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How to determine a curve singularity

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Abstract. We characterize the finite codimension sub-$\mathbf{k}$-algebras of $\mathbf{k}[[t]]$ as the solutions of a computable finite family of higher differential operators. For this end, we establish a duality between such a sub-algebras and the finite codimension $\mathbf{k}$-vector spaces of $\mathbf{k}[u]$, this ring acts on $\mathbf{k}[[t]]$ by differentiation.

1 Introduction

It is well-known that the normalization of a curve X is a non-singular curve Y. Serre considers in [26, Chapter IV] the opposite direction, he showed how to construct a curve X from a given non-singular curve Y such that this curve is the normalization of X. This idea appears in several different contexts. For instance, in [17, 18, 23] and the references therein, is studied how to determine the finite codimension sub-$\mathbf{k}$-algebras B of $\mathbf{k}[t]$. Notice that, in this case, $X = \operatorname{Spec}(B)$ is an algebraic curve and the affine line $Y = \operatorname{Spec}(\mathbf{k}[t])$ is its normalization. These sub-algebras are defined recursively on the codimension by linear and higher differential conditions. Only for low codimensions, explicit conditions are known. Since not all higher differential conditions define sub-algebras of $\mathbf{k}[t]$, it is an open problem for the characterization of families of linear higher differential operators defining finite codimension sub-$\mathbf{k}$-algebras of $\mathbf{k}[t]$ (see [18]).

In the search of one-dimensional reduced local rings with locally decreasing Hilbert function, Roberts constructed such a local rings as connex, finite codimension sub-$\mathbf{k}$-algebras of $\prod_{i=1}^{r} \mathbf{k}[t_i]$ defined by linear and first-order differentials conditions (see [19]). See [11] for the proof of Sally's conjecture on the monotony of Hilbert functions of one-dimensional Cohen–Macaulay local rings.

In this paper, we consider the local complete case. We characterize the finite codimension sub-$\mathbf{k}$-algebras B of $\Gamma = \mathbf{k}[[t]]$ as the solutions of a computable finite codimension $\mathbf{k}$-vector space $B^{\perp} \subset \Delta = \mathbf{k}[u]$ of higher differential operators (see Theorem 3.9). For this purpose, we establish a Macaulay-like duality between finite codimension sub-$\mathbf{k}$-algebras B of Γ and finite codimension $\mathbf{k}$-vector subspaces $B^{\perp}$, so-called algebra-forming vector spaces, of the polynomial ring Δ. The polynomial ring Δ acts on Γ by differentiation as in Macaulay's duality (see [14–16, 20]). At the end

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of Section 3, we describe the linear maps $B_2^\perp \to B_1^\perp$ induced by **k**-algebra morphisms $B_1 \to B_2$ between two finite codimension **k**-algebras B_1, B_2.

In Section 4, we study the algebra-forming vector spaces, showing that such a condition can be checked effectively (see Proposition 4.1). After this, we prove that for any finite codimension δ **k**-algebra B there exist a finite filtration of **k**-algebras, so-called standard filtration of B, $B = B_0 \subset B_1 \subset \cdots \subset B_\delta = \Gamma$ such that $\dim_{\mathbf{k}}(B_{i+1}/B_i) = 1$ for $i = 0, \ldots, \delta - 1$. As corollary of this construction, we get that we only need to consider algebra-forming single elements in order to define recursively a finite codimension **k**-algebras. Moreover, we show how to recover the standard filtration by considering recursively derivations of the local rings appearing in the filtration (see Corollary 4.6).

Section 5 is devoted to study the inverse system of monomial **k**-algebras and the special case of monomial Gorenstein algebras. We end the section relating the inverse system of a curve singularity with its generic plane projection and its saturation.

In the last section, we link $B^\perp$ with the canonical module of B (see Proposition 6.1).

The computations of this paper are performed by using the computer algebra system singular (see [8]).

2 Preliminaries

Let R denote the power series ring $\mathbf{k}[[x_1, \ldots, x_n]]$ over an algebraically closed characteristic zero field **k** and we denote by $\max = (x_1, \ldots, x_n)$ its maximal ideal.

Let A be a one-dimensional local ring with maximal ideal max. We denote by HF_A the Hilbert function of A, i.e., $\mathrm{HF}_A(i) = \mathrm{Length}_A(\max^i / \max^{i+1})$, $i \geq 0$. It is well-known that $\mathrm{HF}_A^0(i) = e_0(A)$, $i \gg 0$, where $e_0(A)$ is the multiplicity of A. The first integral of HF_A is defined by, $i \geq 0$,

$$\mathrm{HF}_A^1(i) = \sum_{j=0}^{i} \mathrm{HF}_A(j) = \mathrm{Length}_A(A/\max^{i+1}).$$

We write $\mathrm{HF}_A^0 = \mathrm{HF}_A$. There exists an integer $e_1(A)$ such that $\mathrm{HF}_A^1(i) = e_0(A)(i+1) - e_1(A)$ for $i \gg 0$; the (first) Hilbert polynomial is $\mathrm{HP}_A^1(T) = e_0(A)(T+1) - e_1(A)$. See [22, Chapter XII] for the basic properties of the Hilbert functions of one-dimensional Cohen–Macaulay local rings.

A branch X is an irreducible curve singularity of $(\mathbf{k}^n, 0) = \mathrm{Spec}(R)$, i.e., X is a one-dimensional, integral scheme $X = \mathrm{Spec}(R/I)$; we write $\mathcal{O}_X = R/I$ and $I(X) = I$.

Let $\nu : \overline{X} = \mathrm{Spec}(\overline{\mathcal{O}_X}) \longrightarrow (X, 0)$ be the normalization of $(X, 0)$, where $\overline{\mathcal{O}_X} \cong \mathbf{k}[[t]]$ is the integral closure of $\mathcal{O}_X$ on its full field of fractions $\mathrm{tot}(\mathcal{O}_X)$. The singularity order of X is $\delta(X) = \dim_{\mathbf{k}}\left(\mathcal{O}_{\overline{X}}/\mathcal{O}_X\right)$. We denote by $\mathcal{C}$ the conductor of the finite extension $\nu^* : \mathcal{O}_X \hookrightarrow \overline{\mathcal{O}_X}$ and by $c(X)$ the dimension of $\overline{\mathcal{O}_X}/\mathcal{C}$.

