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ON A PROBLEM OF P. HALL FOR ENGEL WORDS II

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Abstract

The present paper is related to some recent studies in Abdollahi and Russo ['On a problem of P. Hall for Engel words', *Arch. Math. (Basel)* **97** (2011), 407–412] and Fernández-Alcober *et al.* ['A note on conciseness of Engel words', *Comm. Algebra* **40** (2012), 2570–2576] on the position of the n-Engel marginal subgroup $E^*_n(G)$ of a group G, when $n = 3, 4$. Describing the size of $E^*_n(G)$ for $n = 3, 4$, we show some generalisations of classical results on the partial margins of $E^*_3(G)$ and $E^*_4(G)$.

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1. Statement of the main results

Given a group G, two elements $x, y \in G$ and an integer $n \geq 1$, the *left n-Engel commutator word* of G is $e_n(x, y) = [x,_n y]$, the *verbal subgroup*, determined by $e_n(x, y)$, is $E_n(G) = \langle e_n(x, y) \mid x, y \in G\rangle$ and the *marginal subgroup*, determined by $e_n(x, y)$, is $E^*_n(G) = \{a \in G \mid e_n(x, y) = e_n(ax, y) = e_n(x, ay)\ \forall\, x, y \in G\}$. Terminology and notation are standard and follow [1–4, 8, 10, 11, 14–17, 21]. Since we concentrate mostly on $n = 3, 4$, it may be useful to introduce

$$E_3(G) = \langle [x, y, y, y] \mid \forall x, y \in G\rangle, \quad E_4(G) = \langle [x, y, y, y, y] \mid \forall x, y \in G\rangle,$$

$$E^*_3(G) = \{a \in G \mid [x, y, y, y] = [ax, y, y, y] = [x, ay, ay, ay]\ \forall x, y \in G\},$$

$$E^*_4(G) = \{a \in G \mid [x, y, y, y, y] = [ax, y, y, y, y] = [x, ay, ay, ay, ay]\ \forall x, y \in G\}.$$

These are always characteristic subgroups of G (see [12–14, 21]) and dual in the sense of [19, Theorems 1.1 and 1.2]. We also use the sets

$$R_3(G) = \{x \in G \mid [x, y, y, y] = 1\ \forall y \in G\},\ R_4(G) = \{x \in G \mid [x, y, y, y, y] = 1\ \forall y \in G\},$$

$$L_3(G) = \{x \in G \mid [y, x, x, x] = 1\ \forall y \in G\}, L_4(G) = \{x \in G \mid [y, x, x, x, x] = 1\ \forall y \in G\},$$

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called the *set of right 3-Engel (respectively, 4-Engel) elements* of G and the *set of left 3-Engel (respectively, 4-Engel) elements* of G. Note that $R_3(G), R_4(G), L_3(G), L_4(G)$ have been largely studied in [1–4, 7, 8, 10, 11, 14–16, 21] and are not necessarily subgroups. These references show interesting connections with respect to the usual terms of the upper central series $Z_i(G)$ of G and of the lower central series $\gamma_i(G)$ of G.

A form of duality between $E_2(G), E_2^*(G), E_3(G), E_3^*(G), E_4(G), E_4^*(G)$ appears in our first main theorem. This reflects some well-known conditions between $Z_i(G)$ and $\gamma_i(G)$. (Here $d(G)$ is the minimal number of generators of G.)

