J. Aust. Math. Soc. **89** (2010), 165–179
doi:10.1017/S1446788710001576

NONVANISHING OF JACOBI POINCARÉ SERIES

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(Received 4 October 2009; accepted 8 November 2010)

Communicated by F. Calegari

Abstract

We prove that, under suitable conditions, a Jacobi Poincaré series of exponential type of integer weight and matrix index does not vanish identically. For the classical Jacobi forms, we construct a basis consisting of the 'first' few Poincaré series, and also give conditions, both dependent on and independent of the weight, that ensure the nonvanishing of a classical Jacobi Poincaré series. We also obtain a result on the nonvanishing of a Jacobi Poincaré series when an odd prime divides the index.

2010 *Mathematics subject classification*: primary 11F50; secondary 11F30.
Keywords and phrases: Jacobi forms, Poincaré series, Kloosterman sums.

1. Introduction

Rankin [9] proved that the mth Poincaré series P_m^k of weight k for the full modular group $\mathrm{SL}(2, \mathbb{Z})$ is not identically zero for sufficiently large positive integers k and finitely many positive integers m depending on k. Mozzochi extended Rankin's result to integral weight modular forms for congruence subgroups in [7].

In this paper we prove similar results for higher-degree Jacobi Poincaré series defined on the full Jacobi group $\Gamma_g^J = \mathrm{SL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}^g \times \mathbb{Z}^g)$, where g is a positive integer called the degree of the Jacobi group. The Jacobi group operates on $\mathcal{H} \times \mathbb{C}^g$ and also on functions $\phi : \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$, where $\mathcal{H}$ denotes the upper half plane. We write $|_{k,m}$ for the action on functions. See Section 2 for the definitions.

Let $\mathrm{PSym}(g, \frac{1}{2}\mathbb{Z})$ be the set of symmetric, positive definite, half integral $(g \times g)$ matrices. We define $A[B] := B^t AB$ for matrices A and B of appropriate sizes, where B^t is the transpose of the matrix B, and $e(z) := e^{2\pi i z}$. More generally, we will use the standard notation $e_a(z) := e^{2\pi i z/a}$.

Let $k, g \in \mathbb{Z}$, and $m \in \mathrm{PSym}(g, \frac{1}{2}\mathbb{Z})$. The vector space of Jacobi cusp forms of weight k, index m and degree g, denoted by $J_{k,m,g}^{\mathrm{cusp}}$, is defined to be the space of all holomorphic functions $\phi : \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ that satisfy $\phi|_{k,m}\gamma = \phi$, for all $\gamma \in \Gamma_g^J$, and have a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}^+, r\in\mathbb{Z}^g \\ 4n>m^{-1}[r^t]}} c_\phi(n, r)e(n\tau + rz).$$

We denote $J_{k,m,1}^{\mathrm{cusp}}$ by $J_{k,m}^{\mathrm{cusp}}$.

For $n \in \mathbb{Z}^+$, $r \in \mathbb{Z}^g$ and $m \in \mathrm{PSym}(g, \frac{1}{2}\mathbb{Z})$ such that $4n > m^{-1}[r^t]$, let $P_{n,r}^{k,m}$ be the (n, r)th Poincaré series of weight k and index m (of exponential type) defined when $k > g + 2$, as in [1] (see Section 2 for the definition). It is well known that the Poincaré series $P_{n,r}^{k,m}$, where $n \in \mathbb{Z}$ and $r \in \mathbb{Z}^g$, span $J_{k,m,g}^{\mathrm{cusp}}$. It is then natural to ask whether such Poincaré series vanish identically or not. We prove the following theorem, which gives a partial answer to this question.

Let $k' := k - g/2 - 1$ and define

$$D := \det\begin{pmatrix} 2n & r \\ r^t & 2m \end{pmatrix}.$$

THEOREM 1.1. *Suppose that $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$. Then there exist an integer k_0 that depends only on g, and a constant $B > 3\log 2$ such that for all even $k \geq k_0$, the Jacobi Poincaré series $P_{n,r}^{k,m}$ does not vanish identically when*

$$k' \leq \frac{\pi D}{\det(2m)} \leq k'^{1+\alpha(g)} \exp\left(-\frac{B \log k'}{\log\log k'}\right),$$

where

$$\alpha(g) = \begin{cases} \dfrac{2}{3(g+2)} & \text{if } 1 \leq g \leq 4, \\ \dfrac{2}{3g} & \text{if } g \geq 5. \end{cases}$$

We construct a basis of $J_{k,m}^{\mathrm{cusp}}$ consisting of the 'first' $\dim(J_{k,m}^{\mathrm{cusp}})$ Poincaré series (see Theorem 6.1 in Section 6). We also give conditions for the nonvanishing of Poincaré series independent of the weight for the classical Jacobi forms, where $g = 1$.

As in [9], define

$$M(x) := \exp\left(\frac{B_1 \log x}{\log\log 2x}\right) \quad \forall x \geq 2,$$

where B_1 is a constant as in [9] ($B_1 > \log 2$).

THEOREM 1.2. *Suppose that $g = 1$ and $\pi D > 2m$. Then $P_{D,r}^{k,m} \not\equiv 0$ provided that*

$$M\left(\frac{\pi D}{m}\right)\sigma_0(D)D < \frac{m^{8/7}}{2^{9/4}\pi}\left(\frac{2}{6^{2/3}} + \frac{54}{2^{5/6}} + \frac{16}{2^{3/4}}\right)^{-3/2},$$

where $\sigma_0(D) := \sum_{d|D} 1$.

https://doi.org/10.1017/S1446788710001576 Published online by Cambridge University Press

Finally, following [9], we give conditional statements on the nonvanishing of Jacobi Poincaré series, based on the relation between g-dimensional Kloosterman sums and the corresponding one-dimensional sums and identities involving these.

