

Variational model with nonstandard growth conditions for restoration of satellite optical images using synthetic aperture radar

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In this paper, the problem of restoration of cloud contaminated optical images is studied in the case when we have no information about brightness of such images in the damage region. We propose a new variational approach for exact restoration of optical multi-band images utilising Synthetic Aperture Radar (EOS – Spatial Data Analytics, GIS Software, Satellite Imagery – is a cloud-based platform to derive remote sensing data and analyse satellite imagery for business and science purposes) images of the same regions. We prove existence of solutions, propose an alternating minimisation method for computing them, prove convergence of this method to weak solutions of the original problem and derive optimality conditions.

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1 Introduction

It is well-known that optical satellite multi-band images have a high resolution and can be easily captured by low-cost cameras. However, they are often corrupted because of poor weather conditions, such as rain, clouds, fog and dust conditions. Moreover, it is a typical situation that the measure of degradation of optical images is such that it cannot even rely one brightness value being available inside the damaged regions. As a result, some subdomains of such images become absolutely invisible. Thus, in spite of the fact that in the literature there are many approaches to the reconstruction of images when information of the colours is not everywhere available (see, for instance, [23, 31, 37, 38, 40]), the traditional approaches to the exact restoration of damaged optical images are no longer applicable in this case and it makes this problem challenging.

However, in contrast to optical observations, radar images do not depend on reflected sunlight and they can be used at night and under poor weather conditions. In the vegetation case, instead of giving an indication on biophysical processes in the plant, the radar backscatter rather contains information on the structure and moisture content of vegetation and the underlying soil. Therefore, the fusion of Synthetic Aperture Radar (SAR) and optical images is very important for classification of land cover [34] and estimation of soil moisture to remove vegetation cover effects from radar backscattering coefficient [19, 20, 36]. At the same time, because of the distinct natures of SAR and optical images, there exist a huge radiometric difference between optical and synthetic aperture of radar images.

It is well-known that different wavelengths encode different object properties, therefore leading to significant intensity differences between SAR and optical satellite images for the same object. Because of this, it would be naive to suppose that the intensities in all spectral channels of cloud contaminated optical images can be successfully recovered inside the damaged regions at high level of accuracy through the corresponding intensities of the SAR images of this region. At the same time, when dealing with optical and SAR images of some agricultural areas of various shapes or/and farmland, it can be observed that the textures of such images have many common features (see, for instance, [22]). Mainly inspired by this observation, we propose a new variational model for exact restoration of the damaged multi-band optical satellite images using results of their co-registration with SAR images of the same regions.

Let $\Omega \subset \mathbb{R}^2$ be a bounded image domain with Lipschitz boundary $\partial\Omega$, and let $D \subset \Omega$ be a Borel set with non-empty interior and sufficiently regular boundary and such that $|\Omega \setminus D| > 0$. We call D the damage region of a given multi-band image $\vec{u}_0 = [u_{1,0}, u_{2,0}, \dots, u_{M,0}]^t \in L^2(\Omega \setminus D; \mathbb{R}^M)$ where the optical image \vec{u}_0 is corrupted by clouds. As it was mentioned before, we deal with the case where we have no information about the original image \vec{u}_0 inside D . Instead of this, we assume that a SAR image $u_{SAR} : \Omega \rightarrow \mathbb{R}$ of the same region is given, and this image is well co-registered with \vec{u}_0 in $\Omega \setminus D$. The problem is to reconstruct the intensities $u_{i,0}(x)$, $i = 1, \dots, M$, of the original multi-band image \vec{u}_0 through a variational model starting from the knowledge of SAR image on the subset D (the damaged region) together with the exact information of \vec{u}_0 on $\Omega \setminus D$ (the undamaged region).