Theorem 1.1. *Let G be a group and h, k, r, s positive integers.*

(i) $\gamma_4(G) \leq E_2(G) \leq \gamma_3(G)$.
(ii) $Z_2(G) \leq E_2^*(G) \leq Z_3(G)$.
(iii) *If $r = d(G/E_3(G)) \geq 2$, then $\gamma_{2r}(G) \leq E_3(G) \leq \gamma_4(G)$.*
(iv) *If $s = d(E_3^*(G)) \geq 2$, then $Z_3(G) \leq E_3^*(G) \leq Z_{2s-1}(G)$.*
(v) *If $h = d(G/E_4(G)) \geq 2$, then $\gamma_{4h+1}(G) \leq E_4(G) \leq \gamma_5(G)$.*
(vi) *If $k = d(E_4^*(G)) \geq 2$, then $Z_4(G) \leq E_4^*(G) \leq Z_{4k}(G)$.*

Theorem 1.1 improves [12, Theorem 3], showing that it is possible to avoid the assumption of being metabelian. We note that the conclusions of Theorem 1.1 allow us to extend the converse of Schur's theorem that a finitely generated group G with finite $G/Z(G)$ has finite G'. Generalisations can be found in [5] and [19, Theorems 4.2, 4.3, 4.5 and 4.6].

The second main theorem describes the exact position of the set $T_3(G) = R_3(G) \cap L_3(G) = \{a \in G \mid [a, x, x, x] = [x, a, a, a] = 1, \ \forall x \in G\}$ with respect to $E_n^*(G)$ for $n = 2, 3$ and the Fitting subgroup $\mathrm{Fit}(G)$. Note that $T_3(G)$ is not necessarily a subgroup of G, but is so in the following situation.

Theorem 1.2. *Let G be a group with $d(E_3^*(G)) = 2$. If $T_3(G)$ contains only elements of finite odd order, then there exists a series of G*

$$1 \lhd Z(G) \lhd Z_2(G) \lhd E_2^*(G) \lhd Z_3(G) \lhd T_3(G) \lhd \mathrm{Fit}(G) \lhd G$$

whose terms are characteristic subgroups such that:

(i) *$Z(G)$, $Z_2(G)/Z(G)$, $E_2^*(G)/Z_2(G)$, $Z_3(G)/E_2^*(G)$ are Abelian;*
(ii) *$E_3^*(G)/Z_3(G)$ is trivial, that is, $E_3^*(G) = Z_3(G)$;*
(iii) *$\langle tE_3^*(G)\rangle^{G/E_3^*(G)}$ is nilpotent of class at most three for all $tE_3^*(G) \in T_3(G)/E_3^*(G)$;*
(iv) *$\mathrm{Fit}(G)/T_3(G)$ is a Fitting group without 3-Engel elements;*
(v) *$G/\mathrm{Fit}(G)$ has no normal nilpotent subgroups.*

We recall some notions from [12, 13, 19, 21] to formulate our last main theorem. The *first partial margin of* $E_3(G)$ is the set $A(G) = \{a \in G \mid [x, y, y, y] = [ax, y, y, y] \ \forall x, y \in G\}$ and the *first partial margin of* $E_4(G)$ is the set $B(G) = \{a \in G \mid [x, y, y, y, y] = [ax, y, y, y, y] \ \forall x, y \in G\}$. Both are characteristic subgroups of G and their properties are described in [1, 3, 12, 13, 19, 21].

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Theorem 1.3. *Let G be a group.*

(i) $E_3^*(G) \le A(G) = \{a \in G \mid [a,x]^{y^{-1}}\ [a,x]^{y^x} \in Z(G)\ \forall x,y \in G\}$. *Moreover,* b^G *is nilpotent of class at most three for all* $b \in E_3^*(G)$.

(ii) $E_4^*(G) \le B(G) = \{a \in G \mid [[a,x],x^{y^{-1}}]^{y^x} \in Z(G)\ \forall x,y \in G\}$. *Moreover, if G has no element of order divisible by* 2, 3, 5 *and all the 3-generated subgroups* $\langle u,v,x\rangle$ *with* $u,v \in R_4(G)$ *and* $x \in G$ *are nilpotent, then* c^G *is nilpotent of class at most seven for all* $c \in E_4^*(G)$.

2. Proofs of the main theorems

We begin by proving our first main theorem.