Given a set of nonnegative integers $1 \leq a_1 < \cdots < a_n$, we consider the monomial curve singularity $X(a_1, \ldots, a_n)$ defined by the parameterization

$$\begin{array}{cccc} \gamma : & R & \longrightarrow & \mathbf{k}[[t]] \\ & x_i & \mapsto & t^{a_i}, \end{array}$$

i.e., $I(X(a_1,\ldots,a_n)) = \ker(\gamma)$. If $gcd(a_1,\ldots,a_n) = 1$, then the induced map

$$\gamma : R/I(X(a_1,\ldots,a_n)) \longrightarrow \mathbf{k}[[t]]$$

is the normalization map of $\mathcal{O}_{X(a_1,\ldots,a_n)} = R/I(X(a_1,\ldots,a_n)) = \mathbf{k}[[t^{a_1},\ldots,t^{a_n}]]$.

We denote by D_X the semigroup of values of X: the set of integers $v_t(f) = ord_t(t)$ where $f \in \mathcal{O}_X \setminus \{0\}$. It is easy to see that $\delta(X) = \#(\mathbb{N} \setminus D_X)$. If B is a finite codimension sub-$\mathbf{k}$-algebra of Γ then $X = \operatorname{Spec}(B)$ is branch. We write $D_B = D_X$.

Let ω_X be the dualizing module of X; we can consider the composition of $\mathcal{O}_X$-module morphisms

$$\gamma_X : \Omega_X \longrightarrow \nu_*\Omega_{\overline{X}} \cong \nu_*\omega_{\overline{X}} \longrightarrow \omega_X.$$

Let $d : \mathcal{O}_X \longrightarrow \Omega_X$ the universal derivation, then we have a $\mathbf{k}$-linear map $\gamma_X d$ that we also denote by $d : \mathcal{O}_X \longrightarrow \omega_X$. Recall that the Milnor number of X is $\mu(X) = \dim_{\mathbf{k}}(\omega_X/d\mathcal{O}_X)$, [5]. Since we only consider branches we have that $\mu(X) = 2\delta(X)$ (see [5, Proposition 1.2.1]). Notice that X is non-singular iff $\mu(X) = 0$ iff $\delta(X) = 0$ iff $c(X) = 0$.

We denote by $\pi : Bl(X) \longrightarrow X$ the blowing-up of X on its closed point. The fiber of the closed point of X has a finite number of closed points: the so-called points of the first neighborhood of X. We can iterate the process of blowing-up until we get the normalization of X (see [7, 24]). We denote by $\operatorname{Inf}(X)$ the set of infinitely near points of X. The curve singularity defined by an infinitely point p of X will be denote by (X,p); we set $(X,0) = X$.

Proposition 2.1 *Let X be a branch. Then*
(i)

$$\delta(X) = \sum_{p\in \operatorname{Inf}(X)} e_i(X,p).$$

(ii) *It holds*

$$e_0(X) - 1 \le e_1(X) \le \delta(X) \le \mu(X)$$

and $e_1(X) \le \binom{e_0(X)}{2} - \binom{n-1}{2}$.
(iii) *If X is singular, then* $\delta(X) + 1 \le c(X) \le 2\delta(X)$, *and* $c(X) = 2\delta(X)$ *if and only if* $\mathcal{O}_X$ *is a Gorenstein ring.*

Proof (i) [25]. (ii) [5, Proposition 1.2.4(i)] and [10, 12, 25]. (iii) [26, Proposition 7, page 80] and [2]. ∎

3 Macaulay-like duality

In this section, we establish a Macaulay-like duality for the family of sub-$\mathbf{k}$-algebras B of $\Gamma = \mathbf{k}[[t]]$ of finite codimension. For the classical Macaulay's duality, see [20], [14], and for the generalization to higher dimension of Macaulay's duality, see [15]. Recall that Macaulay's duality is a particular case of Matlis' duality (see [4]).

We write $\Delta = \mathbf{k}[u]$; Γ is a Δ-module with Δ acting on Γ by derivation. This action denoted by $\circ$ is defined by

$$\begin{array}{cccc} \circ : \Delta \times \Gamma & \longrightarrow & \Gamma \\ (g, f) & \to & g \circ f = g(\partial_t)(f), \end{array}$$

where ∂_t denotes the derivative with respect to t. This action induces a non-singular $\mathbf{k}$-bilinear perfect pairing:

$$\begin{array}{cccc} \perp: & \Delta \times \Gamma & \longrightarrow & \mathbf{k} \\ & (g, f) & \mapsto & g \perp f = (g \circ f)(0). \end{array} \tag{1}$$

Definition 3.1 Given a sub-$\mathbf{k}$-algebra B of $\Gamma = \mathbf{k}[[t]]$ we define $B^\perp$ as the set of $g \in \Delta$ such that $g \perp f = 0$ for all $f \in B$. Notice that $B^\perp$ is a $\mathbf{k}$-vector subspace of Δ, this is, following the classic Macaulay's duality terminology, the inverse system of B. Given a $\mathbf{k}$-vector subspace $V \subset \Delta$ we consider $\mathrm{Ann}(V) \subset \Gamma$ as the set of power series $f \in \Gamma$ such that $g \perp f = 0$ for all $g \in V$.

Let B be a finite codimension sub-$\mathbf{k}$-algebra of Γ. Then we have a non-singular $\mathbf{k}$-bilinear perfect pairing:

$$\begin{array}{cccc} \perp: & B^\perp \times \frac{\Gamma}{B} & \longrightarrow & \mathbf{k} \\ & (g, \overline{f}) & \mapsto & g \perp f. \end{array} \tag{2}$$

We denote by $Perp(B)$, the $\mathbf{k}$-vector space of maps

$$\begin{array}{cccc} g^\perp : & B & \longrightarrow & \mathbf{k} \\ & f & \mapsto & g \perp f \end{array}$$

for all $g \in \Delta$. These maps are the elements of the dual space of B with finite support: $g^\perp(\max_B^d) = 0$ for $d > \deg(g)$. We denote by $Der_{\mathbf{k}}(B)$ the $\mathbf{k}$-vector space of $\mathbf{k}$-derivations of B. Since $Der_{\mathbf{k}}(B) \cong (\max_B / \max_B^2)^*$, we can identify $Der_{\mathbf{k}}(B)$ with the $\mathbf{k}$-vector space of elements σ of the dual space of B such that $\sigma(\max_B^2) = 0$.

We have $Der_{\mathbf{k}}(B) \subset Perp(B)$, this inclusion is strict. Let us consider the codimension 8 algebra $B = \mathbf{k}[[t^4, t^7, t^{17}]]$. The linear map $(u^{11})^\perp : B \longrightarrow \mathbf{k}$ is not a derivation since $t^{11} \in \max_B^2$ and $(u^{11}) \perp (t^{11}) = 11! \neq 0$.

