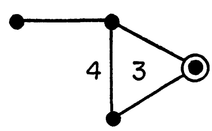
We see from the graph that $[p\ 0\ r^l]^m$ (including $[p\ 0\ r]^m$ or $[p\ r\ 0]^m$ as the special case when $l = 3$) is the symmetric group

$$[3, 3, \ldots, 3] \simeq \mathfrak{S}_{p+r+1},$$

represented by a simple chain of $p + r - 1$ unmarked branches. For instance, the six-dimensional polytope $(2\ 1_1\ 3)^3$ has

$$\frac{108 \cdot 9!}{6!} = 54432$$

vertices. Similarly, by removing the ringed node from



$$(2\ 1\ 1_1{}^4)^3,$$

we see that $[2\ 1\ 0^4]^3$ is the extended octahedral group [3, 4], of order 48 (**14**, pp. 82, 85).

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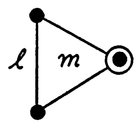
The only remaining case of $[p\ q\ 0^l]^m$ with $l > 3$ is $[1\ 1\ 0^l]^m$, which is the dihedral group $[l]$, of order $2l$, defined by

$$R_1^2 = R_2^2 = (R_1R_2)^l = E.$$

This is the case of the polyhedron

10.1 

$$(1\ 1\ 1_1{}^l)^m,$$

whose faces are equilateral triangles. The initial vertex P is taken on the line of intersection of the first two reflecting planes ("mirrors"). The third reflection transforms P into another vertex Q, such that PQ is an edge (symbolized by the ringed node alone). The first two reflections, whose product is of period l, transform PQ into $2l$ edges PQ, PQ', etc. (see Fig. 1). Hence this is a polyhedron of type $\{3, 2l\}$: the faces are triangles such as PQQ', and every vertex belongs to $2l$ of them. By 6.4, the order of the group $[1\ 1\ 1^l]^m$ is

$$g = (m_1 + 1)\ (m_2 + 1)\ (m_3 + 1),$$

where m_1, m_2, m_3 are computed from the roots $e^{2\pi i m_j/h}$ of the equation 6.2 (see Table II). Since there is one element of the group for every half-edge of $(1\ 1\ 1_1{}^l)^m$, this polyhedron of type $\{3, 2l\}$ has $\frac{1}{2}g$ edges, $\frac{1}{3}g$ triangular faces, and $g/2l$ vertices.

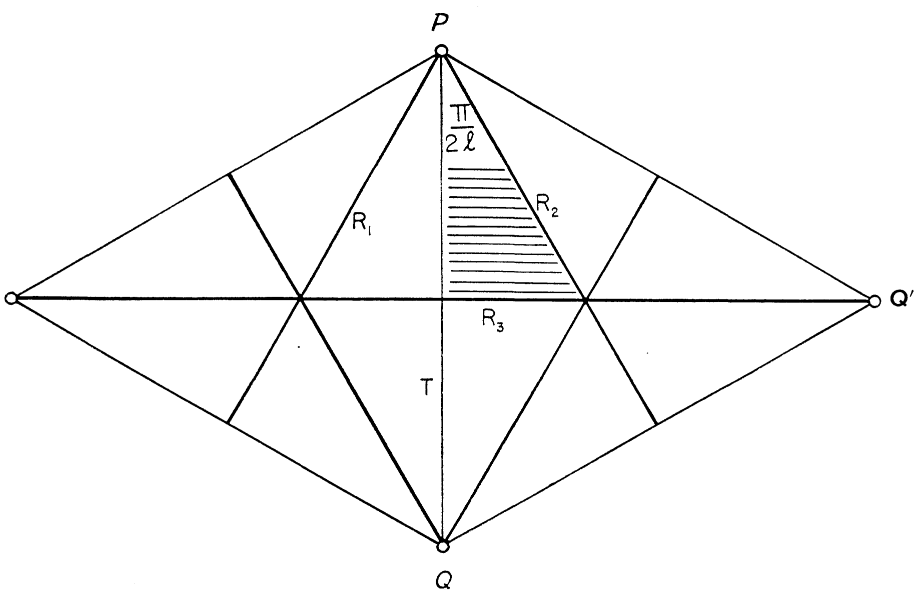


FIG. 1: A face PQQ' of $(1\ 1\ 1_1{}^l)^m$

When $l = 2$ (and therefore, by 3.3, $m = 3$), we have the octahedron $\{3, 4\}$. In other cases we may obtain a real interpretation by regarding the real and imaginary parts of the ξ's as Cartesian coordinates in a Euclidean space of twice as many dimensions. When so interpreted, $(1\ 1\ 1_1{}^l)^m$ appears as a regular *skew* polyhedron **(11)** in Euclidean 6-space. Shephard **(28)** did not include it in his list of regular complex polyhedra because, as we shall see in § 11, the symmetry operation interchanging two adjacent faces is not, in general, a unitary reflection.