Proof of Theorem 1.1. (i) The upper inclusion follows easily from the definitions of $E_2(G)$ and $\gamma_3(G) = [[G,G],G]$. If $xE_2(G), yE_2(G) \in G/E_2(G)$, then

$$[xE_2(G), yE_2(G), yE_2(G)] = [x,y,y]E_2(G) = E_2(G),$$

because $E_2(G)$ contains all elements $[x,y,y]$ of G. Then $G/E_2(G)$ is a 2-Engel group and there are some classical results which show that $G/E_2(G)$ is nilpotent of class at most three. Thus, $\gamma_4(G) \le E_2(G)$.

(ii) The lower inclusion follows from [19, Theorem 2.3(a)] and the upper inclusion from [19, Corollary 2.8].

(iii) The upper inclusion follows clearly from the definitions. Then we proceed to prove the lower inclusion. An argument as in (i) above can be applied. If $xE_3(G), yE_3(G) \in G/E_3(G)$, then

$$[xE_3(G), yE_3(G), yE_3(G), yE_3(G)] = [x,y,y,y]E_3(G) = E_3(G).$$

Now [8] shows that an r-generated 4-Engel group is nilpotent of class at most $2r-1$. Therefore, $\gamma_{2r}(G) \le E_3(G)$.

(iv) Since $Z_3(G) = \{g \in G \mid [g,x,y,z] = 1\ \forall x,y,z \in G\}$, the lower inclusion follows from the definitions. On the other hand, again [8] allows us to conclude that $E_3^*(G) \le Z_{2s-1}(G)$.

(v) The upper bound follows from the definitions. For the lower bound, the arguments of (i) and (iii) imply that $G/E_4(G)$ is a 4-Engel group. From [10, 22], we know that $G/E_4(G)$ is nilpotent of class at most $4h$. Then $\gamma_{4h+1}(G) \le E_4(G)$.

(vi) The lower bound follows from the definitions. Applying again the results in [10, 22], we may argue as in (iv) and find the required upper bound. □

To understand the importance of Theorem 1.1, we need to recall some results on the topic. A classical problem, which does not occur for right 2-Engel words, appears in fact for left 2-Engel words and for left or right Engel words of length at least three. While $R_2(G)$ is a characteristic subgroup of G (see [19, Lemma 2.2(a)] or [13, Theorem 2.1 and (2.2.1)]), $R_3(G), R_4(G), L_2(G), L_3(G), L_4(G)$ are not subgroups in general, as Nickel noted in [16] and [2, Proposition 3.2.5]. The situation is clearer in the metabelian case, thanks to the following result.

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Theorem 2.1 (See [12, Corollary at page 1968]). *The sets $R_3(G)$ and $R_4(G)$ are subgroups, provided G is a metabelian group.*

Another remarkable case is illustrated by the following result.

Theorem 2.2 (See [4, Theorems 2.2 and 3.1]). *Let G be a group.*

(i) *If $\gamma_5(G)$ has no element of order two, then $R_3(G)$ is a subgroup of G.*
(ii) *If no element of G is of order divisible by $2, 3, 5$ and $\langle a, b, x\rangle$ is nilpotent for all $a, b \in R_4(G)$ and $x \in G$, then $R_4(G)$ is a subgroup of G.*

Unfortunately, the situation is more complicated for $L_3(G)$ and $L_4(G)$ (see [1–3, 12]), where several open questions remain. Even with the best hypothesis that $R_3(G), R_4(G), L_2(G), L_3(G), L_4(G)$ are subgroups of G, not enough is known on their position in the subgroup lattice of G. Among the main results of [19] there are in fact a series of connections among $\gamma_i(G)$, $Z_i(G)$, $R_2(G)$, $L_2(G)$, $E_2(G)$, $E_2^*(G)$ and one might expect that generalisations would be true for Engel words of length at least three. Again the metabelian case gives evidence.