Next step is to characterize the vector $\mathbf{k}$-vector subspaces $B^\perp$ of Δ, where B ranges the family of finite codimension sub-$\mathbf{k}$-algebras of Γ. First, we give some properties of $B^\perp$ that we will use along the paper.

Given a polynomial $g = \sum_{i=0}^d a_i u^i \in \Delta$ we denote by $\mathrm{Supp}(g)$ the support of g: the finite set of integers i such that $a_i \neq 0$.

Proposition 3.2 *Let $B \subset \Gamma$ be a codimension δ sub-$\mathbf{k}$-algebra B of Γ, and let $\mathcal{C} = (t^c)$ be the conductor of the extension $B \subset \Gamma$. Then:*

(1) $\dim_{\mathbf{k}}(B^\perp) = \delta$.

(2) *For all $g \in B^\perp$, we have $\mathrm{Supp}(g) \subset [1, c-1]$, and*

$$u^{[1, e_0(B)-1]} = \{u^i; i \in [1, e_0(B) - 1]\} \subset B^\perp \subset \langle u, u^2, \ldots, u^{c-1} \rangle.$$

(3) *The following conditions are equivalent:*

(i) $\delta = 0$,
(ii) $B = \Gamma$,
(iii) $B^\perp = 0$,
(iv) $B^\perp \subset \langle u^2, u^3, \dots \rangle$.

Proof (1) Since $\perp$ is a $\mathbf{k}$-bilinear perfect pairing, we get $\dim_{\mathbf{k}}(B^\perp) = \delta$, see the equation (2).

(2) Since B is a $\mathbf{k}$-algebra, we have $1 \in B$, so if $g = \sum_{j\geq 0} a_i u^i \in B^\perp$, then $0 = g \perp 1 = a_0$. Hence $B^\perp \subset \langle u, u^2, \dots \rangle$. We know that $(t^c) \subset B$ so for all $g = \sum_{j\geq 0} a_i u^i \in B^\perp$, we have

$$0 = g \perp t^{c+i} = (c+i)! a_{c+i}$$

$i \geq 0$. Hence, if $g \in B^\perp$, then $\deg(g) \leq c-1$. From this, we deduce that $B^\perp \subset \langle u, u^2, \dots, u^{c-1} \rangle$.

Notice that $v_t(f) \geq e_0(B)$ for all $f \in B \setminus \{1\}$, so given $i \in [1, e_0(B)-1]$ we have $u^i \perp f = 0$. Hence $u^i \in B^\perp$ and then $u^{[1,e_0(B)-1]} \subset B^\perp$.

(3) The condition of (i) is equivalent to (ii). (ii) trivially implies (iii) and this implies (iv). If $B^\perp \subset \langle u^2, u^3, \dots \rangle$, then $t \in B$, since B is a $\mathbf{k}$-algebra, we get (ii). ∎

For all power series $f = \sum_{i\geq 0} b_i t^i \in \Gamma$ and given a nonnegative integer $s \in \mathbb{N}$, we denote by $[f]_{\leq s}$ the truncated polynomial $[f]_{\leq s} = \sum_{i\geq 0}^{s} b_i t^i$.

Let B be a finite codimension sub-$\mathbf{k}$-algebra of Γ with conductor c. Then B is a finitely generated $\mathbf{k}$-algebra; let $f_1, \dots, f_r$ be a system of generators of B as $\mathbf{k}$-algebra. We denote by $\natural_{B,d}$, $d \geq c-1$, the finite set of polynomials $[f_1^{l_1} \dots f_r^{l_r}]_{\leq d}$ with $l_i \geq 0$, $i = 1, \dots, r$, and $l_1 + \dots + l_r \leq d$. We denote by $W(\{f_1, \dots, f_r\}, d) \subset \Delta$ the $\mathbf{k}$-vector space generated by the polynomials of $\natural_{B,d}$. Notice that $W(\{f_1, \dots, f_r\}, d) + \langle t^{d+1} \rangle = W(\{f_1, \dots, f_r\}, d+1)$.

Proposition 3.3 *Let B be a finite codimension sub-$\mathbf{k}$-algebra of Γ with conductor c. Then $B^\perp$ is the set of $g \in \Delta$ of degree at most $c-1$ and such that $g \perp h = 0$ for all $h \in \natural_{B,c-1}$.*

Proof Let $f_1, \dots, f_r$ be a system of generators of B as $\mathbf{k}$-algebra, and let $\natural_{B,c-1}$ be the associated set of polynomials.

If $g \in B^\perp$, then $\deg(g) \leq c-1$, Proposition 3.2(2), so

$$0 = g \perp (f_1^{l_1} \dots f_r^{l_r}) = g \perp [f_1^{l_1} \dots f_r^{l_r}]_{\leq c-1}.$$

Hence, $g \perp h = 0$ for all $h \in \natural_{B,c-1}$.

Let $g \in \Delta$ be a polynomial with $\deg(g) \leq c-1$ and such that $g \perp h = 0$ for all $h \in \natural_{B,c-1}$. Any $f \in B$ can be written as

$$f = \sum_{l_1,\dots,l_r \in \mathbb{N}} c_{l_1,\dots,l_r} f_1^{l_1} \dots f_r^{l_r}$$

with $c_{l_1,\dots,l_r} \in \mathbf{k}$. Since $\deg(g) \leq c-1$, we have

$$g \perp f = \sum_{l_1,\dots,l_r \in \mathbb{N}} c_{l_1,\dots,l_r} (g \perp f_1^{l_1} \dots f_r^{l_r}) = \sum_{l_1,\dots,l_r \in \mathbb{N}} c_{l_1,\dots,l_r} (g \perp [f_1^{l_1} \dots f_r^{l_r}]_{\leq c-1}) = 0,$$

so $g \in B^\perp$. ∎

Remark 3.4 Notice that Proposition 3.3 shows that the computation of $B^\perp$ is effective. In fact, in the set $\natural_{B,c-1}$, there are involved a finite number of monomials and we only have to consider polynomials g of degree at most $c-1$.

Remark 3.5 Although $B^\perp$ is a $\mathbf{k}$-vector subspace of Δ for any sub-$\mathbf{k}$-algebra B of Γ, not all $\operatorname{Ann}(V)$ is a $\mathbf{k}$-algebra for a given $\mathbf{k}$-vector subspace $V \subset \Delta$. In fact, let us consider the $\mathbf{k}$-vector subspace $V \subset \Delta$ generated by u^2. Then $\operatorname{Ann}(V)$ is the set of $f = \sum_{i\geq 0} a_i t^i \in \Gamma$ such that $a_2 = 0$. This is not a $\mathbf{k}$-algebra because $u^2 \perp t = 0$, so $t \in \operatorname{Ann}(V)$ and $u^2 \perp t^2 = 2 \neq 0$, so $t^2 \notin \operatorname{Ann}(V)$.