When $l = 3$ we have $(1\ 1\ 1_1)^m$, and 9.2 shows that the first two mirrors meet along the line $\xi_1 = \xi_2 = \xi_3$, so we take P to be the point $(1, 1, 1)$. Applying the reflections, we find Q to be $(c, \bar{c}, 1)$, and Q' $(1, \bar{c}, c)$; altogether we obtain the m^2 points

$$(c^{k_1}, c^{k_2}, c^{k_3}), \qquad c = e^{2\pi i/m}, \qquad k_1 + k_2 + k_3 \equiv 0 \pmod{m}.$$

Topologically, $(1\ 1\ 1_1)^m$ is a triangulation of the torus, namely the regular map

$$\{3, 6\}_{m,0}$$

(15, p. 421**)**, which thus has a metrical realization as a skew polyhedron in Euclidean 6-space. Fig. 2 shows the case $m = 4$, with the values of $k_1k_2k_3$ marked at each vertex (so that P, Q, Q' are 000, 130, 031).

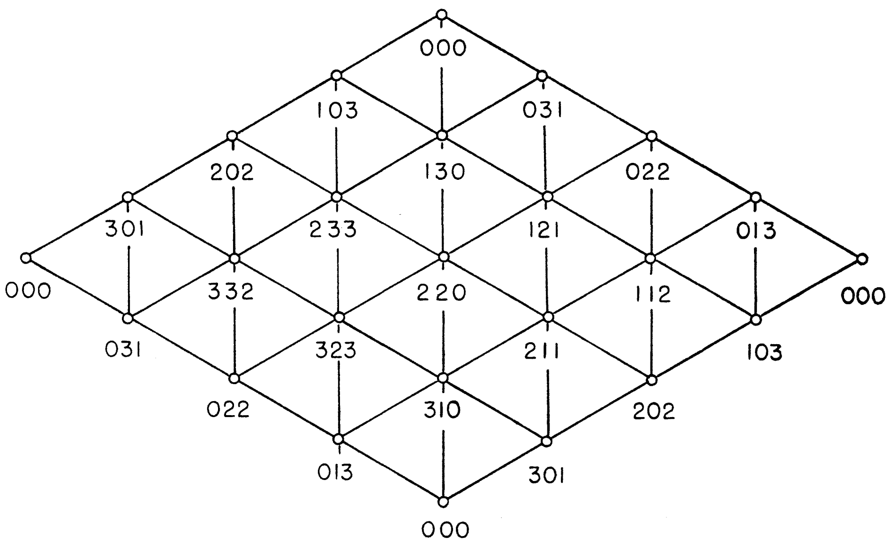


FIG. 2: $(1\ 1\ 1_1)^4 = \{3, 6\}_{4,0}$.

Similarly in n dimensions, reflections in the n hyperplanes

$$\xi_2 = \xi_3,\ \xi_3 = \xi_4,\ \ldots,\ \xi_n = \xi_1,\ \xi_1 = c\xi_2 \qquad (c = e^{2\pi i/m})$$

generate the group

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$$[1\ 1\ \ldots\ 1]^m,$$

which transforms the point $(1, 1, \ldots, 1)$ into the m^{n-1} points

$$(c^{k_1}, c^{k_2}, \ldots, c^{k_n}), \qquad \sum k_j \equiv 0 \pmod{m}.$$

These were shown by Shephard (**29**, p. 378) to be the vertices of the "fractional γ polytope"

$$(1_1\ 1\ r^m)^3 = \tfrac{1}{m}\gamma_n^m, \qquad r = n-2.$$

We now recognize the same m^{n-1} points as the vertices of the new polytope

$$(1_1\ 1\ \ldots\ 1)^m,$$

whose $(n-1)$-dimensional elements are the various truncations of the regular simplex α_{n-1}, just as they are in the limiting case of the real honeycomb

$$(1_1\ 1\ \ldots\ 1)^\infty = \alpha_{n-1}\mathrm{h}$$

(**7**, p. 366; **14**, p. 205, footnote).