Theorem 2.3 (See [12, Theorem 3]). *Let G be a metabelian group, p a prime and $n \geq 2$ an integer.*

(i) *If $n + 1$ is not a prime and G contains no elements of order $p \leq n - 1$, or if $n + 1$ is prime and G contains no elements of order $p \leq n + 1$, then $Z_n(G) = E_n^*(G)$.*
(ii) *If $q = n + 1$ is prime and G contains no elements of order $p \leq n - 1$, then $Z_{q-1}(G) \leq E_{q-1}^*(G) \leq Z_q(G)$, and there exists a metabelian q-group K with $Z_{q-1}(K)$ being a proper subgroup of $E_{q-1}^*(K)$.*
(iii) *If G contains elements of order $p \leq n - 1$, then no integer $k \geq 1$ can be found such that $E_n^*(G) \leq Z_k(G)$.*

If we specialise Theorem 2.3 in our context, we have the following statement.

Corollary 2.4. *Let G be a metabelian group.*

(i) *If G contains no elements of order two, then $Z_3(G) = E_3^*(G)$.*
(ii) *If G contains no elements of order two, three or five, then $Z_4(G) = E_4^*(G)$.*
(iii) *$Z_2(G) \leq E_2^*(G) \leq Z_3(G)$.*
(iv) *If G contains no elements of order two or three, then $Z_4(G) \leq E_4^*(G) \leq Z_5(G)$.*
(v) *If G contains elements of order two, then no integer $k \geq 1$ can be found such that $E_3^*(G) \leq Z_k(G)$.*
(vi) *If G contains elements of order two or three, then no integer $k \geq 1$ can be found such that $E_4^*(G) \leq Z_k(G)$.*

Then it is clear that Theorem 1.1 improves Corollary 2.4 and [19, Theorem 2.3 and Corollary 2.8], showing that the assumption of being metabelian is not necessary for $n = 2, 3, 4$.

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Remark 2.5. Given a group G and an arbitrary word θ in n variables on G, it is possible to introduce the verbal subgroup $\theta(G)$ of G and the marginal subgroup $\theta^*(G)$ of G, determined by θ (see [17, 18]). Therefore, $E_n(G)$ and $E_n^*(G)$ turn out to be specialisations for $e_n(x, y)$ (see [9, 17, 18]). The following question:

'If $|G : \theta^*(G)|$ is finite, then is $\theta(G)$ also finite?'

is problematic and may have a positive or a negative answer. Originally, it is attributed to P. Hall and known as *The First Hall's Problem for Words* (see [3, 7, 17]). Schur's theorem is a particular case for the commutator word $e_1(x, y) = [x, y]$. See [5, 6] for a positive answer to this question for the word $e_n(x, y)$ (with $1 \leq n \leq 4$).

Before proving Theorem 1.2, some observations are useful. For instance, we may use the definitions and check that $A(G) \leq \langle R_3(G)\rangle$ is always true, but the following example shows that $\langle R_3(G)\rangle \nleq A(G)$.

Example 2.6. Let G be the largest group generated by three elements a, b, c such that $a, b, c \in A(G)$ and H is the largest group generated by three elements which are right 3-Engel elements in H; if $A(G) = \langle R_3(G)\rangle$ were true, then we should have the torsion subgroups $T(G)$ and $T(H)$ of equal orders (actually G must be isomorphic to H). Note that both G and H are 3-generated nilpotent groups of class five of derived length three. The computation by GAP [20] shows that $|T(G)| = 128000$ but $|T(H)| = 8192000$, where $T(G)$ and $T(H)$ are torsion subgroups of G and H, respectively. Below there are details concerning the program we wrote.

```
LoadPackage("nq");
f:=FreeGroup(3);
x:=f.1;y:=f.2;a:=f.3;b:=f.4;c:=f.5;
g:=f/[LeftNormedComm([x,y,y,y]) * LeftNormedComm([a*x,y,y,y])^-1,
LeftNormedComm([x,y,y,y]) * LeftNormedComm([b*x,y,y,y])^-1],
LeftNormedComm([x,y,y,y]) * LeftNormedComm([c*x,y,y,y])^-1]);
G:=NilpotentQuotient(g,[x,y]); h:=f/[LeftNormedComm([a,y,y,y]),
LeftNormedComm([b,y,y,y]),LeftNormedComm([c,y,y,y]));
H:=NilpotentQuotient(h,[x,y]);
TG:=TorsionSubgroup(G);
TH:=TorsionSubgroup(H);
Size(TG);
Size(TH)
```