Definition 3.6 A finite dimensional $\mathbf{k}$-vector subspace $V \subset \Delta$ is so-called algebra-forming with respect to a $\mathbf{k}$-algebra $B \subset \Gamma$ iff the following conditions hold:

(a) $g(0) = 0$ for all $g \in V$ and,
(b) for all $f \in B$ such that $g \perp f = 0$ for all $g \in V$ it holds $g \perp f^2 = 0$ for all $g \in V$.

An element $g \in \Delta$ is so-called algebra-forming with respect to B if $V = \langle g \rangle$ is algebra-forming with respect to B.

Example 3.7 Let us consider the codimension $\delta = 4$ algebra $B = \mathbf{k}[[t^3 + t^4, t^5]]$ of Γ. The conductor of B is $c = 8$. Then $B^\perp$ is the set of polynomials $g \in \Delta$ of degree at most 7 such that $g \perp f = 0$ for $f \in \natural_{B,c-1} = \{t^3 + t^4, t^5, t^6 + 2t^7\}$. A simple computation shows that $B^\perp$ is the $\mathbf{k}$-vector space generated by the four linear independent polynomials $u, u^2, u^3 - \frac{1}{4}u^4, u^6 - \frac{1}{2.7}u^7$. Let us consider

$$B_2 = \mathbf{k}[[t^3, t^4, t^5]] \subset B_3 = \mathbf{k}[[t^2, t^3]],$$

then we have $B_2 = \operatorname{Ann}\langle u^2 \rangle \cap B_3$, i.e., u^2 is an algebra-forming element with respect to B_2.

In the following result, we prove that, in fact, if $V \subset \Delta$ is algebra-forming with respect to a $\mathbf{k}$ algebra $B \subset \Gamma$, then $\operatorname{Ann}(V) \cap B$ is a sub-$\mathbf{k}$-algebra of Γ.

Proposition 3.8 *Let $V \subset \Delta$ be an algebra-forming $\mathbf{k}$-vector subspace with respect to a $\mathbf{k}$-algebra $B \subset \Gamma$. Then $\operatorname{Ann}(V) \cap B$ is a sub-$\mathbf{k}$-algebra of Γ.*

Proof Clearly $C = \operatorname{Ann}(V) \cap B$ is a $\mathbf{k}$-vector subspace of Γ. Given $f_1, f_2 \in C$ we have that $f_1 + f_2 \in C$ and from

$$f_1 f_2 = \frac{1}{2}((f_1 + f_2)^2 - f_1^2 - f_2^2),$$

we deduce that $g \perp (f_1 f_2) = 0$, i.e., $f_1 f_2 \in C$. Since $g(0) = 0$ for all $g \in V$ we get $1 \in C$, so C is a sub-$\mathbf{k}$-algebra of Γ. ∎

The following result is an extension of Macaulay's duality to finite codimension sub-$\mathbf{k}$-algebras $B \subset \Gamma$.

Theorem 3.9 *Given a nonnegative integers $\delta > 0$ and $c \geq \delta + 1$, there is a one-to-one correspondence $\perp$ between the following sets:*

(1) *sub-$\mathbf{k}$-algebras B of Γ of codimension δ as $\mathbf{k}$-vector spaces such that the conductor of $B \subset \Gamma$ is (t^c),*

(2) *algebra forming, with respect to Γ, $\mathbf{k}$-vector subspace $V \subset \Delta$ of dimension δ, generated by polynomials of degree at most $c-1$ and such that there is a polynomial $g \in V$ with $\deg(g) = c-1$.*

This correspondence is inclusion reversing: given two sub-$\mathbf{k}$-algebras B_1 and B_2 of Γ, $B_1 \subset B_2$ if and only if $B_2^\perp \subset B_1^\perp$.

Proof Let B be a sub-$\mathbf{k}$-algebra B of Γ. Since we have a non-singular $\mathbf{k}$-bilinear pairing:

$$\begin{array}{cccc} \perp: & B^\perp \times \frac{\Gamma}{B} & \longrightarrow & \mathbf{k} \\ & (g, \overline{f}) & \mapsto & g \perp f, \end{array}$$

we get that $B^\perp$ is a $\mathbf{k}$-vector subspace of dimension δ of Δ. By definition $B^\perp$ is algebra-forming with respect to Γ. Being c the conductor we have $(t^c) \subset B$, so $\deg(g) \leq c-1$ for all $g \in B^\perp$ and there exist $g \in B^\perp$ of degree $c-1$.

Let V be an algebra forming, with respect to Γ, $\mathbf{k}$-vector subspace satisfying the conditions of (2). Let us consider the $\mathbf{k}$-algebra $B = \mathrm{Ann}(V)$. From the perfect pairing (1), we get that the codimension of B in Γ is δ. Since V is generated by polynomials of degree at most $c-1$ we have that $(t^c) \subset B$, so the conductor of B is at most c. Furthermore, since there is $g \in V$ with $\deg(g) = c-1$ we deduce that c is the conductor of B.

It is straightforward to prove the inclusion reversing from the definition of the inverse system $B^\perp$. ∎

We end this section by describing the $\mathbf{k}$-linear maps $B_2^\perp \longrightarrow B_1^\perp$ induced by $\mathbf{k}$-algebra isomorphisms $B_1 \longrightarrow B_2$ between two finite codimension $\mathbf{k}$-algebras B_1 and B_2 of Γ. Let c be an integer bigger than the conductors of B_1 and B_2.

The perfect pairing (1) induce a perfect pairing

$$\begin{array}{cccc} \perp: & \Delta_{\leq c-1} \times \frac{\Gamma}{(t^c)} & \longrightarrow & \mathbf{k} \\ & (g, \overline{f}) & \mapsto & g \perp f = (g \circ f)(0), \end{array}$$

where $\Delta_{\leq c-1}$ is the $\mathbf{k}$-vector space of polynomials of degree at most $c-1$. We consider the usual $\mathbf{k}$-vector basis of $\Gamma/(t^c)$ of the cosets of t^i, $i = 0, \ldots, c-1$. Its dual basis is $\frac{1}{i!}u^i$, $i = 0, \ldots, c-1$, since

$$\left(\frac{1}{i!}u^i\right) \perp t^j = \delta_{i,j}$$

$1 \leq i, j \leq c-1$.

The $\mathbf{k}$-algebra B_i has conductor at most c so we can consider that $B_i \subset \Gamma/(t^c)$, $i = 1, 2$. On the other hand, from Proposition 3.2, we have that $B_i^\perp \subset \Delta_{\leq c-1}$, $i = 1, 2$.