In the case of $(1\ 1\ 1_1{}^l)^3$, 9.3 shows that the first two mirrors meet along the line $\xi_1 = \xi_2 = 0$, so we take P to be $(0, 0, 1)$ and obtain the $3l$ points whose coordinates are the permutations of

$$(b^k, 0, 0), \qquad b = e^{2\pi i/l}, \qquad k = 0, 1, \ldots, l-1$$

(**29**, p. 377). Accordingly, $(1\ 1\ 1_1{}^l)^3$ is the "generalized octahedron" β_3^l.

Similarly in n dimensions, reflections in the n hyperplanes

$$\xi_1 = b\xi_2,\ \ \xi_1 = \xi_2,\ \ \xi_2 = \xi_3,\ \ \ldots,\ \ \xi_{n-1} = \xi_n$$

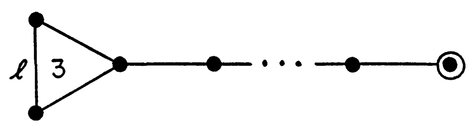
generate the group

$$[1\ 1\ r^l]^3, \qquad r = n-2;$$

and the nl points obtained by permuting

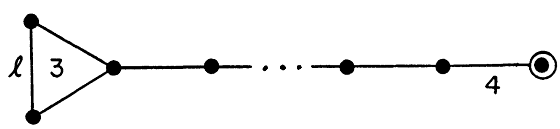
$$(b^k, 0, 0, \ldots, 0) \qquad (k = 0, 1, \ldots, l-1)$$

are the vertices of the generalized cross polytope



$$(1\ 1\ r_r^l)^3 = \beta_n^l$$

(**10**, p. 287). When $l = 2, 3, 4$ or 6, this polytope β_n^l is the vertex figure of the honeycomb



which is derived from the origin by applying the group $[1\ 1\ r^l]^3$ along with the $(n+1)$th reflection

$$(\xi_1, \ldots, \xi_{n-1}, \xi_n) \quad \rightarrow \quad (\xi_1, \ldots, \xi_{n-1}, 1 - \xi_n).$$

The reason for restricting l to the "crystallographic" values

$$2, 3, 4, 6$$

may already be seen when $n = 1$ (although the graphical symbol applies only when $n \geqslant 3$). The one-dimensional honeycomb consists of the points (ξ) whose single coordinate ξ is derived from 0 by the two operations of multiplying by b and subtracting from 1. In the Argand plane, these operations appear as rotation through $2\pi/l$ about the origin and reflection in the line $x = \frac{1}{2}$. The group so generated is not discrete when $l = 5$, nor when $l \geqslant 7$. (The proof, ascribed to Wigner in **(14**, p. 65**)**, is really due to Barlow **(3**, p. 17**)**.) When $l = 2$, the values of ξ are the real integers; when $l = 4$ they are the Gaussian integers $x + yi$; when $l = 6$ they are the Eisenstein integers $u + v\omega$; when $l = 3$ they are two-thirds of the Eisenstein integers, namely those for which $u + v \equiv 0$ or 1 (mod 3) **(14**, p. 64**)**. The corresponding honeycombs in the Argand plane are

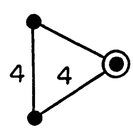
$$\{\infty\}, \quad \{4, 4\}, \quad \{3, 6\}, \quad \{6, 3\}.$$

For greater values of n, we have points $(\xi_1, \ldots, \xi_n)$ where each of the n coordinates is restricted to the appropriate one of the four classes. The corresponding real honeycombs are the "rectangular products"

$$\{\infty\}^n = \delta_{n+1}, \quad \{4, 4\}^n = \delta_{2n+1}, \quad \{3, 6\}^n, \quad \{6, 3\}^n$$

(7, pp. 353–354; **14**, pp. 123–124**)**.