THEOREM 1.3. *Suppose that $\mu \in \mathbb{Z}^+$ and that p is an odd prime such that $p \mid (m, r)$ but $p \nmid n$. If $P^{k,p^\mu m}_{p^\mu n, p^\mu r} \not\equiv 0$, then either $P^{k,mp^{\mu-1}}_{np^{\mu-1},rp^{\mu-1}} \not\equiv 0$ or both $P^{k,p^{2\mu}m}_{np^{2\mu},rp^{2\mu}} \not\equiv 0$ and $P^{k,p^{2\mu}m}_{n,rp^\mu} \not\equiv 0$.*

(Here $p \mid m$ means that p divides every entry of m; this makes sense since $2m$ is a $(g \times g)$ matrix with integer entries and p is odd.)

REMARK 1.4. In Section 3, we first prove that the Poincaré series $P^{k,m}_{n,r}$ does not vanish when

$$\frac{\pi D}{\det(2m)} = 2\pi\left(n - \frac{1}{4}m^{-1}[r^t]\right) \leq Ck',$$

giving the constant C explicitly, and pointing out for which k this is valid. This follows from Proposition 3.1 for arbitrary g and also from Theorem 6.1 in the case where $g = 1$ (recall that $\dim J^{\text{cusp}}_{k,m,1} \leq C(\frac{1}{12}k(m+1))$, where the constant C may be taken to be 1 when k is large enough).

REMARK 1.5. Theorem 1.1 improves the trivial case mentioned in the previous remark. However, achieving the 'order of $k^{2-\epsilon}$ when $\epsilon > 0$' as in [9] in the case of Jacobi Poincaré series using Rankin's methods seems difficult, mainly because of the presence of the factor $\gcd(c, D)$ instead of $\gcd(c, D)^{1/2}$ in the estimate of Kloosterman sums of degree g (even for small g); see Section 3.

REMARK 1.6. The condition that k be even when $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$ in Theorem 1.1 is necessary, as the (n, r)th Poincaré series vanish when k is odd and $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$. The restriction $k' \leq \pi D/\det(2m)$ in Theorem 1.1 is natural, since we know the result in the complement (see Proposition 3.1). The same is true for the condition $\pi D > 2m$ in Theorem 1.2.

2. Notation and preliminaries

The Jacobi group Γ^J_g operates on $\mathcal{H} \times \mathbb{C}^g$ in the usual way by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu)\right) \circ (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-1}(z + \lambda\tau + \mu)\right).$$

Let $k \in \mathbb{Z}$ and $m \in \mathrm{PSym}(g, \frac{1}{2}\mathbb{Z})$. Then the action of Γ^J_g on functions $\phi : \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ is given by

$$\begin{aligned}\phi|_{k,m}\gamma(\tau, z) := (c\tau + d)^{-k} e(&-c(c\tau + d)^{-1} m[z + \lambda\tau + \mu] \\ &+ m[\lambda]\tau + 2\lambda^t m z)\phi(\gamma \circ (\tau, z)).\end{aligned}$$

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The vector space of Jacobi cusp forms of weight k, index m and degree g, denoted by $J^{\mathrm{cusp}}_{k,m,g}$, is defined to be the space of all holomorphic functions $\phi : \mathcal{H} \times \mathbb{C}^g \to \mathbb{C}$ that satisfy $\phi|_{k,m}\gamma = \phi$, for all $\gamma \in \Gamma^J_g$, and have a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}^+, r\in\mathbb{Z}^g\\ 4n>m^{-1}[r^t]}} c_\phi(n, r)e(n\tau + rz).$$

For $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}^g$ such that $4n > m^{-1}[r^t]$, let $P^{k,m}_{n,r}$ be the (n, r)th Poincaré series of weight k and index m (of exponential type) defined when $k > g + 2$ by

$$P^{k,m}_{n,r}(\tau, z) := \sum_{\gamma\in\Gamma^J_{g,\infty}\backslash\Gamma^J_g} e(n\tau + rz)|_{k,m}\gamma(\tau, z) \quad \forall\tau \in \mathcal{H}, z \in \mathbb{C}^g,$$

where

$$\Gamma^J_{g,\infty} := \left\{ \left(\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu) \right) \,\middle|\, n \in \mathbb{Z}, \mu \in \mathbb{Z}^g \right\}.$$

It is well known that $J^{\mathrm{cusp}}_{k,m,g}$ is finite-dimensional and the family of Poincaré series $P^{k,m}_{n,r}$, where $n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}^g$, generate the space $J^{\mathrm{cusp}}_{k,m,g}$. In [1, Lemma 1], Böcherer and Kohnen found the following Fourier expansion of $P^{k,m}_{n,r}$.

