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On the Notion of Visibility of Torsors

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Abstract. Let J be an abelian variety and A be an abelian subvariety of J, both defined over $\mathbf{Q}$. Let x be an element of $H^1(\mathbf{Q}, A)$. Then there are at least two definitions of x being visible in J: one asks that the torsor corresponding to x be isomorphic over $\mathbf{Q}$ to a subvariety of J, and the other asks that x be in the kernel of the natural map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. In this article, we clarify the relation between the two definitions.

1 Introduction and Definitions

Let J be an abelian variety and A be an abelian subvariety of J, both defined over $\mathbf{Q}$. The concept of visibility of torsors of A (*i.e.,* elements of $H^1(\mathbf{Q}, A)$) was introduced by Mazur [9] in the context where J is the Jacobian of a modular curve and A is an elliptic curve. He was interested in visualizing elements of the Shafarevich-Tate group of A, which is a subgroup of $H^1(\mathbf{Q}, A)$, as subvarieties in an ambient space (*i.e.,* describing them geometrically, as opposed to just algebraically). Apart from $\mathbf{P}^n$ for some n, the other natural choice for the ambient space is the abelian variety J, where A is already a subvariety. The theory that the notion of visibility led to has provided much computational and theoretical evidence for the second part of the Birch and Swinnerton-Dyer conjecture (see [2–5, 7, 8]).

Following Mazur's original motivation, we give the following definition.

Definition 1.1 An element of of $H^1(\mathbf{Q}, A)$ is said to be *visible as a variety* in J if it is isomorphic over $\mathbf{Q}$ to a subvariety of J.

In the applications of the notion of visibility to the Birch and Swinnerton-Dyer conjecture (*e.g.,* [7]), the following definition of visibility has been used, which has become the standard definition.

Definition 1.2 We say that an element of $H^1(\mathbf{Q}, A)$ is *visible* in J if it is in the kernel of the map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$ induced by the inclusion of A in J.

Note that Definition 1.2 is algebraic in nature, while Definition 1.1 is geometric. The first goal of this article is to relate these two definitions and thus give a geometric interpretation of visible elements (which also explains the use of the word "visible" in Definition 1.2 above). In order to do so, we introduce yet another notion of visibility.

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Definition 1.3 Let x be an element of $H^1(\mathbf{Q}, A)$ and let V denote the corresponding torsor. We say that x (or V) is *visible as a torsor* in J if there is a subvariety V' of J and an isomorphism of varieties $\iota\colon V \xrightarrow{\cong} V'$ which respects the action of A, where the action of A on V' is via the group law of J (note that this makes V' into an A-torsor).

We show in Proposition 2.1 that an element of $H^1(\mathbf{Q}, A)$ is visible in J if and only if it is visible as a torsor. It is clear that if an element of $H^1(\mathbf{Q}, A)$ is visible as a torsor in J, then it is visible as a variety in J; in particular, if it is visible, then it is visible as a variety. We do not know if the converse is true in general; however we do give some conditions under which the converse holds; see Proposition 3.1.

2 Visibility as a Torsor

The goal of this section is a proof of the following proposition.

Proposition 2.1 *Recall that J is an abelian variety and A is an abelian subvariety of J, both defined over* $\mathbf{Q}$. *Let V be an A-torsor. Then V is visible as a torsor in J if and only if it is visible in J* (i.e., *the cocycle class corresponding to V is in the kernel of the natural map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$*).

It is convenient to use the notion of sheaf torsors (see [10, § III.4]). If A is an abelian variety over $\mathbf{Q}$, let $\mathrm{ST}(A)$ denote the equivalence classes of sheaf torsors of A. If V is a sheaf torsor, pick $P \in V(\overline{\mathbf{Q}})$. Corresponding to P, we have a cocycle given by $\sigma \mapsto \sigma(P) - P \in A(\overline{\mathbf{Q}})$ for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. One can show that this gives an element of $H^1(\mathbf{Q}, A)$ that is independent of the choice of the point P above. Thus we get a canonical map $\mathrm{ST}(A) \to H^1(\mathbf{Q}, A)$. By Theorems 1.7, 3.9, 2.10, and 4.6 in Chapter III of [10], this map is an isomorphism.