Returning to unitary 3-space, let us investigate the polyhedron



$$(1\ 1\ 1_1^4)^4,$$

for which 9.6 yields the line $\xi_1 = \xi_2 = 0$ again. To avoid fractions, we now take P to be $(0, 0, 2)$, whose images are the $6 + 12 + 24 = 42$ permutations of

$$(0, 0, \pm 2), \ (\pm c, \pm c, 0), \ (\pm 1, \pm 1, \pm \bar{c}),$$

where c is given by 9.4. Since each of these 42 vertices belongs to 8 edges, there are altogether 168 edges and 112 triangular faces. This polyhedron $(1\ 1\ 1_1^4)^4$ is the vertex figure of the honeycomb

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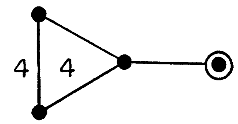

$(1\ 1\ 2_2{}^4)^4$,

whose group $[1\ 1\ 2^4]^4$ is derived from $[1\ 1\ 1^4]^4$ by adjoining the fourth reflection

$$\xi_1' = \xi_1, \quad \xi_2' = \xi_2, \quad \xi_3' = 2 - \xi_3.$$

Applying this group to the origin, we obtain the points whose coordinates ξ_1, ξ_2, ξ_3 satisfy the congruences

10.2 $$\xi_1 \equiv \xi_2 \equiv \xi_3 \pmod{c}, \qquad \xi_1 + \xi_2 + \xi_3 \equiv 0 \pmod{\bar{c}}$$

in the domain of algebraic integers generated by the roots (c and $\bar{c}$) of the equation 9.5. From the nature of this prescription, we see that the vertices of $(1\ 1\ 2_2{}^4)^4$, like those of $(2\ 1\ 4_4)^3$ **(18**, p. 386**)**, form a *lattice*.

The polyhedron $(1_1\ 1\ 1^4)^4$ (see Table III) is less interesting than $(1\ 1\ 1_1{}^4)^4$ because it is not regular but only "quasi-regular" **(14**, p. 18**)**. In fact, the graph shows that each of its $336/6 = 56$ vertices is surrounded by three triangles and three squares, arranged alternately.

11. The symmetry group of the polyhedron $(1\ 1\ 1_1{}^l)^m$. The group 3.3 has an involutory automorphism which interchanges R_1 and R_2. Adjoining a new element T which transforms the group in this manner (so that $T^2 = E$, $R_1 = TR_2T$ and $TR_3 = R_3T$), we obtain the larger group

11.1 $$T^2 = R_2{}^2 = R_3{}^2 = (R_2R_3)^3 = (TR_2)^{2l} = (TR_3)^2 = (TR_2R_3)^{2m} = E,$$

which is $G^{3,2l,2m}$ in the notation of **(12**, pp. 104–105**)**. As a transformation of the contravariant coordinates, leaving 3.1 invariant, we may take T to be

$$(x^1, x^2, x^3) \quad \rightarrow \quad (\bar{x}^2, \bar{x}^1, \bar{x}^3).$$

Although this is not a unitary reflection but an "anti-projectivity", its effect on the polyhedron 10.1 is like that of reflecting in the common edge PQ of two adjacent faces. Since the fundamental region of the symmetry group of the polyhedron is one-sixth of a face (shaded in Fig. 1), this symmetry group is precisely $G^{3,2l,2m}$, and $(1\ 1\ 1_1{}^l)^m$ is the regular skew polyhedron

$$\{3, 2l\}_{2m}$$

(12, p. 127**)** or $\{3, 2l\,|, m\}$ **(11**, p. 59**)**. In particular, we see again that the polyhedron

$$(1\ 1\ 1_1{}^4)^4 = \{3, 8\}_8 = \{3, 8\,|, 4\}$$

has 42 vertices, 168 edges, and 112 triangular faces.

Setting $l = \frac{1}{2}n$ and $m = \frac{1}{2}p$ in 3.7, we deduce that the even values of n and p for which the group $G^{3,n,p}$ is finite are given by

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11.2 $$\cos\frac{4\pi}{n} + \cos\frac{4\pi}{p} < \tfrac{1}{2},$$

in agreement with (**12**, p. 120, Fig. 3), where, however, the locus drawn is not

$$\cos\frac{4\pi}{n} + \cos\frac{4\pi}{p} = \tfrac{1}{2} \quad \text{but} \quad \cos\frac{2\pi}{n} + \cos\frac{2\pi}{p} = \tfrac{3}{2}.$$

One is tempted to conjecture that the new criterion 11.2 will remain valid when n and p are not both even. It is satisfied by all the known finite groups $G^{3,n,p}$, including

$$G^{3,9,9} \simeq LF(2,19), \quad G^{3,8,11} \simeq PGL(2,23), \quad G^{3,7,15} \simeq LF(2,29).$$

The only unknown group that should be finite, according to 11.2, is $G^{3,7,16}$, whose subgroup (2, 3, 7; 8), of index 2, is defined by

$$\mathrm{S}^3 = \mathrm{T}^2 = (\mathrm{ST})^7 = (\mathrm{S}^{-1}\mathrm{TST})^8 = \mathrm{E}.$$

Perhaps some enterprising reader will test this with an electronic computer.