The following observation helps us to understand when $T_3(G)$ is a group.

Lemma 2.7 (See [11, Hauptsatz 3]). *Let G be a group. Suppose that one of the following conditions is satisfied.*

(i) *G has no elements of order two.*
(ii) *$T_3(G)$ contains only elements of finite odd order.*

Then $T_3(G)$ is a subgroup of G.

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Some remarks of a historical nature may be appropriate here.

REMARK 2.8. Historically, Heineken [11, page 682] introduced $A = \{a \in G \mid g^{(3)} \circ a = g^{(3)} \circ a^{-1} = 1, \forall g \in G\}$ in the context of 3-Engel groups, where $g \circ a = [g, a]$, $g^{(2)} \circ a = g \circ (g \circ a) = [g, [g, a]]$, $g^{(3)} \circ a = g \circ (g \circ (g \circ a)) = [g, [g, [g, a]]]$ and so on. It was noted in [11, page 682] that $g^{(3)} \circ a = 1$ for all $g \in G$ if and only if $a \in R_3(G)$ and that $g^{(3)} \circ a^{-1} = 1$ if and only if $a \in L_3(G)$. This means that $A = T_3(G)$.

The following two applications of Theorem 1.1 are recalled here because they will be used in the proof of our third main theorem.

COROLLARY 2.9. *Let G be a group.*

(i) *If $d(G/E_3(G)) = 2$, then $E_3(G)/\gamma_5(G)$ is Abelian.*
(ii) *If $d(E_3^*(G)) = 2$, then $E_3^*(G) = Z_3(G)$.*

There is another way to get to the conclusions of Corollary 2.9 because Heineken [11, Hauptsatz 1] proved that all 3-Engel groups are in fact nilpotent of class at most four, when no elements of orders two and five are present.

COROLLARY 2.10. *A group G without elements of orders 2 and 5 has $\gamma_5(G) \le E_3(G) \le \gamma_4(G)$ and $Z_3(G) \le E_3^*(G) \le Z_4(G)$. In particular, $E_3(G)/\gamma_5(G)$ and $E_3^*(G)/Z_3(G)$ are Abelian.*

Now we can prove the second main theorem.

PROOF OF THEOREM 1.2. From [2, 11–13], we know that $E_2^*(G)$ and $E_3^*(G)$ are characteristic subgroups of G. Of course, this is true also for $Z(G)$, $Z_2(G)$, $Z_3(G)$ and $\mathrm{Fit}(G)$. From Lemma 2.7, $T_3(G)$ is a subgroup of G and it is fixed under the action of inner automorphisms of G by results in [11–13].

Since $E_3^*(G) \le L_3(G)$ and $E_3^*(G) \le R_3(G)$, we have $E_3^*(G) \le L_3(G) \cap R_3(G) = T_3(G)$. Note that the result of Newell [15] implies $R_3(G) \le \mathrm{Fit}(G)$; therefore, $T_3(G) \le R_3(G) \le \mathrm{Fit}(G)$. Theorem 1.1(ii) shows that $Z_2(G) \le E_2^*(G) \le Z_3(G)$ and Theorem 1.1(iv) that $Z_3(G) \le E_3^*(G)$. We conclude that $Z_2(G) \triangleleft E_2^*(G) \triangleleft Z_3(G) \triangleleft E_3^*(G)$ is always true. Therefore, $1 \triangleleft Z(G) \triangleleft Z_2(G) \triangleleft E_2^*(G) \triangleleft Z_3(G) \triangleleft E_3^*(G) \triangleleft T_3(G) \triangleleft \mathrm{Fit}(G) \triangleleft G$ is a well-defined series with characteristic terms in G.