Proposition 2.1. *The function $P^{k,m}_{n,r}$ is in $J^{\mathrm{cusp}}_{k,m,g}$. The Fourier expansion of the Poincaré series is given by*

$$P^{k,m}_{n,r}(\tau, z) = \sum_{\substack{n'\in\mathbb{Z}^+, r'\in\mathbb{Z}^g\\ 4n'>m^{-1}[r'^t]}} c^{k,m}_{n,r}(n', r')e(n'\tau + r'z)$$

and

$$\begin{aligned} c^{k,m}_{n,r}(n', r') = \delta_m(n, r, n', r') &+ (-1)^k\delta_m(n, r, n', -r') \\ &+ 2\pi i^k \det(2m)^{-1/2}(D'/D)^{k'/2} \sum_{c\geq1}(H_{m,c}(n, r, n', r') \\ &+ (-1)^k H_{m,c}(n, r, n', -r'))J_{k'}\left(\frac{2\pi\sqrt{DD'}}{c\det(2m)}\right), \end{aligned}$$

where

$$D' := \det\begin{pmatrix} 2n' & r' \\ r'^t & 2m \end{pmatrix},$$

$$\delta_m(n, r, n', r') := \begin{cases} 1 & \textit{if } D = D' \textit{ and } r \equiv r' \bmod \mathbb{Z}^g \cdot 2m, \\ 0 & \textit{otherwise}, \end{cases}$$

$$H_{m,c}(n, r, n', r') := c^{-g/2-1} \sum_{x,y} e_c((m[x] + rx + n)\bar{y} + n'y + r'x)e_{2c}(r'm^{-1}r^t).$$

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In the last sum, x and y run over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ and $(\mathbb{Z}/c\mathbb{Z})^*$ respectively and $\bar{y}$ denotes an inverse of y modulo c, while $e_c(z) := e^{2\pi i z/c}$ and J_ν denotes the Bessel function of order ν.

Further, $\langle \phi, P_{n,r}^{k,m} \rangle = \lambda_{k,m,D} c_\phi(n, r)$, where $\langle \cdot, \cdot \rangle$ is the Petersson inner product on $J_{k,m,g}^{\mathrm{cusp}}$, while $c_\phi(n, r)$ denotes the (n, r)th Fourier coefficient of ϕ and

$$\lambda_{k,m,D} := \frac{2^{g(k'-1)}\Gamma(k')\det(m)^{k'-1/2}}{(2\pi D)^{k'}}.$$

From Proposition 2.1, we conclude that the Poincaré series $P_{n,r}^{k,m}$ is nonzero if and only if its (n, r)th Fourier coefficient $c_{n,r}^{k,m}$ is positive. So it is enough to prove that $c_{n,r}^{k,m}$ is nonzero.

LEMMA 2.2. *The Poincaré series $P_{n,r}^{k,m}$ vanishes if k is odd and $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$.*

PROOF. Suppose that $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$. Then $m^{-1}r^t \in \mathbb{Z}^g$, and we may consider the group element

$$\gamma_{m,r} := \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, (m^{-1}r^t, 0) \right) \in \Gamma_g^J.$$

We note that

$$e(n\tau + rz)|_{k,m}\gamma_{m,r} = (-1)^k e(n\tau + rz).$$

It follows that $P_{n,r}^{k,m} = (-1)^k P_{n,r}^{k,m}$, and the proof of the lemma is complete. □

We also note that the (n, r)th coefficient $c(n, r)$ of a general Jacobi form of degree g is zero if k is odd when $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$. This is an easy consequence of the transformation property of Jacobi forms. See, for instance, [4] for the case where $g = 1$. For the rest of this paper, we suppose that k is even when $2r \equiv 0 \bmod \mathbb{Z}^g \cdot 2m$.

Now $2m$ is a positive definite matrix with integer entries, and so $\det(2m) \geq 1$, so we deduce from the Fourier expansion of $P_{n,r}^{k,m}$ that, in order to prove that $P_{n,r}^{k,m}$ is nonzero, it is enough to prove that $|S(n, r)| < (2\pi)^{-1}$, where

$$\begin{aligned} S(n, r) := \det(2m)^{-1/2} \sum_{c \geq 1} (&H_{m,c}(n, r, n, r) \\ &+ (-1)^k H_{m,c}(n, r, n, -r)) J_{k'}\left(\frac{2\pi D}{c \det(2m)}\right). \end{aligned} \tag{2.1}$$

We will need the following estimates:

$$|J_\nu(x)| \leq \min\left\{1, \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu\right\} \quad \forall x > 0,\ \nu \geq 2, \tag{2.2}$$

$$|H_{m,c}(n, r, n, \pm r)| \leq 2^{\omega(c)} c^{g/2-1} \gcd(D, c), \tag{2.3}$$

where $\omega(c)$ is the number of distinct prime divisors of c. See [1, 12, 13] respectively for the details.

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3. Some simple bounds

In this section, we will first establish the following proposition and lemma, using trivial estimates of Bessel functions.

PROPOSITION 3.1. *There exists an integer k_0 such that the (n, r)th Poincaré series $P_{n,r}^{k,m}$ does not vanish identically whenever $k \geq k_0$, $(n, r) \in \mathbb{Z}^+ \times \mathbb{Z}^g$ and $e\pi D \leq k' \det(2m)$. If $k > g + 3$, then one may take $k_0 = \max(g + 4, \lfloor \frac{1}{2} g \rfloor + 69)$.*

The Poincaré series $P_{n,r}^{k,m}$ does not vanish identically whenever $\pi D < \det(2m)$ and either $k > g + 3$ if $g \geq 2$ or $k > 5$ if $g = 1$.