12. Barnes's new extreme senary form. Finally, we will show how, when the complex 3-space is regarded as a real 6-space, the vertices of the honeycomb $(1\ 1\ 2_2{}^4)^4$ yield the lattice representing Barnes's new quadratic form.

As a basis for the lattice 10.2, where $c = \frac{1}{2}(-1 + i\sqrt{7})$, we may use the six complex vectors

$$\begin{aligned} \mathbf{t}_1 &= (1, \bar{c}, 1), & \mathbf{t}_2 &= (2, 0, 0),\\ \mathbf{t}_3 &= (1, 1, \bar{c}), & \mathbf{t}_4 &= (0, 2, 0),\\ \mathbf{t}_5 &= (\bar{c}, 1, 1), & \mathbf{t}_6 &= (0, 0, 2) \end{aligned}$$

(cf. **18**, p. 397), which combine to yield

$$\begin{aligned} (c, \ \ c, 0) &= -\mathbf{t}_1 - \mathbf{t}_5 + \mathbf{t}_6,\\ (c, -c, 0) &= \mathbf{t}_1 - \mathbf{t}_2 + \mathbf{t}_4 - \mathbf{t}_5, \end{aligned}$$

and so on. Thus the general vector of the lattice is

$$\begin{aligned} \mathbf{z} &= \sum u_j \mathbf{t}_j\\ &= (u_1 + 2u_2 + u_3 + \bar{c}u_5, \ \ \bar{c}u_1 + u_3 + 2u_4 + u_5, \ \ u_1 + \bar{c}u_3 + u_5 + 2u_6), \end{aligned}$$

where the u's are real integers; and its norm is

$$\begin{aligned} \mathbf{z}\bar{\mathbf{z}} &= (u_1 + 2u_2 + u_3 + \bar{c}u_5)(u_1 + 2u_2 + u_3 + cu_5)\\ &\quad + (\bar{c}u_1 + u_3 + 2u_4 + u_5)(cu_1 + u_3 + 2u_4 + u_5)\\ &\quad + (u_1 + \bar{c}u_3 + u_5 + 2u_6)(u_1 + cu_3 + u_5 + 2u_6)\\ &= 4(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2\\ &\quad + u_1u_2 + u_2u_3 + u_3u_4 + u_4u_5 + u_5u_6 + u_6u_1)\\ &\qquad - 2(u_1u_4 + u_2u_5 + u_3u_6)\\ &= (2u_1 + u_2 - \tfrac{1}{2}u_4 + u_6)^2 + (2u_3 + u_4 - \tfrac{1}{2}u_6 + u_2)^2\\ &\quad + (2u_5 + u_6 - \tfrac{1}{2}u_2 + u_4)^2 + \tfrac{7}{4}(u_2^2 + u_4^2 + u_6^2). \end{aligned} \tag{12.1}$$

Since this definite senary form represents 4 and has determinant 7^3, one naturally looks to see whether it is equivalent to the new form $4\phi_6$ discovered by Barnes **(4)**. To establish this equivalence, we note that Barnes represented his form by a real six-dimensional lattice consisting of the points in Euclidean 7-space whose coordinates

$$y_0, y_1, y_2, y_3, y_4, y_5, y_6$$

are integers satisfying

$$\sum_0^6 y_j = 0, \quad \sum_0^6 j y_j \equiv 0 \pmod{7}.$$

In terms of unit vectors $\mathbf{e}_j$ along the seven orthogonal axes, he observed that the minimal vectors are

$$\mathbf{e}_a + \mathbf{e}_b - \mathbf{e}_c - \mathbf{e}_d,$$

with distinct suffixes satisfying

$$a + b \equiv c + d \pmod{7}$$

(5, §7**)**, and that this set of 42 vectors $\sum y_j \mathbf{e}_j$ is transformed into itself by the permutations

$$y_j \to y_{j+1} \quad \text{and} \quad y_j \to y_{3j},$$

where the suffixes are reduced modulo 7. The latter permutation (of period 6), applied repeatedly to the vector

$$\mathbf{t}_1 = -\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_5 - \mathbf{e}_6,$$