In this section, the letter R will stand for a $\mathbf{Q}$-algebra of finite type. If V is an A-sheaf torsor, then recall that the *pushout* $V \times^A J$ is the sheaf whose sections over R are the set of orbits of $V(R) \times J(R)$ under the action of $A(R)$, where $A(R)$ acts on $V(R)$ in the usual way, but on $J(R)$ the action is by the inverse of the group law on $J(R)$. Also $V(R) \times J(R)$ has an action of $J(R)$ on the second component, which is compatible with the $A(R)$ action. Thus we have an action of $J(R)$ on $(V \times^A J)(R)$, and so $V \times^A J$ is a J-torsor.

The map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$ induces a map $\mathrm{ST}(A) \to \mathrm{ST}(J)$. We first claim that the image of the sheaf torsor corresponding to V under this induced map is the pushout $V \times^A J$.

Proof of the claim Pick $P \in V(\overline{\mathbf{Q}})$ and let $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Just for the proof of this claim, we shall write the torsor action as a function, *i.e.,* if $a \in A(\overline{\mathbf{Q}})$ and $x \in V(\overline{\mathbf{Q}})$, then $a(x)$ stands for the image of a acting on x under the action of A on V. The cocycle in $H^1(\mathbf{Q}, A)$ corresponding to V maps σ to a_σ, where a_σ is the unique element of $A(\overline{\mathbf{Q}})$ such that $\sigma(P) = a_\sigma(P)$. Now consider the point $(P, 0) \in V(\overline{\mathbf{Q}}) \times J(\overline{\mathbf{Q}})$, and let Q be its image in $(V \times^A J)(\overline{\mathbf{Q}})$. Then an easy check shows that $\sigma(Q) = a_\sigma(Q)$, where a_σ is now considered an element of $J(\overline{\mathbf{Q}})$. So the cocycle in $H^1(\mathbf{Q}, J)$ corresponding to $V \times^A J$ maps σ to $a_\sigma \in J(\overline{\mathbf{Q}})$. This is exactly the image of V under the map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. This proves the claim. ■

Proof of Proposition 2.1 Suppose V is visible as a torsor in J and let i denote the composite map $V \xrightarrow{\iota} V' \hookrightarrow J$, where ι and V' are as in Definition 1.3. Then consider the map of sheaf torsors $j\colon V \to V \times^A J$ induced by the map on sections $V(R) \to V(R) \times J(R)$ given by $v \mapsto (v, -i(v))$. Let v_1 and v_2 be elements of $V(R)$. Then v_1 and v_2 differ by translation by an element of $A(R)$, and so $-i(v_1)$ and $-i(v_2)$ differ by translation by the same element of $A(R)$. Hence the images of v_1 and v_2 under the map j are the same. Thus the image of the map $V(R) \to (V \times^A J)(R)$ is a point. This point is also invariant under the action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ (since the map j is defined over $\mathbf{Q}$). Hence this gives us a point of $V \times^A J$ over $\mathbf{Q}$. But that makes $V \times^A J$ the trivial torsor. Hence by the claim above, the cocycle class corresponding to V in $H^1(\mathbf{Q}, A)$ maps to the trivial element of $H^1(\mathbf{Q}, J)$, which proves the "only if" part of Proposition 2.1.

In the other direction, suppose the cocycle class corresponding to V is in the kernel of the map $H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. By the claim above, this means that there is an isomorphism $\phi\colon V \times^A J \xrightarrow{\simeq} J$ over $\mathbf{Q}$. Recall that R denotes a $\mathbf{Q}$-algebra of finite type and consider the map $\psi\colon V(R) \to (V \times^A J)(R)$ induced by the map $V(R) \to V(R) \times J(R)$ given by $v \mapsto (v, 0)$. An easy check shows that the composite

$$V(R) \xrightarrow{\psi} (V \times^A J)(R) \xrightarrow{\phi} J(R)$$

is an injection and respects the action of $A(R)$. By Yoneda's lemma, we have a monomorphism (*i.e.,* a closed immersion) $V \to J$ that respects the action of A. This shows that V is visible as a torsor in J and completes the proof of Proposition 2.1. ■

3 Visibility as a Variety

This section is a generalization of some results from [9].