LEMMA 3.2. *When $g = 1$, for all positive integers n no greater than $(m + 3)/36$, there exists r such that $r^2 < 4mn$ and $\pi D < \det(2m)$. So the condition $\pi D < \det(2m)$ in the second part of Proposition 3.1 is nontrivial.*

PROOF. Suppose that $D < 2m/\pi < 2m/3$. Then $2m(2n - 1/3) < r^2 < 4mn$. Note that there is an integral square in the interval $[x, y]$ when $2\sqrt{x} + 1 < y - x$, so we need

$$2(2m(2n - \tfrac{1}{3}))^{1/2} + 1 < \tfrac{2}{3} m,$$

that is,

$$n < \frac{(2m + 3)^2}{144m} = \frac{m + 3}{36} + \frac{1}{16m},$$

which, since n is an integer, is equivalent to requiring that

$$n \leq \frac{m + 3}{36}.$$

Thus, in the case where $g = 1$, the Poincaré series $P_{k,m}^{n,r}$ does not vanish identically when $k > 4$ and $n \leq (m + 3)/36$. □

PROOF OF PROPOSITION 3.1. Let $S := \pi D/\det(2m)$. In a straightforward manner, using estimates (2.2) and (2.3), we get

$$|S(n, r)| \leq \frac{2S^{k'}}{\Gamma(k' + 1)} \sum_{c \geq 1} \frac{2^{\omega(c)}}{c^{k-g-1}}. \tag{3.1}$$

Recall Stirling's formula,

$$n! = \sqrt{2\pi n}(n/e)^n e^{\lambda_n},$$

where $(12n + 1)^{-1} < \lambda_n < (12n)^{-1}$ for all $n \in \mathbb{Z}^+$. We let $\xi := \sum_{c \geq 1} 2^{\omega(c)}/c^{k-g-1}$. Since $\Gamma(x)$ and $(x/e)^x$ are increasing functions of x in the intervals $[\frac{3}{2}, \infty)$ and $(1, \infty)$ respectively and $k' \geq \frac{5}{2}$, the hypothesis of the theorem gives

$$|S(n, r)| \leq \frac{2\xi}{\Gamma(\lfloor k' \rfloor + 1)} \left(\frac{\lfloor k' \rfloor + 1}{e} \right)^{\lfloor k' \rfloor + 1} < \frac{2\xi}{\sqrt{2\pi(\lfloor k' \rfloor + 1)}}.$$

Therefore by choosing k large enough, we get the first assertion of Proposition 3.1. For the second assertion, when $k \geq g+4$, note that $\xi < 1/\pi^2$, which follows from the trivial estimate $2^{\omega(c)} \leq c$.

For the second part of Proposition 3.1, we refer to equation (3.1). In this case, $S < 1$. We need to have $\Gamma(k'+1) > \frac{2}{3}\pi^2$, since $\xi < \zeta(2)$ from the conditions on k. The result follows by noting that $k' \geq 5$. □

4. Poincaré series for small weights

When $\mathrm{Re}(s) > \frac{1}{2}(1-k')$, the Jacobi Poincaré series is defined using the 'Hecke trick' as in [2] by

$$P^{k,m}_{n,r;s}(\tau, z) := \sum_{\gamma \in \Gamma^J_{g,\infty}\backslash\Gamma^J_g} \left(\frac{v}{|c\tau+d|^2}\right)^s e(n\tau + rz)|_{k,m}\gamma(\tau, z)$$

for all $\tau = u + iv \in \mathcal{H}$, all $z \in \mathbb{C}^g$, and all $s \in \mathbb{C}$. If $k > \frac{1}{2}g + 2$, then $P^{k,m}_{n,r;0} \in J^{\mathrm{cusp}}_{k,m}$ and has the same Fourier properties as $P^{k,m}_{n,r}$. We also consider conditions on its nonvanishing in the following proposition.

PROPOSITION 4.1. *There exists an integer $c(m)$ such that the Poincaré series $P^{k,m}_{n,r;0}$ does not vanish identically when $k \geq \max\{c(m), \frac{1}{2}(g+7)\}$ while $(n, r) \in \mathbb{Z}^+ \times \mathbb{Z}^g$ and $e\pi D \leq k' \det(2m)$.*

PROOF. This proposition follows from the arguments of the proof of Proposition 3.1. Here we use the following estimate for Kloosterman sums of degree g (see [1, pp. 508 and 512]):

$$H_{m,c}(n, r, n, \pm r) \leq 2^{\omega(c)} c^{-1/2} \gcd(D, c) \quad \forall c \geq C(m),$$

where $C(m)$ is a constant. As in the proof of Proposition 3.1, put $S := \pi D/\det 2m$, and recall the definition (2.1) of $S(n, r)$. Then for some positive constant $C_1(m)$,

$$\begin{aligned}
|S(n,r)| &\leq \sum_{1\leq c\leq C(m)} \frac{2^{\omega(c)+1} c^{g/2-1} \gcd(D, c)}{\Gamma(k'+1)} \left(\frac{S}{c}\right)^{k'} \\
&\quad + \sum_{c > C(m)} \frac{2^{\omega(c)+1} c^{-1/2} \gcd(D, c)}{\Gamma(k'+1)} \left(\frac{S}{c}\right)^{k'} \\
&\leq C(m)^{(g-1)/2} \sum_{1\leq c\leq C(m)} \frac{2^{\omega(c)+1} c^{-1/2} \gcd(D, c)}{\Gamma(k'+1)} \left(\frac{S}{c}\right)^{k'} \\
&\quad + \sum_{c > C(m)} \frac{2^{\omega(c)+1} c^{-1/2} \gcd(D, c)}{\Gamma(k'+1)} \left(\frac{S}{c}\right)^{k'} \\
&\leq \frac{2C_1(m) S^{k'}}{\Gamma(k'+1)} \sum_{c} \frac{2^{\omega(c)}}{c^{k'-1/2}}.
\end{aligned}$$

The condition $k > (g+7)/2$ precisely guarantees convergence of the series above. The rest of the proof is identical to that of Proposition 3.1. □

5. Proof of Theorem 1.1

We now come to the main result of this paper. For convenience of notation we will drop (k, m) in the calculations.