yields

$$\begin{aligned}
\mathbf{t}_2 &= -\mathbf{e}_0 + \mathbf{e}_3 + \mathbf{e}_1 - \mathbf{e}_4,\\
\mathbf{t}_3 &= -\mathbf{e}_0 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_5,\\
\mathbf{t}_4 &= -\mathbf{e}_0 + \mathbf{e}_6 + \mathbf{e}_2 - \mathbf{e}_1,\\
\mathbf{t}_5 &= -\mathbf{e}_0 + \mathbf{e}_4 + \mathbf{e}_6 - \mathbf{e}_3,\\
\mathbf{t}_6 &= -\mathbf{e}_0 + \mathbf{e}_5 + \mathbf{e}_4 - \mathbf{e}_2.
\end{aligned}$$

The six $\mathbf{t}$'s, being independent, constitute a basis for the lattice, and enable us to construct the form

$$\begin{aligned}
\left(\sum_1^6 u_j \mathbf{t}_j\right)^2 = \{&-\sum_1^6 u_j \mathbf{e}_0 + (u_1 + u_2 - u_4)\mathbf{e}_1 + (u_3 + u_4 - u_6)\mathbf{e}_2\\
&+ (u_2 + u_3 - u_5)\mathbf{e}_3 + (u_5 + u_6 - u_2)\mathbf{e}_4\\
&+ (u_6 + u_1 - u_3)\mathbf{e}_5 + (u_4 + u_5 - u_1)\mathbf{e}_6\}^2\\
= \left(\sum_1^6 u_j\right)^2 &+ \sum_1^6 (u_j + u_{j+1} - u_{j+3})^2\\
= 4\sum_1^6 u_j^2 &+ 4\sum_1^6 u_j u_{j+1} - 2\sum_1^3 u_k u_{k+3},
\end{aligned}$$

which is 12.1.

We observe that this form (equivalent to Barnes's has the elegant matrix

$$\begin{pmatrix} 4 & 2 & 0 & -1 & 0 & 2 \\ 2 & 4 & 2 & 0 & -1 & 0 \\ 0 & 2 & 4 & 2 & 0 & -1 \\ -1 & 0 & 2 & 4 & 2 & 0 \\ 0 & -1 & 0 & 2 & 4 & 2 \\ 2 & 0 & -1 & 0 & 2 & 4 \end{pmatrix}$$

Barnes (**4**, p. 240) has shown that its group of automorphs is

$$\mathfrak{C}_2 \times PGL(2, 7).$$

In view of §11, we now recognize this as the symmetry group $G^{3,8,8}$ of the complex polyhedron $(1\ 1\ 1_1{}^4)^4$.

TABLE I

THE COMPUTATION OF c FOR $[1\ 1\ 1^l]^m$ (see 3.5)

l	m	$2\cos\frac{\pi}{l}$	$2\cos\frac{\pi}{m}$	$2\cos\frac{\pi}{s}$	c
3	m	1	$2\cos\pi/m$	$2\cos 2\pi/m$	$e^{2\pi i/m}$
l	3	$2\cos\pi/l$	1	$-2\cos\pi/l$	$-1-e^{-2\pi i/l}$
4	4	$\sqrt{2}$	$\sqrt{2}$	$-1/\sqrt{2}$	$(-1+i\sqrt{7})/2$
4	5	$\sqrt{2}$	τ	$-\tau^{-2}/\sqrt{2}$	$-1-\tau\bar{\omega}$
5	4	τ	$\sqrt{2}$	-1	$\tau\omega$
4	6	$\sqrt{2}$	$\sqrt{3}$	0	$i\sqrt{2}$
6	4	$\sqrt{3}$	$\sqrt{2}$	$-2/\sqrt{3}$	$-1+i\sqrt{2}$