If B_1 is isomorphic to B_2 by ϕ, then their normalizations are isomorphic:

$$\Gamma = \overline{B_1} \overset{\overline{\phi}}{\cong} \overline{B_2} = \Gamma.$$

This automorphism is determined by a power series $h(t) \in (t)$ such that $u \perp h \neq 0$ and

$$\begin{array}{cccc} \overline{\phi}: & \Gamma & \longrightarrow & \Gamma \\ & f & \mapsto & f(h). \end{array}$$

Then we have an isomorphism of $\mathbf{k}$-vector spaces

$$\frac{\Gamma}{B_1} \xrightarrow{\overline{\phi}} \frac{\Gamma}{B_2}$$

and the perfect pairing induces a $\mathbf{k}$-vector isomorphism

$$\phi^* : B_2^{\perp} \longrightarrow B_1^{\perp}.$$

The matrix M_ϕ associated with ϕ in the basis t^i, $i = 0, \ldots, c-1$, is the $c \times c$ matrix whose columns are the coefficients of $\phi(t^i) = h^i$, $i = 0, \ldots, c-1$, with respect to this basis. Hence, the matrix of $\phi^* : B_2^* = B_2^{\perp} \longrightarrow B_1^* = B_1^{\perp}$ with respect to the basis $\frac{1}{i!}u^i$, $i = 0, \ldots, c-1$, is the transpose matrix ${}^{\tau}M_\phi$ of M_ϕ.

Example 3.10 Let $B_2 \subset \Gamma$ be a $\mathbf{k}$-algebra generated by two elements f_1, f_2 with $v_t(f_1) = 2$ and $v_t(f_2) = 7$. We may assume that $f_1 = t^2$+monomials of higher degree. Then B_2 is of finite codimension $\delta = 3$ and conductor $c = 6$.

Since Γ is complete there exist a power series $h \in (t)$ such that $h^2 = f_1$; we write $h = t + h_2 t^2 + \cdots + h_5 t^5 + \ldots$. Notice that $\Gamma = \mathbf{k}[[h]]$.

Let ϕ the automorphism of Γ defined by h, i.e., $\phi(f) = f(h)$. Then $\phi^{-1}(B_2)$ is a $\mathbf{k}$-algebra B_1 generated by $f_1' = t^2$ and $f_2'(h)$ such that $v_h(f_2') = 7$. After a change of generators B_1 is generated by $f_1' = t^2$ and $f_2' = t^7$.

The induced isomorphism $\phi : B_1 \longrightarrow B_2$ has the following 6×6 associated matrix with respect the basis t^i, $i = 0, \ldots, 5$,

$$M_\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 1 & 0 & 0 & 0 \\ 0 & h_3 & 2h_2 & 1 & 0 & 0 \\ 0 & h_4 & 2h_3 + h_2^2 & 3h_2 & 1 & 0 \\ 0 & h_5 & 2b_4 + 2h_2 h_3 & 3h_3 + 3h_2^2 & 4h_2 & 1 \end{pmatrix}.$$

Then the matrix of the isomorphism $\phi^* : B_2^{\perp} \longrightarrow B_1^{\perp}$ with respect to $\frac{1}{i!}u^i$, $i = 0, \ldots, 5$, is M_ϕ^τ. Since B_1 is the monomial $\mathbf{k}$-algebra $\mathbf{k}[[t^2, t^7]]$, the $\mathbf{k}$-vector space $B_1^{\perp}$ is generated by u, u^3, u^5. From this, we can compute $B_2^{\perp}$ by considering $({}^{\tau}M_\phi)^{-1}$.

4 Algebra-forming vector spaces

The first goal of this section is to characterize the algebra-forming $\mathbf{k}$-vector spaces.

Proposition 4.1 *Let B be a $\mathbf{k}$-sub-algebra of finite codimension of Γ with conductor c, and let $f_1, \ldots, f_s$ be a system of generators of B. Given an integer $d \geq c-1$, let $h_1, \ldots, h_m$ be a system of generators of $W(\{f_1, \ldots, f_s\}, d)$.*

Let V be a dimension δ $\mathbf{k}$-vector subspace of $(u) \subset \Delta$ generated by polynomials of degree at most $d-1$. Let $g_1, \ldots, g_\delta \in V$ be a basis of V.

Then V is algebra-forming with respect to B iff for all r-upla $(\lambda_1, \ldots, \lambda_m) \in \mathbf{k}^m$ such that

$$\sum_{j=1}^{m} \lambda_j (g_i \perp h_j) = 0 \tag{3}$$

for all $i = 1, \ldots, \delta$, then

$$\sum_{j=1}^{m} \lambda_j^2 (g_i \perp h_j^2) + 2 \sum_{j=1, l=1, j \neq l}^{m} \lambda_j \lambda_j (g_i \perp h_j h_l) = 0 \tag{4}$$

for all $i = 1, \ldots, \delta$.

Proof From Proposition 3.2, we have to prove that for all $f \in B$ such that $g \perp f = 0$ for all $g \in V$ we have that $g \perp f^2 = 0$ for all $g \in V$. Since the polynomials of V are of degree at most $d-1$ we only have to prove that for all $f \in W = W(\{f_1, \ldots, f_s\}, d)$ such that $g \perp f = 0$ for all $g \in V$, we have that $g \perp f^2 = 0$ for all $g \in V$.

A general element of W can be written as $f = \sum_{j=1}^{m} \lambda_j h_j$. Hence the condition $g_i \perp f = 0$ is equivalent to

$$\sum_{j=1}^{m} \lambda_j (g_i \perp h_j) = 0$$

for all $i = 1, \ldots, \delta$. Similarly, the condition $g_i \perp f^2 = 0$ is equivalent to

$$\sum_{j=1}^{m} \lambda_j^2 (g_i \perp h_j^2) + 2 \sum_{j=1, l=1, j \neq l}^{m} \lambda_j \lambda_j (g_i \perp h_j h_l) = 0$$

for all $i = 1, \ldots, \delta$. ∎

Remark 4.2 The set of points $(\lambda_1, \ldots, \lambda_m) \in \mathbb{P}_{\mathbf{k}}^{m-1}$ satisfying the identities of (3) form a linear subvariety L, and the points satisfying the identities of (4) defines a subvariety $Q \subset \mathbb{P}_{\mathbf{k}}^{m-1}$ intersection of δ quadrics. Hence, V is algebra forming with respect to B iff $L \subset Q$. This is a computable condition.