PROOF OF THEOREM 1.1. We use Rankin's method, as in [9]. With $S(n, r)$ as in (2.1), we need to prove that $|S(n, r)| < 1/2\pi$. Define

$$\sigma := k'^{-1/6}, \quad Q^* := \frac{2\pi D}{k' \det(2m)}, \quad M(D) := \exp\left(\frac{B_1 \log D}{\log\log 2D}\right),$$

and

$$H^{\pm}_{m,c}(n, r, n, r) := H_{m,c}(n, r, n, r) + (-1)^k H_{m,c}(n, r, n, -r).$$

Then

$$|S(n, r)| \le \det(2m)^{-1/2}|S_1(n, r)| + \det(2m)^{-1/2}|S_2(n, r)|,$$

where

$$|S_1(n, r)| := \sum_{1\le c\le Q^*} |H^{\pm}_{m,c}(n, r, n, r)| \left|J_{k'}\left(\frac{k'Q^*}{c}\right)\right|,$$
$$|S_2(n, r)| := \sum_{c>Q^*} |H^{\pm}_{m,c}(n, r, n, r)| \left|J_{k'}\left(\frac{k'Q^*}{c}\right)\right|.$$

After calculations similar to [9] (see also [7]), we get

$$\begin{aligned} |S_1(n, r)| &\le A_1 M(D) Q^{*g/2-1} \sum_{d|D, d\le Q^*} 2^{\omega(d)}(Q^*\sigma^3 + 3d\sigma^2) \\ &\le A_2 M(D)^3 \frac{Q^{*g/2}}{k'^{1/2}} + A_3 M(D)^3 \frac{Q^{*g/2}}{k'^{1/3}} \\ &\le A_4 M(D)^3 \frac{(\pi D)^{g/2}}{\det(2m)^{g/2} k'^{g/2+1/2}} + A_5 M(D)^3 \frac{(\pi D)^{g/2}}{\det(2m)^{g/2} k'^{g/2+1/3}}. \end{aligned} \tag{5.1}$$

However, the sum $S_2(n, r)$ needs to be handled differently. We have

$$\begin{aligned} |S_2(n, r)| \le &\sum_{Q^*<c\le k'Q^*} 2^{\omega(c)+1} c^{g/2-1} \gcd(D, c) \left|J_{k'}\left(\frac{2\pi D}{c \det(2m)}\right)\right| \\ &+ \sum_{c>k'Q^*} 2^{\omega(c)+1} c^{g/2-1} \gcd(D, c) \left|J_{k'}\left(\frac{2\pi D}{c \det(2m)}\right)\right| \end{aligned}$$

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$$
\begin{aligned}
&\leq 2M(D)\frac{(\pi D)^{k'}}{\det(2m)^{k'}\Gamma(k'+1)}\sum_{Q^*<c\leq k'Q^*}\frac{1}{c^{k'-g/2}}\\
&\quad+2\sum_{c>k'Q^*}c^{g/2}\left|J_{k'}\left(\frac{2\pi D}{c\det(2m)}\right)\right|\\
&\leq\frac{2M(D)}{Q^{*k'-g/2-1-\epsilon}}\frac{(\pi D)^{k'}}{\det(2m)^{k'}\Gamma(k'+1)}\sum_{Q^*<c\leq k'Q^*}\frac{1}{c^{1+\epsilon}} \qquad (5.2)\\
&\quad+\frac{2(\pi D)^{g/2+1+\delta}}{\det(2m)^{k'}\Gamma(k'+1)}\sum_{c>k'Q^*}\frac{1}{c^{1+\delta}}\\
&\leq a_0\frac{M(D)}{k'^{g/2+3/2+\epsilon}}\left(\frac{\pi D}{\det(2m)}\right)^{g/2+1+\epsilon}+a_1\frac{(\pi D/\det(2m))^{g/2+1+\delta}}{k'^{k'+1/2}},
\end{aligned}
$$

where the a_i and A_j are constants, and $0<\epsilon,\delta<1$. Now let $\alpha(g)=2(3g+2)^{-1}$. For all $g\geq 1$, choose $0<\epsilon,\delta<1/2$. We find that $S_1(n,r)$ and $S_2(n,r)$ are small if k is chosen large enough. If $g\geq 5$, then we find that a better bound, $\alpha(g)=2(3g)^{-1}$, also works. This completes the proof. □

6. An explicit basis for $J_{k,m}^{\mathrm{cusp}}$ and the proof of Theorem 1.2

It is well known that the space of cusp forms S_k for $\mathrm{SL}(2,\mathbb{Z})$ has an explicit basis of the form $\{\Delta E_4^a E_6^b \mid 4a+6b=k-12\}$. Here Δ is the discriminant cusp form of weight 12, while E_4 and E_6 are the Eisenstein series of weight 4 and 6. Such an explicit result is not available for $J_{k,m}^{\mathrm{cusp}}$. Petersson proved that the set of Poincaré series $\{P_1^k,\ldots,P_{d_k}^k\}$ is a basis for S_k, where $d_k=\dim S_k$. We prove the corresponding result for Jacobi forms. The proof is based on the dimension formula in [4, p. 121].