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TABLE II

FINITE NON-REAL GROUPS GENERATED BY n INVOLUTORY REFLECTIONS

Group	Number of dimensions n	$\det(2a_{jk})$	Exponents $m_1, m_2, \ldots, m_n$	Number of reflections $\Sigma\, m_j$	Order of group $\Pi(m_j+1)$	Order of centre k	$\frac{h}{k}$	Symbol of Shephard (29)	Symbol of Todd (31)
$[1\ 1\ r^l]^3$ $[1\ 1 \ldots 1]^l$	$r+2$	$4\sin^2\frac{\pi}{l}$	$r+1, l-1, 2l-1, \ldots, (r+1)l-1$	$l\binom{n}{2}$	$l^{n-1}n!$	(l, n)	$\frac{l(n-1)}{(l, n)}$	$[1\ 1;\ r]^l$	$G(l, l, n)$
$[1\ 1\ 1^4]^4$	3	1	3, 5, 13	21	336	2	7	—	No. 24
$[1\ 1\ 1^5]^4$ $[1\ 1\ 1^4]^5$	3	τ^{-2}	5, 11, 29	45	2160	6	5	—	No. 27
$[2\ 1\ 1^4]^3$ $[2\ 1\ 1]^4$	4	1	3, 7, 11, 29	40	$64\cdot 5!$	4	5	$[2\ 1;\ 1]^4$	No. 29
$[2\ 1\ 2]^3$	5	2	3, 5, 9, 11, 17	45	$72\cdot 6!$	2	9	$[2\ 1;\ 2]^3$	No. 33
$[2\ 1\ 3]^3$	6	1	5, 11, 17, 23, 29, 41	126	$108\cdot 9!$	6	7	$[2\ 1;\ 3]^3$	No. 34

TABLE III

SOME COMPLEX POLYTOPES

Symbol defined in §10	Number of dimensions	Number of vertices	Symbol of Shephard **(29)** or of Coxeter **(11; 12)**	Reference
$(1\ 1\ r_r{}^l)^3$	$n = r + 2$	ln	$(1\ 1;\ r_r)^l = \beta^l{}_n$	**(29**, p. 377**)**
$(1_1\ 1\ r^l)^3$	$n = r + 2$	l^{n-1}	$(1_1\ 1;\ r)^l = \frac{1}{l}\ \gamma^l{}_n$	**(29**, p. 378**)**
$(1_1 1 \ldots 1)^l$	n	l^{n-1}	—	p. 251
$(1\ 1\ 1_1{}^4)^4$	3	42	$\{3, 8\|, 4\} = \{3, 8\}_8$	p. 265
$(1_1\ 1\ 1^4)^4$	3	56	—	p. 266
$(1\ 1\ 1_1{}^5)^4$	3	216	$\{3, 10\|, 4\} = \{3, 10\}_8$	**(11**, p. 61**)**
$(1\ 1\ 1_1{}^4)^5$	3	270	$\{3, 8\|, 5\} = \{3, 8\}_{10}$	**(11**, p. 61**)**
$(1_1\ 1\ 1^5)^4$	3	360	—	—
$(1_1\ 1\ 1^4)^5$	3	360	—	—
$(2_2\ 1\ 1^4)^3$	4	80	$(2_2\ 1;\ 1)^4 = (\frac{1}{4}\gamma^4{}_3)^{+1}$	**(29**, p. 382**)**
$(2_2\ 1\ 1)^4$	4	80	—	—
$(2\ 1\ 1_1{}^4)^3$	4	160	$(2\ 1;\ 1_1)^4$	p. 261
$(2\ 1\ 1_1)^4$	4	320	—	—
$(2\ 1\ 2_2)^3$	5	80	$(2\ 1;\ 2_2)^3 = (\frac{1}{3}\ \gamma^3{}_4)^{+1}$	**(29**, p. 381**)**
$(2\ 1_1\ 2)^3$	5	432	$(2\ 1_1;\ 2)^3$	**(29**, p. 381**)**
$(2\ 1\ 3_3)^3$	6	756	$(2\ 1;\ 3_3)^3 = (\frac{1}{3}\ \gamma^3{}_4)^{+2}$	**(18**, p. 391**)**
$(2_2\ 1\ 3)^3$	6	4032	$(2_2\ 1;\ 3)^3 = (\frac{1}{3}\ \gamma^3{}_5)^{+1}$	**(18**, p. 391**)**
$(2\ 1_1\ 3)^3$	6	54432	$(2\ 1_1;\ 3)^3$	—