Definition 4.3 Let B be a sub-$\mathbf{k}$-algebra of finite codimension δ of Γ and conductor c. Let D be the semigroup of B; we write the set $t^{\mathbb{N} \setminus D_B} = \{t^i; i \in \mathbb{N} \setminus D_B\}$ as $g_1 = t^{c-1}, \ldots, g_\delta = t$. Then we define the so-called standard filtration of B as follows: B_i is the $\mathbf{k}$-algebra generated by B and $g_1, \ldots, g_i$ for $i = 1, \ldots, \delta$; we set $B_0 = B$. Notice that $B_\delta = \Gamma$ and that we have

$$B = B_0 \subset B_1 \subset \cdots \subset B_\delta = \Gamma$$

and $\dim_{\mathbf{k}}(B_{i+1}/B_i) = 1$, $i = 0, \ldots, \delta - 1$.

After the definition of standard filtration, we only have to consider algebra-forming elements $g \in \Delta$, with respect a suitable sub-$\mathbf{k}$-algebras of Γ, in order to define a $\mathbf{k}$-algebra recursively. The algebra-forming elements are not unique as the following example shows.

Example 4.4 Let us consider the Example 3.7. The standard filtration of B is

$$B = \mathbf{k}[[t^3 + t^4, t^5]] \subset B_1 = \mathbf{k}[[t^3 + t^4, t^5, t^7]] \subset B_2 = \mathbf{k}[[t^3, t^4, t^5]] \subset B_3 = \mathbf{k}[[t^2, t^3]] \subset \Gamma.$$

The chain of $\mathbf{k}$-algebras is defined as follows. The cosets of t, t^2, t^4, t^7 in Γ/B form a basis of Γ/B as $\mathbf{k}$-vector space. Then B_1 is the $\mathbf{k}$-algebra generated by B and t^7, B_2 is the $\mathbf{k}$-algebra generated by B_1 and t^4, B_3 is the $\mathbf{k}$-algebra generated by B and t^2, and finally Γ is the $\mathbf{k}$-algebra generated by B and t.

We know that $B^\perp$ is a four-dimensional $\mathbf{k}$-vector space generated by $u, u^2, u^3 - \frac{1}{4}u^4, u^6 - \frac{1}{2.7}u^7$; we have $B_3 = \operatorname{Ann}\langle u\rangle$, $B_2 = \operatorname{Ann}\langle u^2\rangle \cap B_3$, $B_1 = \operatorname{Ann}\langle u^3 - \frac{1}{4}u^4\rangle \cap B_2$, $B = \operatorname{Ann}\langle u^6 - \frac{1}{2.7}u^7\rangle \cap B_1$. On the other hand, the $\mathbf{k}$-algebra $C_1 = \mathbf{k}[[t^3 + t^5, t^4]] \subset B_1$ can be obtained as

$$C_1 = \operatorname{Ann}\langle u^3 - \frac{1}{4.5}u^5\rangle \cap B_2,$$

i.e., $u^3 - \frac{1}{4.5}u^5$ is an algebra-forming element with respect to B_2. Notice that B_1 and C_1 are non analytically isomorphic codimension one $\mathbf{k}$-algebras of B_2 .

Next, we show how to build the standard filtration by using derivations.

Proposition 4.5 *Let $C \subset B$ be two sub-$\mathbf{k}$-algebras of Γ such that $\dim_{\mathbf{k}}(B/C) = 1$. There exist $\alpha \in Der_{\mathbf{k}}(B)$ such that $\ker(\alpha) = C$.*

Proof If we denote by $\max_B$, the maximal ideal of B then $\max_C \subset \max_B$, $\dim_{\mathbf{k}}(\max_B / \max_C) = 1$ and $\max_B^2 \subset \max_C$. Since we have

$$\frac{\max_C}{\max_B^2} \subset \frac{\max_B}{\max_B^2},$$

we deduce that there exists a linear form $\alpha : \frac{\max_B}{\max_B^2} \longrightarrow \mathbf{k}$ such that $\ker(\alpha) = \frac{\max_C}{\max_B^2}$. From this, we get the claim. ■

Corollary 4.6 *Let B be a sub-$\mathbf{k}$-algebra of finite codimension δ of Γ. Let us consider the standard filtration of B:*

$$B = B_0 \subset B_1 \subset \cdots \subset B_\delta = \Gamma.$$

For all $i = 1, \ldots, \delta$, there exists a derivation $\partial_{l_i} \in Der_{\mathbf{k}}(B_i)$, $l_i \in \max_{B_i}$, such that $\ker(\partial_{l_i}) = B_i$.

Example 4.7 Let us consider the Example 4.4. The element $u^\perp$ corresponds to the derivation ∂_t of Γ defined by t, so $B_3 = \ker(\partial_t)$. The maximal ideal of B_3 is minimally generated by t^2, t^3, the element $(u^2)^\perp$ is the derivation $\partial_{t^2} \in Der_{\mathbf{k}}(B_3)$, so $B_2 = \ker(\partial_{t^2})$. The maximal ideal of B_2 is minimally generated by t^3, t^4, t^5. The element $(u^3 - \frac{1}{4}u^4)^\perp$ is the derivation $\partial_{t^3 - \frac{1}{4}t^4} \in Der_{\mathbf{k}}(B_2)$, so $B_1 = \ker(\partial_{t^3 - \frac{1}{4}t^4})$. Finally, $\partial_{t^7} \in Der_{\mathbf{k}}(B_1)$ and $B = \ker(\partial_{t^7})$.

5 Monomial algebras

In this section, we first compute the inverse system of a monomial $\mathbf{k}$-algebra. After this, we characterize monomial Gorenstein curve singularities in terms of its inverse system. We end the section relating the inverse system of a curve singularity with its generic plane projection and its saturation.

The following result it is easy to deduce from the proof of the second part of Proposition 3.2(2).

Proposition 5.1 *Let D be an additive sub-semigroup of $\mathbb{N}$ with finite complement. Then $B^\perp$ is the $\mathbf{k}$-vector space generated by: $g_i = u^i$ for $i \in \mathbb{N} \setminus D$.*

Example 5.2 Let B be a sub-$\mathbf{k}$-algebra of $\mathbf{k}[[t]]$ of codimension $\delta = 1$. Then B is the $\mathbf{k}$-algebra $B = \mathbf{k}[[D]]$, where D is the sub-semigroup of $\mathbb{N}$ generated by $2, 3$. Hence, $B^\perp$ is the $\mathbf{k}$-vector space generated by u, i.e., B is the set of power series $f = \sum_{i \geq 0} b_i t^i \in \mathbf{k}[[t]]$ with $u \perp f = b_1 = 0$ (see [26, Example b, Section 4 of Chapter IV] and [18, Section 22]).