THEOREM 6.1. *Let $k\geq m+12$, and define*

$$D_\mu := 4m\left(\left\lfloor\frac{\mu^2}{4m}\right\rfloor+1\right)-\mu^2.$$

Then $\{P_{D_\mu+4m\lambda_\mu,\mu}^{k,m}\}$ is the classical basis for $J_{k,m,1}^{\mathrm{cusp}}$, where the range of the indices is given as follows:

(1) *if k is even, $\mu=0,1,\ldots,m$; $\lambda_\mu=0,1,\ldots,\dim S_{k+2\mu}-\lfloor\mu^2(4m)^{-1}\rfloor-1$;*
(2) *if k is odd, $\mu=1,\ldots,m-1$; $\lambda_\mu=0,1,\ldots,\dim S_{k+2\mu-1}-\lfloor\mu^2(4m)^{-1}\rfloor‘-1$.*

PROOF. We prove the theorem for even k, the odd case being analogous. The condition that $k\geq m+12$ ensures that $\dim S_{k+2\mu}\geq\lfloor\mu^2(4m)^{-1}\rfloor+1$ (see [4, p. 103]). The proof follows Petersson's argument in the elliptic case (see [8, 10]). Let $d_{k,m}=$

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dim $J_{k,m}^{\mathrm{cusp}}$ and $\{\phi_1, \ldots, \phi_{d_{k,m}}\}$ be an orthonormal basis. We write

$$\phi_j(\tau, z) = \sum_{\substack{D'>0, r\in\mathbb{Z}\\ D'\equiv -r^2 \bmod 4m}} c_j(D', r) e\Big(\frac{D'+r^2}{4m}\tau + rz\Big)$$

and

$$P^{k,m}_{D_\mu+4m\lambda_\mu,\mu} = \lambda^{-1}_{k,m,D_\mu+4m\lambda_\mu} \sum_{j=1}^{d_{k,m}} c_j(D_\mu + 4m\lambda_\mu, \mu)\phi_j,$$

where μ and λ_μ vary as in the statement of the theorem. We get a $d_{k,m} \times d_{k,m}$ matrix indexed by pairs $(D_\mu + 4m\lambda_\mu, \lambda_\mu)$ and j. It suffices to prove that the matrix is invertible. If not, then there would be a linear relation

$$\sum_{j=1}^{d_{k,m}} \xi_j c_j(D_\mu + 4m\lambda_\mu, \mu) = 0$$

for all $(D_\mu + 4m\lambda_\mu, \mu)$, where $(\xi_1, \ldots, \xi_{d_{k,m}}) \neq (0, \ldots, 0)$.

Considering the nonzero Jacobi cusp form $\Phi := \sum_{j=1}^{d_{k,m}} \xi_j\phi_j$, we see that the Fourier coefficients $c_\Phi(D_\mu + 4m\lambda_\mu, \mu)$ of Φ are zero, where μ and λ_μ vary as in the theorem. This implies that $D_{2\mu}\Phi = 0$ when $\mu = 0, \ldots, m$ (see [4, p. 32] for the definition of the operators $D_{2\mu}$), which shows that $\Phi = 0$, a contradiction.

To see that $D_{2\mu}\Phi = 0$ when $\mu = 0, \ldots, m$, we recall the following Fourier expansion of the modular form $D_{2\nu}\Phi$ of weight $k + 2\nu$, where k is even and $\nu = 0, \ldots, m$ (see [4, p. 32]):

$$D_{2\nu}\Phi = A_{k,\nu} \sum_{n\geq 0} \sum_{\substack{r: r^2<4mn\\ \mu: 0\leq\mu\leq\nu}} \frac{(k+2\nu-\mu-2)!\,(-mn)^\mu r^{2\nu-2\mu}}{(k+2\nu-2)!\,\mu!\,(2\nu-2\mu)!} c_\Phi(n_\nu, r_\nu)q^n, \tag{6.1}$$

where $q := e(\tau)$ and

$$A_{k,\nu} := (2\pi i)^{-\nu}\frac{(k+2\nu-2)!\,(2\nu)!}{(k+\nu-2)!}.$$

Let ℓ be an even positive integer, and put $d_\ell := \dim S_\ell$. It is well known that an elliptic cusp form $\sum_{n=1}^{\infty} a(n, f)q^n$ of weight ℓ is determined by the first d_ℓ of its Fourier coefficients $a(1, f), \ldots, a(d_\ell, f)$. Therefore, looking at equation (6.1), we need to prove that

$$c_\Phi(n_\nu, r_\nu) = 0 \tag{6.2}$$

for all $\nu = 0, \ldots, m$, all r_ν such that $r_\nu^2 < 4mn_\nu$ and $0 \leq r_\nu \leq m$, and all n_ν such that $\lfloor r_\nu^2(4m)^{-1}\rfloor + 1 \leq n_\nu \leq d_{k+2\nu}$. Let ℓ denote $k + 2\nu$, where $\nu = 0, \ldots, m$, and for convenience, we drop the suffix ν in the terms n_ν and r_ν. To see (6.2), if $|r| > m$ in

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equation (6.1), we can find suitable integers x and $n' \geq 1$ such that

$$-m \leq r' = r - 2mx \leq m \quad \text{and} \quad 4mn' - r'^2 = 4mn - r^2. \tag{6.3}$$

We then use the facts that $c_\Phi(n', r') = c_\Phi(n, r)$ and $n \geq n' \geq 1$, implying that n' satisfies the same upper bound as n, namely, $\lfloor r'^2(4m)^{-1} \rfloor + 1 \leq n' \leq d_\ell$. We may finally reduce to the case where $0 \leq r \leq m$ since $c_\Phi(n, r) = c_\Phi(n, -r)$ as ℓ is even.