(i) Of course, $Z(G)$ and $Z_2(G)/Z(G)$ are Abelian, but $Z_3(G)/E_2^*(G)$ follows from Theorem 1.1(ii).

(ii) See Corollary 2.9.

(iii) The fact that $T_3(G) \le R_3(G)$ and a result of Newell [15] show that g^G is nilpotent of class at most three for all $g \in T_3(G)$. This property is invariant under quotients.

(iv) Clearly, $\mathrm{Fit}(G)/T_3(G)$ has no 3-Engel elements and $\mathrm{Fit}(\mathrm{Fit}(G)/T_3(G)) = \mathrm{Fit}(G)T_3(G)/T_3(G) = \mathrm{Fit}(G)/T_3(G)$.

(v) This is straightforward from the maximality of $\mathrm{Fit}(G)$. □

Our last theorem is proved below. We note that it shows a condition of embedding of $E_3^*(G)$ and $E_4^*(G)$ in their own first margins.

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Proof of Theorem 1.3. (i) The inclusion $E_3^*(G) \leq A(G)$ is straightforward. Then we proceed to prove equality between $A(G)$ and the set

$$S = \{a \in G \mid [a, x]^{y^{-1}} [a, x]^{y^x} \in Z(G) \ \forall x, y \in G\}.$$

Replacing the role of x with that of y, the elements of $A(G)$ may be characterised:

$$\begin{aligned}
[y, x, x, x] = [ay, x, x, x] &\Leftrightarrow [[ay, x, x], x] = [[y, x, x], x] \Leftrightarrow [ay, x, x]\,[y, x, x]^{-1} \in C_G(x) \\
&\Leftrightarrow [ay, x, x]\,[x, [y, x]] \in C_G(x) \Leftrightarrow [[a, x]^y\,[y, x], x]\,[x, [y, x]] \in C_G(x) \\
&\Leftrightarrow \Big(([a, x]^y\,[y, x])^{-1} x^{-1} [a, x]^y [y, x] x\Big)\Big(x^{-1}[y, x]^{-1} x [y, x]\Big) \in C_G(x) \\
&\Leftrightarrow [y, x]^{-1}\Big([a, x]^{y^{-1}}\, x^{-1} [a, x]^y\, x\Big)[y, x] \in C_G(x) \\
&\Leftrightarrow \Big([a, x]^{y^{-1}}\, x^{-1}[a, x]^y\, x\Big)^{[y,x]} \in C_G(x) \Leftrightarrow \Big([a, x]^{y^{-1}}\,[a, x]^{y^x}\Big)^{[y,x]} \in C_G(x) \\
&\Leftrightarrow [y, x]^{-1}\Big([a, x]^{y^{-1}}\,[a, x]^{y^x}\Big)[y, x]x = x[y, x]^{-1}\Big([a, x]^{y^{-1}}\,[a, x]^{y^x}\Big)[y, x] \\
&\Leftrightarrow \Big([a, x]^{y^{-1}}\,[a, x]^{y^x}\Big)[y, x]x[y, x]^{-1} = [y, x]x[y, x]^{-1}\Big([a, x]^{y^{-1}}\,[a, x]^{y^x}\Big) \\
&\Leftrightarrow \Big([a, x]^{y^{-1}}\,[a, x]^{y^x}\Big) \in C_G([y, x]x[y, x]^{-1}) = C_G(x^{[y,x]^{-1}}) = C_G(x^{[x,y]}) \leq C_G(x).
\end{aligned}$$

We deduce that

$$A(G) = \{a \in G \mid [a, x]^{y^{-1}}\,[a, x]^{y^x} \in C_G(x) \ \forall x, y \in G\},$$

but, for all $x, y \in G$,

$$[a, x]^{y^{-1}}[a, x]^{y^x} \in C_G(x) \Leftrightarrow [a, x]^{y^{-1}}[a, x]^{y^x} \in \bigcap_{x \in G} C_G(x) = Z(G),$$

so that the result follows.