Let J be an abelian variety and A be an abelian subvariety of J, both defined over $\mathbf{Q}$. Consider the following condition on the pair (J, A):

($*$) if $J \sim A \times B$ is an isogeny over $\overline{\mathbf{Q}}$, then no simple factor of A (over $\overline{\mathbf{Q}}$) is isogenous (over $\overline{\mathbf{Q}}$) to a simple factor (over $\overline{\mathbf{Q}}$) of B.

The following result was stated without proof in [1, Lemma 3.1].

Proposition 3.1 *Let A be an abelian subvariety of J satisfying ($*$). Let V be an A-torsor that is visible as a variety in J. Let i denote the embedding of A in J and consider the natural map $\tilde{i}\colon H^1(\mathbf{Q}, A) \to H^1(\mathbf{Q}, J)$. Then there exists an automorphism ϕ of A (defined over $\mathbf{Q}$) such that $\tilde{i}(\tilde{\phi}(V))$ is trivial, where $\tilde{\phi}$ is the automorphism of $H^1(\mathbf{Q}, A)$ induced by ϕ.*

Thus if the condition ($*$) holds, then a torsor is visible *as a variety* if and only if it is visible "up to an automorphism of A". The condition ($*$) is satisfied for example if J is the Jacobian of the modular curve $X_0(N)$ for some positive integer N and A is the abelian subvariety of J associated with a newform on $\Gamma_0(N)$ (see, *e.g.,* the proof of [6, Lemma 3.1]). This is the most important case for the application of the notion of visibility so far. In [8], the same situation was considered, with the added restriction that A is a semistable elliptic curve; in this case, the only automorphisms of A are multiplication by ± 1, and so all definitions of visibility coincide (cf. [8, Remark 2]).

Proof of Proposition 3.1 Suppose V is an A-torsor visible as a variety in J and let V' be the subvariety of J isomorphic to V over $\mathbf{Q}$ given by Definition 1.1. Let $\iota\colon V \to V'$ denote the isomorphism between V and V' (over $\mathbf{Q}$). Since V is an A-torsor, we have an isomorphism $\psi\colon A \stackrel{\simeq}{\to} V$ over $\overline{\mathbf{Q}}$. Consider the composite map

$$A \stackrel{\psi}{\longrightarrow} V \stackrel{\iota}{\longrightarrow} V' \longrightarrow J/A,$$

defined over $\overline{\mathbf{Q}}$. Up to translation, it is a homomorphism of abelian varieties. Its image has to be a point, because otherwise that would violate $(*)$. Hence the image of $V' \to J/A$ is also a point. Thus V' is a translate of A (over $\overline{\mathbf{Q}}$) and hence has an action of A by translation. As a torsor in $H^1(\mathbf{Q}, A)$, it is given by $\sigma \mapsto \sigma(Q) - Q$ for any $Q \in V'(\overline{\mathbf{Q}})$, where the subtraction is the usual subtraction in J. But this is the zero element in $H^1(\mathbf{Q}, J)$ (under $\tilde{\imath}$), since $Q \in V'(\overline{\mathbf{Q}}) \subseteq J(\overline{\mathbf{Q}})$. Thus $\tilde{\imath}(V') = 0$.

Next, let $P \in V(\overline{\mathbf{Q}})$. Then the element of $H^1(\mathbf{Q}, A)$ corresponding to V is $\sigma \mapsto \sigma(P) -_V P$ where we will be using subscripts to distinguish different actions of A. Then the element of $H^1(\mathbf{Q}, A)$ corresponding to V' is given by $\sigma \mapsto \sigma(\iota(P)) -_{V'} \iota(P)$. Consider the map $\phi\colon A \to A$ given by $a \mapsto \iota(P +_V a) -_{V'} \iota(P)$. It is defined over $\mathbf{Q}$, and it is a homomorphism of abelian varieties, since it takes the identity element of A to itself. It takes the torsor V to V' and thus $\tilde{\imath}(\tilde{\phi}(V)) = \tilde{\imath}(V')$. But as shown above, $\tilde{\imath}(V') = 0$, and so $\tilde{\imath}(\tilde{\phi}(V)) = 0$. Also, ϕ is an automorphism since it has an inverse given by $a \mapsto \iota^{-1}(\iota(P) +_{V'} a) -_V P$. This finishes the proof. ■

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