But any such n_ν may be written

$$n_\nu = \lfloor \nu^2(4m)^{-1} \rfloor + 1 + \lambda_\nu = \frac{D_\nu + \nu^2 + 4m\lambda_\nu}{4m},$$

where $0 \leq \nu \leq m$ and D_ν and λ_ν are as in the statement of the theorem. This completes the proof. □

The Eichler–Zagier map $Z_1 : J_{k,1} \to M^+_{k-1/2}$ for Jacobi forms of integral weight and index 1 ($M^+_{k-1/2}$ denotes the Kohnen + space for $\Gamma_0(4)$, as in [6]) is defined by

$$Z_1 : \sum_{\substack{D \in \mathbb{Z}^+, r \in \mathbb{Z} \\ D \equiv -r^2 \bmod 4}} c(D) e\Big(\frac{D + r^2}{4}\tau + rz\Big) \mapsto \sum_{D \in \mathbb{Z}^+} c(D) e(D\tau),$$

where the Fourier coefficient $c(D)$ does not depend on r.

Suppose that k is even and $(-1)^{k-1}D \equiv 0, 1 \bmod 4$. Following the notation in [6], let $P_{k-1,4,D}$ be the Dth Poincaré series in $M^+_{k-1/2}$. By comparing the Fourier developments of $P_{D,r}$ from [1] and of $P_{k-1,4,D}$ from [6], we get the following result.

PROPOSITION 6.2. *The Eichler–Zagier map Z_1 takes $P_{D,r} \in J^{\text{cusp}}_{k,1}$ to $3P_{k-1,4,D} \in M^+_{k-1/2}$.*

PROOF. The proof involves an easy calculation of Gauss sums, which may be found in [3]. □

PROPOSITION 6.3. *There exist positive constants k_0 and B, where $B > 4 \log 2$, such that, for all even $k \geq k_0$ and all positive integers $D \leq k^2 \exp(-B \log k / \log\log k)$, the Poincaré series $P_{k-1,4,D}$ and hence also the Poincaré series $P^{k,1}_{D,r}$ do not vanish identically.*

PROOF. From the Fourier expansion of $P_{k-1,4,D}$ given in [6], we see that the proof is the same as in the case of integral weight Poincaré series for congruence subgroups of $\mathrm{SL}(2, \mathbb{Z})$ given in [7]; we omit it. □

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PROOF OF THEOREM 1.2. We write $S(n,r) = S_1(n,r) + S_2(n,r)$, where

$$S_1(n,r) := i^k \pi \sqrt{2} m^{-1/2} \sum_{1 \le c \le \pi D/m} H^{\pm}_{m,c}(n,r,n,r) J_{k'}\left(\frac{\pi D}{mc}\right),$$
$$S_2(n,r) := i^k \pi \sqrt{2} m^{-1/2} \sum_{c > \pi D/m} H^{\pm}_{m,c}(n,r,n,r) J_{k'}\left(\frac{\pi D}{mc}\right).$$

To estimate $S_1(n,r)$, we use the following estimate of Bessel functions:

$$|J_\nu(r)| \le C r^{-1/3},$$

when $\nu \ge 0$ and $r \ge 1$; see [5, Lemma 3.4]. The constant C in this lemma may be computed to be the constant A in Theorem 1.2 of this paper using [11, p. 333]. Then

$$\begin{aligned} |S_1(n,r)| &\le \frac{2^{3/2}\pi}{m^{1/2}} \sum_{1 \le c \le \pi D/m} \frac{2^{\omega(c)} \gcd(D,c)}{c^{1/2}} \left| J_{k'}\left(\frac{\pi D}{mc}\right)\right| \\ &\le \frac{2^{3/2} m^{1/3} \pi^{2/3}}{D^{1/3} m^{1/2}} M(\pi D/m) \sum_{1 \le c \le \pi D/m} \frac{\gcd(D,c)}{c^{1/6}} \\ &\le \frac{2^{3/2}\pi^{2/3}}{D^{1/3} m^{1/6}} M(\pi D/m) \sum_{d \mid D, d < \pi D/m} d \\ &\le \frac{2^{3/2} D^{2/3} \pi^{5/3}}{m^{7/6}} M(\pi D/m) \sigma_0(D), \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} |S_2(n,r)| &\le \frac{2^{3/2}\pi}{m^{1/2}} \sum_{c > \pi D/m} c^{3/2} \left| J_{k'}\left(\frac{\pi D}{mc}\right)\right| \\ &\le \frac{2^{3/2}\pi}{\Gamma(k'+1) m^{1/2}} \sum_{c > \pi D/m} c^{3/2} \left(\frac{\pi D}{mc}\right)^{3/2+2} \\ &\le \frac{2^{3/2}\pi^{9/2} D^{7/2}}{\Gamma(k'+1) m^4} \sum_{c > \pi D/m} \frac{1}{c^2} \\ &\le \frac{2^{3/2}\pi^{13/2} D^{7/2}}{6\Gamma(k'+1) m^4}. \end{aligned} \tag{6.5}$$

From the bound given in Theorem 1.2, it follows from estimates (6.4) and (6.5) that S_1 and S_2 are both less than $\frac{1}{2}$ in absolute value. Finally, from the expression for the (n,r)th Fourier coefficient of $P^{k,m}_{n,r}$ given in Proposition 2.1, we get the theorem. □

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7. Further results

Recall the one-dimensional Kloosterman sum for a positive integer c,

$$S(r, m; c) = \sum_{\substack{1 \le h \le c \\ (h,c)=1}} e_c(rh + mh'), \tag{7.1}$$

where $hh' \equiv 1 \bmod c$. It is well known (see, for example, [9, Section 3]) that the following relation holds for a prime p:

$$S(rp^{\rho}, mp^{\mu}; cp) = S(r, mp^{\rho+\mu}; cp) + pS(rp^{\rho-1}, mp^{\mu-1}; c), \tag{7.2}$$

where $p \nmid rm$ and $\rho, \mu \ge 1$.