Finally, a result of Newell [15] allows us to conclude that b^G is nilpotent of class at most three for each $b \in E_3^*(G) \leq A(G) \leq R_3(G)$.

(ii) Arguing as in step (i), $E_4^*(G) \leq B(G)$ and we begin by proving equality between $B(G)$ and the set $T = \{a \in G \mid [[a, x], x^{y^{-1}}]^{y^x} \in Z(G) \ \forall x, y \in G\}$. We have

$$\begin{aligned}
[y, x, x, x, x] = [ay, x, x, x, x] &\Leftrightarrow [[y, x, x, x], x] = [[ay, x, x, x], x] \\
&\Leftrightarrow [ay, x, x, x]\,[y, x, x, x]^{-1} \in C_G(x) \Leftrightarrow [ay, x, x, x]\,[x, [y, x, x]] \in C_G(x) \\
&\Leftrightarrow [[[a, x]^y\,[y, x], x], x]\,[x, [y, x, x]] \in C_G(x) \\
&\Leftrightarrow [[a, x]^y, x]^{[y,x]}\,[y, x, x, x]\,[y, x, x, x]^{-1} \in C_G(x) \Leftrightarrow [[a, x]^y, x]^{[y,x]} \in C_G(x) \\
&\Leftrightarrow [[a, x]^y, x]^{y^{-1}y^x} = [[a, x], x^{y^{-1}}]^{y^x} \in C_G(x).
\end{aligned}$$

Taking this over all $x \in G$, $T = B(G)$.

From [4, Corollary 3.2], which gives more details about the situation described in Theorem 2.2(ii), we know that an element $w \in R_4(H)$ has w^H which is nilpotent of class at most seven, provided H is a $\{2, 3, 5\}'$-group such that all the 3-generated subgroups of the form $\langle u, v, h\rangle$ are nilpotent, where $u, v \in R_4(H)$ and $h \in H$. Using this, [4, Corollary 3.2] shows that $R_4(H)$ is a subgroup of H. Since $E_4^*(G) \leq B(G) \leq R_4(G)$, the result follows. □

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3. Applications and consequences

We may compare Theorems 1.1 and 2.1 and Corollary 2.4 and look for examples for which the inclusions of Theorem 1.1 are satisfied properly. This may be hard to do.

COROLLARY 3.1. *Let G be a group.*

(i) *If G contains involutions and $E_3^*(G) \le Z_{2s}(G)$ for some $s = d(E_3^*(G)) \ge 2$, then G should have derived length at least three.*
(ii) *If G contains elements of order two or three and $E_4^*(G) \le Z_{4k}(G)$ for some $k = d(E_4^*(G)) \ge 2$, then G should have derived length at least three.*

PROOF. (i) follows from Theorem 1.1(iv) and Corollary 2.4(v). (ii) follows from Theorem 1.1(v) and Corollary 2.4(vi). □

Another interesting consideration is the following corollary.

COROLLARY 3.2. *Let G be a finitely generated nilpotent group of class at most four with $d(E_3^*(G)) = 2$. If $G/E_3^*(G)$ is finite, then $E_3(G) \simeq \mathbb{Z}_{2^\alpha} \times \mathbb{Z}_{5^\beta}$ for $\alpha = 0, 1, 2$ and $\beta = 0, 1$.*