Example 5.3 Assume now that B is sub-$\mathbf{k}$-algebra of $\mathbf{k}[[t]]$ of codimension $\delta = 2$. Then its semi-group D_B is $D_1 = \langle 2, 5 \rangle$ or $D_2 = \langle 3, 4 \rangle$. In the first case, B is generated as $\mathbf{k}$-algebra by $f_1 = t^2 + b_3 t^3$ and $f_2 = t^5$. The conductor is $c = 4$. Then $B^\perp$ is generated by $g_1 = u$, $g_2 = 6b_3 u^2 + u^3$. In the second case, B is the monomial $\mathbf{k}$-algebra $B = \mathbf{k}[[D_2]]$ so $B^\perp$ is the sub-$\mathbf{k}$-algebra generated by $g_1 = u$ and $g_2 = u^2$. The conductor is $c = 5$ (see [18, Section 23]). It is known that the algebras of the first case are all analytically isomorphic to $\mathbf{k}[[D_1]]$.

The inverse system of a monomial Gorenstein $\mathbf{k}$-algebra case can be handled. Let us recall the definition of symmetric semi-group and the celebrate result of Kunz.

Definition 5.4 We say that a sub-semigroup D of $\mathbb{N}$ such that $\#(\mathbb{N} \setminus D) < \infty$ and with conductor c is symmetric if the condition $t \in D$ is equivalent to $c - 1 - t \notin D$.

Kunz proved that the ring $\mathbf{k}[[D]]$ is Gorenstein ring if and only if D is a symmetric semigroup,[21]. This symmetry is inherited by $B^\perp$.

Proposition 5.5 *Let D be a sub-semigroup of $\mathbb{N}$ such that $\#(\mathbb{N} \setminus D) < \infty$ and conductor c. The following conditions are equivalent:*

(1) *$\mathbf{k}[[D]]$ is Gorenstein,*
(2) *for all $g \in \mathbf{k}[[D]]^\perp$ it holds $t^{c-1} g(1/t) \in \mathbf{k}[[D]]$.*

Proof Since $B = \mathbf{k}[[D]]$ is a monomial $\mathbf{k}$-algebra we know that $B^{\perp}$ is generated by $g = \sum_{i=1}^{c-1} a_i u^i$ such that $a_i = 0$ for $i \in D$ (see Proposition 5.1). Then the exponents of the nonzero terms of $t^{c-1}g(1/t)$ are $c-1-i$ with $i \notin D$. Then the claim is equivalent to the symmetry of D, i.e., the Gorensteinness of B. ■

Example 5.6 Let D be the semigroup generated by 4, 6, and 9. This is a symmetric semigroup with conductor $c = 12$. The algebra $B = \mathbf{k}[[D]]$ is Gorenstein and isomorphic to $\mathbf{k}[[x, y, z]]/I$, where $I = (x^3 - y^2, y^3 - z^2)$. Then $B^{\perp}$ is generated by the polynomials $g = a_1u + a_2u^2 + a_3u^3 + a_5u^5 + a_7u^7$, $a_i \in \mathbf{k}$. The polynomials $t^{11}g(1/t) = a_1t^{10} + a_2t^9 + a_3u^8 + a_4u^6 + a_5u^4$ have all exponents in D. The $\mathbf{k}$-vector space $B^{\perp}$ is generated by the following elements $g_1 = u, g_2 = u^2, g_3 = u^3, g_4 = u^5, g_5 = u^7$.

Given a finite codimension subalgebra B of Γ, we consider the curve singularity $X = \operatorname{Spec}(B)$ defined by B. Let X' be the generic plane projection of X, [3], and let $\widetilde{X}$ be the saturation of X, [28] and the references therein. We have

$$\mathcal{O}_{X'} \subset \mathcal{O}_X = B \subset \mathcal{O}_{\widetilde{X}} \subset \Gamma,$$

and then

$$\mathcal{O}_{\widetilde{X}}^{\perp} \subset B^{\perp} \subset \mathcal{O}_{X'}^{\perp}.$$

We have, [9],

$$\delta(\widetilde{X}) \leq \delta(X) \leq \delta(X') \leq (e_0(X) - 1)\delta(\widetilde{X}) - \binom{e_0(X) - 1}{2}.$$

From [27, Proposition 1.6, page 971], we know that $\widetilde{X}$ is also the saturation of X'.

On the other hand, $\widetilde{X}$ is a monomial curve singularity. Assume that the coset of x_1 in B is t^{e_0} with e_0 the multiplicity of B. Since the rings are complete and the ground field is algebraically closed, we can assumed it after a suitable election of the uniformization parameter of Γ. Let $\{e_0; \beta_1, \dots, \beta_g\}$ be the characteristic of X', [28, Section 3, page 993], then $\mathcal{O}_{\widetilde{X}}$ is the monomial subalgebra with generators:

$$\begin{cases} t^{e_0}, & \\ t^{s_\nu n_{\nu+1} \dots n_g}, & m_\nu \leq s_\nu \leq [m_{\nu+1}/n_{\nu+1}], \nu = 1, \dots, g-1, \\ t^{m_g + i}, & 0 \leq i \leq e_0 - 1, \end{cases}$$

where $\beta_\nu / e_0 = m_\nu / n_1 \dots n_\nu$ is the νth characteristic exponent of X', $\nu = 1, \dots, g-1$, and $\gcd(m_i, n_i) = 1$ for all $i = 1, \dots, g$ (see [28, Section 3, page 995]).

The facts $\mathcal{O}_{\widetilde{X}}^{\perp} \subset B^{\perp}$ and Proposition 5.2 can be useful in order to simplify the computation of $B^{\perp}$ as the next example shows.

Example 5.7 Let us consider the $\mathbf{k}$-algebra $B = \mathbf{k}[[t^6, t^8 + t^{11}, t^{10} + t^{13}]]$; its saturation is $\widetilde{B} = \mathbf{k}[[t^6, t^8, t^{10}, t^{11}, t^{13}, t^{15}]]$ (see [6, Example 2.5.1]). The sequence of multiplicities of the resolution of $X = \operatorname{Spec}(B)$ is $\{6, 2, 2, 2, 2, 1, \dots\}$. We can compute $\delta(X)$ by computing $e_1(C)$, where C ranges the local rings of the resolution process, in this case, we get $\{8, 1, 1, 1, 1, 0, \dots\}$, so $\delta(X) = 12$. The semigroup of B is $D = \{0, 6, 8, 10, 12, 14, 16, 18, 19, 20, 22 \rightarrow\}$, i.e., the conductor of D is 22.