DEFINITION 7.1. Suppose that $n, n' \in \mathbb{Z}^+$ and $r, r' \in \mathbb{Z}^g$. We let

$$\begin{aligned} K_{m,c}(n, r, n', r') &= \sum_{x,y} e_c((m[x] + rx + n)\bar{y} + n'y + r'x) \\ &= c^{g/2+1} e_{2c}(-r'm^{-1}r^t) H_{m,c}(n, r, n', r'), \end{aligned}$$

where in the sum, x and y run over a complete set of representatives for $\mathbb{Z}^{(g,1)}/c\mathbb{Z}^{(g,1)}$ and $(\mathbb{Z}/c\mathbb{Z})^*$ respectively and $\bar{y}$ denotes an inverse of y modulo c.

LEMMA 7.2. *Let p be a odd prime such that $p \mid (m, r)$ and $p \nmid nn'$, and let $\mu' = \mu - 1$ and $\rho' = \rho - 1$. Then the following identity holds:*

$$\begin{aligned} K_{mp^{\mu},cp}(p^{\mu}n, p^{\mu}r, p^{\rho}n', r'p) = K_{mp^{\rho+\mu},cp}(p^{\rho+\mu}n, p^{\rho+\mu}r, n', r'p) \\ + p^2 K_{mp^{\mu'},c}(p^{\mu'}n, p^{\mu'}r, p^{\rho'}n', r'). \end{aligned} \tag{7.3}$$

PROOF. The proof follows by noting that

$$K_{m,cp}(n, r, n', r'p) = \sum_{x \bmod cp} e_c(r'x) S(n', m[x] + rx + n; cp), \tag{7.4}$$

from which the left-hand side and the first term on the right-hand side in (7.3) are taken care of by summing both sides of equation (7.2) with appropriate arguments over x modulo cp. For the last term, we split the summation in equation (7.4) as $x = cx_1 + x_2$, where x_1 and x_2 range over $\mathbb{Z}^g/p\mathbb{Z}^g$ and $\mathbb{Z}^g/c\mathbb{Z}^g$, respectively. After replacing $(m, n, r, n'; cp)$ by $(p^{\mu'}m, p^{\mu'}n, p^{\mu'}r, p^{\rho'}n'; c)$ and summing, we have

$$\begin{aligned} &\sum_{x \bmod c} e_{cp}(r'x) S(p^{\rho'}n', p^{\mu'}(m[x] + rx + n); c) \\ &\quad = \sum_{x_1,x_2} e_c(r'(cx_1 + x_2)) S(p^{\rho'}n', p^{\mu'}(m[cx_1 + x_2] + p^{\mu'}(r(cx_1 + x_2) + n)); c) \\ &\quad = \sum_{x_1} e(r'cx_1) \sum_{x_2} e_c(r', x_2) S(p^{\rho'}n', p^{\mu'}(m[x_2] + rx_2 + n); c) \\ &\quad = pK_{mp^{\mu'},c}(p^{\mu'}n, p^{\mu'}r, p^{\rho'}n', r'). \end{aligned}$$

Therefore the lemma follows from (7.2). □

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PROOF OF THEOREM 1.3. From Lemma 7.2, we easily deduce that under the conditions of the lemma,

$$\begin{aligned} H_{mp^{\mu},cp}&(p^{\mu}n,\ p^{\mu}r,\ p^{\rho}n',\ r'p) \\ &= H_{mp^{\rho+\mu},cp}(p^{\rho+\mu}n,\ p^{\rho+\mu}r,\ n',\ r'p) \\ &\quad + p^{-g/2+1}H_{mp^{\mu-1},c}(p^{\mu-1}n,\ p^{\mu-1}r,\ p^{\rho-1}n',\ r'). \end{aligned} \tag{7.5}$$

The following equality also follows from the definition, when $p \nmid c$:

$$H_{mp^{\mu},c}(p^{\mu}n,\ p^{\mu}r,\ p^{\rho}n',\ r'p) = H_{mp^{\rho+\mu},c}(p^{\rho+\mu}n,\ p^{\rho+\mu}r,\ n',\ r'p). \tag{7.6}$$

We sum equation (7.5) over all $c \geq 1$, and equation (7.6) over all $c \geq 1$ coprime to p, after multiplying them by the appropriate Bessel functions, and add them; see Proposition 2.1. Gathering all of the terms, putting $\rho = \mu$ and $n' = n$ and $r'p = p^{\mu}r$, in all three sums, we find positive constants α_1 and α_2 such that

$$c^{k,p^{\mu}m}(p^{\mu}n,\ p^{\mu}r) = \alpha_1 c^{k,p^{2\mu}m}(p^{2\mu}n,\ p^{2\mu}r;\ n,\ p^{\mu}r) + \alpha_2 c^{k,p^{\mu-1}m}(p^{\mu-1}n,\ p^{\mu-1}r)$$

(we have used the notation $c_{P^{k,m}_{n,r}}(n, r) := c^{k,m}(n, r; n, r) = c^{k,m}(n, r)$). This immediately implies the conclusion of Theorem 1.3. □

Acknowledgements

The author would like to thank the referee for his comments on the paper. The author also wishes to thank Professor B. Ramakrishnan for going through the manuscript and for his support and encouragement. This paper is a part of the author's PhD thesis at the Harish Chandra Research Institute, Allahabad, India.

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