REFERENCES

1. G. Bagnera, *I gruppi finiti di trasformazioni lineari dello spazio che contengono omologie*, Rend. Circ. Mat. Palermo, *19* (1905), 1–56.
2. H. F. Baker, *A Locus with* 25920 *Linear Self Transformations* (Cambridge, 1946).
3. W. Barlow, *Crystal symmetry*: *The actual basis of the thirty-two classes*, Phil. Mag. (6), *1* (1901), 1–36.
4. E. S. Barnes, *The perfect and extreme senary forms*, Can. J. Math., *9* (1957), 235–242.
5. ———, *The complete enumeration of extreme senary forms*, Phil. Trans. Royal Soc., London, A *249* (1957), 461–506.
6. W. Burnside, *Theory of Groups of Finite Order* (2nd ed., Cambridge 1911).
7. H. S. M. Coxeter, *The polytopes with regular-prismatic vertex figures*, Phil. Trans. Royal Soc. London, A *229* (1931), 329–425.
8. ———, *Discrete groups generated by reflections*, Ann. Math., *35* (1934), 588–621.
9. ———, *Wythoff's construction for uniform polytopes*, Proc. London Math. Soc. (2), *38* (1935), 327–339.
10. ———, *The abstract groups* $R^m = S^m = (R^jS^j)^{p_j} = 1$, *etc.*, Proc. London Math. Soc. (2), *41* (1936), 278–301.
11. ———, *Regular skew polyhedra in three and four dimensions, and their topological analogues*, Proc. London Math. Soc. (2), *43* (1937), 33–62.
12. ———, *The groups* $G^{m,n,p}$, Trans. Amer. Math. Soc., *45* (1939), 73–150.
13. ———, *A method for proving certain abstract groups to be infinite*, Bull. Amer. Math. Soc., *46* (1940), 246–251.
14. ———, *Regular Polytopes* (London, 1948; New York, 1949).

15. ———, *Self-dual configurations and regular graphs*, Bull. Amer. Math. Soc., *56* (1950), 413–455.
16. ———, *The product of the generators of a finite group generated by reflections*, Duke Math. J., *18* (1951), 765–782.
17. ———, *Extreme forms*, Can. J. Math., *3* (1951), 391–441.
18. H. S. M. Coxeter and J. A. Todd, *An extreme duodenary form*, Can. J. Math., *5* (1953), 384–392.
19. L. E. Dickson, *Linear Groups, with an Exposition of the Galois Field Theory* (Leipzig, 1901).
20. J. Dieudonné, *La géométrie des groupes classiques*, Ergeb. Math. (Neue Folge), *5* (1955).
20a. W. L. Edge, *The Kummer quartic and the tetrahedroid based on the Maschke forms*, Proc. Cambridge Philos. Soc., *45* (1949), 519–535.
21. W. E. Edington, *Abstract group definitions and applications*, Trans. Amer. Math. Soc., *25* (1923), 193–210.
22. F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, *3* (Bologna, 1924).
23. C. M. Hamill, *On a finite group of order* 6,531,840, Proc. London Math. Soc. (2), *52* (1951), 401–454.
24. R. W. H. T. Hudson, *Kummer's Quartic Surface* (Cambridge, 1905).
25. W. Killing, *Die Zusammensetzung der stetigen endlichen Transformationsgruppen*, II, Math. Ann. *33* (1888), 1–48.
26. F. Klein, *Ueber die Transformation siebenter Ordnung der elliptischen Functionen*, Math. Ann., *14* (1879), 428–471.
26a. ———, *Lectures on the Icosahedron* (London, 1913).
27. H. Maschke, *Ueber die quaternäre, endliche, lineare Substitutionsgruppe der Borchardt'schen Moduln*, Math. Ann., *30* (1887), 496–515.
28. G. C. Shephard, *Regular complex polytopes*, Proc. London Math. Soc. (3), *2* (1952), 82–97.
29. ———, *Unitary groups generated by reflections*, Can. J. Math. *5* (1953), 364–383.
30. ———, *Some problems on finite reflection groups*, L'Enseignement Math. (2), *2* (1956), 42–48.
31. G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Can. J. Math., *6* (1954), 274–304.
32. W. Threlfall, *Gruppenbilder*, Abh. Sächs. Akad. Wiss. Math.-phys. Kl., *41* (1932), 1–59.
33. J. A. Todd, *On the simple group of order* 25920, Proc. Royal Soc. London, A *189* (1947), 326–358.
34. ———, *The invariants of a finite collineation group in* 5 *dimensions*, Proc. Cambridge Philos. Soc. *46* (1950), 73–90.
35. H. Valentiner, *Théorie des groupes de transformations finis*, K. danske vidensk. selsk. (6), *5* (1889), 205–235.
36. O. Veblen and J. W. Young, *Projective Geometry*, *2* (Boston, 1918).
37. A. Wiman, *Ueber eine einfache Gruppe von* 360 *ebenen Collineationen*, Math. Ann. *47* (1896), 531–556.

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