On the other hand, the semigroup of $\mathcal{O}_{\widetilde{X}}$ is $\{0, 6, 8, 10 \longrightarrow\}$, its conductor is 10. Hence, $\mathcal{O}_{\widetilde{X}}^{\perp}$ is generated by u^i with $i \in \{1, 2, 3, 4, 5, 7, 9\}$, and $B^{\perp}$ is the set of polynomials $g = \sum_{i=0}^{21} a_i u^i$ such that $a_6 = 0$, $990a_{11} - a_8 = 0$, $a_{12} = 0$, $1716a_{13} - a_{10} = 0$, $a_{16} = 0$, $4080a_{17} - a_{14} = 0$, $a_{18} = a_{19} = a_{20} = a_{21} = 0$.

6 The canonical module

As in the Artin case, we can relate the canonical module with the inverse system. In that case, we have that if I is an Artinian ideal, then $I^{\perp} \cong E_{R/I}(\mathbf{k}) \cong \omega_{R/I}$ (see [4, 14]). In the case of branches, we can determine the "negative" part of the canonical module.

Let X be a branch of $(\mathbf{k}^n, 0)$ and $\overline{X}$ its normalization. We first describe the canonical module ω_X by using Rosenlicht's regular differential forms (see [26, Chapter IV 9], [5, Section 1], see also [13]). We denote by $\Omega_{\overline{X}}(p)$, the set of meromorphic forms in $\overline{X}$ with a pole at most in $p = \nu^{-1}(0)$. Then Rosenlicht's differential forms are defined as follows: ω_X^R is the set of $\nu_*(\alpha)$, $\alpha \in \Omega_{\overline{X}}(p)$, such that for all $F \in \mathcal{O}_X$,

$$\operatorname{res}_p(F\alpha) = 0.$$

Notice that we have a mapping that we also denote by

$$d_R : \mathcal{O}_X \longrightarrow \Omega_X \longrightarrow \nu_*\Omega_{\overline{X}} \hookrightarrow \omega_X^R.$$

In [1, Chapter VIII], it is proved that $\omega_X \overset{\phi}{\cong} \omega_X^R$ and $d_R = \phi d$, where $d : \mathcal{O}_X \longrightarrow \omega_X$ is the map defined in the Section 1. Since $\mathcal{O}_X$ is a one-dimensional reduced ring, we know that $\omega_{(X,0)}$ is a sub-$\mathcal{O}_X$-module of $\operatorname{tot}(\mathcal{O}_X)$ (see [4, Proposition 3.3.18]). There is a perfect pairing, [26, Chapter IV],

$$\begin{array}{ccccc} \frac{\nu_*\mathcal{O}_{\overline{X}}}{\mathcal{O}_X} & \times & \frac{\omega_{(X,0)}}{\nu_*\Omega_{\overline{X}}} & \xrightarrow{\eta} & \mathbb{C} \\ F & \times & \alpha & \longrightarrow & \operatorname{res}_p(F\alpha) \end{array}$$

notice that for all $\lambda \in R$ it holds $\eta(\lambda F, \alpha) = \operatorname{res}_p(\lambda F\alpha) = \eta(F, \lambda\alpha)$.

Proposition 6.1 *Let X be a branch of $(\mathbf{k}^n, 0)$ and $\overline{X}$ its normalization. Then we have an isomorphism of the $\delta(X)$ dimensional $\mathbf{k}$-vector spaces:*

$$B^{\perp} \overset{\varepsilon}{\cong} \frac{\omega_X}{\nu_*\Omega_{\overline{X}}}$$

such that $\varepsilon(g)$ is the coset defined by $\alpha = \sum_{i=0}^{c-1} i!c_i t^{-i-1}$, for all $g = \sum_{i=0}^{c-1} c_i u^i \in B^{\perp}$.

Proof We write $B = \mathcal{O}_X$, $\Gamma = \nu_*\mathcal{O}_{\overline{X}}$, and $\Omega_{\overline{X}} = \Gamma dt$. Then ε is the composition of the isomorphisms induced by the above two perfect pairings

$$B^{\perp} \overset{\varepsilon_1}{\cong} \left(\frac{\Gamma}{B}\right)^* \overset{\varepsilon_2}{\cong} \frac{\omega_X}{\nu_*\Omega_{\overline{X}}}.$$

Next, we describe both morphisms $\varepsilon_1, \varepsilon_2$. Given $g \in B^{\perp}$, we can write it as

$$g = c_0 + c_1 u + \ldots, c_{c-1} u^{c-1},$$

so $\varepsilon_1(g)$ is the linear form induced by $\xi : \Gamma^* \longrightarrow \mathbf{k}$ defined by: if $f = \sum_{i\geq 0} a_i t^i \in \Gamma$, then

$$\xi(f) = \sum_{i=0}^{c-1} i! a_i c_i.$$

On the other hand, every $\alpha \in \omega_X$ can be written as $\alpha = t^n h(t) dt$ with $n \in \mathbb{Z}$ and $h(t) \in \Gamma$ an invertible series. From [13, Proposition 2.6], we get that $\alpha = \sum_{i\geq -c} e_i t^i$ such that $\operatorname{res}_0(\alpha F) = 0$ for all $f \in B$. Given $f = \sum_{i\geq 0} a_i t^i \in \Gamma$, we have

$$\operatorname{res}_0(f\alpha) = \sum_{i=0}^{c-1} a_i e_{-i-1}$$

so $\varepsilon_2^{-1}(\alpha)$ is the linear form induced by $\xi' : \Gamma^* \longrightarrow \mathbf{k}$ defined by

$$\xi'(f) = \sum_{i=0}^{c-1} a_i e_{-i-1}.$$

From this, we deduce that $e_{-i-1} = i! c_i$ for $i = 0, \dots, c-1$. ■

Example 6.2 [13, Example 2.7] Let us consider the monomial curve X with parameterization $x_1 = t^4, x_2 = t^7, x_3 = t^9$. We have $c = 11$, $\delta = 6$. Then ω_X is the $\mathbf{k}$-vector space spanned by $t^{-11}, t^{-7}, t^{-6}, t^{-4}, t^{-3}, t^{-2}, t^n, n \geq 0$, and the quotient $\omega_X / \nu_* \Omega_{\overline{X}}$ admits as $\mathbf{k}$-vector space base the cosets of $t^{-11}, t^{-7}, t^{-6}, t^{-4}, t^{-3}, t^{-2}$, and $\mathcal{O}_X^{\perp}$ is the $\mathbf{k}$-vector space with basis $u, u^2, u^3, u^5, u^6, u^{10}$.

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