PROOF. From results in [5, 6], we may conclude that $E_3^*(G)$ is finitely generated and it is meaningful to consider $d(E_3^*(G))$; furthermore, $E_3(G)$ is finite. We have $\gamma_5(G) = 1$ and $E_3(G)/\gamma_5(G) = E_3(G)$ is Abelian by Corollary 2.9(i). A celebrated result of Gupta and Newman [8] shows that the fifth term of the lower central series of a finitely generated 3-Engel group has exponent dividing 20. In our case $G/E_3(G)$ is a finitely generated 3-Engel group, so

$$\exp\left(\gamma_5\left(\frac{G}{E_3(G)}\right)\right) = \exp\left(\frac{\gamma_5(G)E_3(G)}{\gamma_5(G)}\right) = \exp\left(\frac{E_3(G)}{\gamma_5(G)}\right) = \exp(E_3(G))$$

divides 20 and the result follows. □

The following example is instructive.

EXAMPLE 3.3. (i) Let $D = \langle a, b \mid a^2 = 1, a^{-1}ba = b^{-1}\rangle = \langle a\rangle \ltimes \langle b\rangle = \mathbb{Z}_2 \ltimes \mathbb{Z}$ be the infinite dihedral group. It is easy to see that $1 = Z_1(D) = Z_2(D) = Z_3(D) = \cdots$, that $D = \gamma_1(D) \ge \gamma_2(D) = D' = \langle b\rangle = \gamma_3(D) = \gamma_4(D) = \cdots$, that $E_3(D) = D'$ and that $E_3^*(D) = 1$.

(ii) Consider the group $G = \mathbb{Z} \times A_5$, where A_5 is the alternating group over five elements. Here $Z(G) = Z(\mathbb{Z}) \times Z(A_5) \simeq \mathbb{Z}$ and we have a nontrivial centre. Now $E_3^*(G) = E_3^*(\mathbb{Z}) \times E_3^*(A_5) = \mathbb{Z} \times 1 \simeq \mathbb{Z}$ is infinite, $G/E_3^*(G) \simeq A_5$ is finite and $E_3(G) = E_3(\mathbb{Z}) \times E_3(A_5) = 1 \times A_5 \simeq A_5$ is finite. A similar conclusion holds if we replace $E_3^*(G)$ with $E_4^*(G)$ and $E_3(G)$ with $E_4(G)$, respectively.

Finally, we note two consequences of Theorem 1.3.

COROLLARY 3.4. *Let G be a metabelian group. Then*

$$E_3^*(G) \le A(G) = \langle R_3(G)\rangle = \{a \in G \mid [a, x]^{y^{-1}}\ [a, x]^{y^x} \in Z(G)\ \forall x, y \in G\}$$

and b^G is nilpotent of class at most three for all $b \in E_3^*(G)$.

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PROOF. A metabelian group G has $A(G) = \langle R_3(G)\rangle$ by [12, Theorem 1]. Then the result follows from Theorem 1.3(i). □

COROLLARY 3.5. *Let G be a metabelian group. Then*

$$E_4^*(G) \le B(G) = \langle R_4(G)\rangle = \{a \in G \mid [[a, x], x^{y^{-1}}]^{y^x} \in Z(G)\ \forall x, y \in G\}.$$

PROOF. A metabelian group G has $B(G) = \langle R_4(G)\rangle$ by [12, Theorem 1]. Then the result follows from Theorem 1.3(ii). □

We end with two observations.

REMARK 3.6. Let G be a group without elements of order two. Looking at Corollary 2.10, we get to the same conclusions in Theorem 1.2 if the assumption $d(E_3^*(G)) = 2$ is replaced by the absence of elements of order five in G. On the other hand, if G is a group in which all the elements of $T_3(G)$ have finite odd order different from five, then Corollary 2.10 continues to be true and we can again omit $d(E_3^*(G)) = 2$.

We do not know a corresponding result for margins of 4-Engel groups.

REMARK 3.7. Even if it is natural to expect a result of the type of Theorem 1.2 for $E_4^*(G)$, there are some difficulties for the commutator calculus and [3, Question 1.2] which prevent us from a variation on the themes which we have used.

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