Mostly motivated by the recent studies in this field [4] (see also [6, 7, 8, 9, 11] for comparison), we propose a new strategy for the restoration problem of cloud contaminated multi-band optical images through their fusion with the corresponding SAR images of the same territory. The variational approach we consider is inspired, in some sense, by the recent paper [4], where the authors consider the minimisation of the following energy functional

$$u \mapsto \frac{1}{2} \| (-\Delta)^{\frac{s}{2}} u \|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \| (-\Delta)^{-\frac{\beta}{2}} (u - g) \|_{L^2(\Omega)}^2 \quad (1.1)$$

with $0 < s < 1$ and $\beta \in [0, 1]$, where $(-\Delta)^s$ denotes the fractional power of the Laplacian with zero Neumann boundary conditions. Then, the first necessary and sufficient optimality condition determines the unique minimiser u via

$$\begin{aligned} (-\Delta)^s u + \alpha (-\Delta)^{-\beta} (u - g) &= 0 \quad \text{in } \Omega, \\ \partial_\nu u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which is a linear elliptic partial differential equation that can be efficiently solved using, for instance, the Fourier spectral method [3] or the Stinga-Torrea extension [35]. Since, from the reconstruction point of view, it is desirable that the regularity of the solution to (1.1) is low in

places in Ω where edges or discontinuities are present in u , and that is high in places where u is smooth or contains homogeneous features, it is of interest to consider (1.1) where $s : \Omega \rightarrow [0, 1]$ is not a constant. So, the choice of parameter s has a direct influence on the global regularity of the solution to problem (1.1). However, the definition of the fractional Laplacian in terms of the Caffarelli–Silvestre [10] or the Stinga–Torrea [39] extension is well-known only for a constant $s \in (0, 1)$, whereas such a result remains open when $s(x) \in (0, 1)$ for $x \in \Omega$. So, there is no obvious way how to correctly define the operator $(-\Delta)^{s(x)}$ and for nowadays it looks as an open question. For the details, we refer to [4].

In contrast to the standard setting of restoring colour damaged images where the starting point is either the knowledge of the grey level of the original colour image \vec{u}_0 on a given open subset D of Ω (the damaged region) together with the exact information of \vec{u}_0 on $\Omega \setminus D$ (the undamaged region) or the grey level information in the damage region $D \subset \Omega$ is modelled as a nonlinear distortion of the colours, in this paper we deal with the case where we do not have any information about \vec{u}_0 inside D but instead we assume that a SAR image $u_{SAR} : \Omega \rightarrow \mathbb{R}$ of the same region is given. So, the purpose of this paper is to study the faithfulness of the reconstruction following the proposed variational model and supply this approach by the rigorous mathematical substantiation.

The paper is structured as follows. Section 2 contains some preliminaries and auxiliary results. For other details related to Orlicz and variable exponent Sobolev spaces, we refer to Appendices A–C. In Section 3, we begin with some key assumptions and after that we give a precise statement of the restoration problem in the form of some constrained minimisation problems with nonstandard growth energy functionals. We discuss the consistency issues of the proposed minimisation problems and the existence of the corresponding minimisers in Section 4. To prove the existence result, mainly because of the specific form of the objective functional, we follow the technique that was recently developed by the authors in [14, 15, 24].

In spite of the fact that the proposed minimisation problems are well-posed and possess good approximating properties, their practical implementation to the restoration of real satellite images is a tricky matter because of the non-convexity and complicated structure of the corresponding optimality conditions. Therefore, our main intention in Section 5 is to present ‘an approximation approach’, which is based on the concept of relaxation of extremal problems and their variational convergence. With that in mind, we propose to pass to some relaxation of the original problem using a special iterative algorithm. We show that at each step of the iteration procedure, we obtain a strictly convex optimisation problem which admits a unique solution. It is established that the so-defined sequence of approximations is precompact in some Hausdorff topology, and each convergent subsequence leads to a weak solution of the original problem. Optimality conditions for the relaxed version of a given restoration problem and their substantiation are studied in Section 6. The experiments undertaken in this study (see Section 7) confirmed the efficacy of the proposed method and revealed that it can acquire plausible visual performance and satisfactory quantitative accuracy for agro scenes with rather complicated texture of background surface.

2 Preliminaries and some auxiliary results

We begin with some notation. For vectors $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$, $(\xi, \eta) = \xi^t \eta$ denotes the standard vector inner product in \mathbb{R}^N , where t denotes the transpose operator. The norm $|\xi|$ is the Euclidean norm given by $|\xi| = \sqrt{(\xi, \xi)}$.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with a Lipschitz boundary $\partial\Omega$. For any subset $E \subset \Omega$, we denote by $|E|$ its 2-dimensional Lebesgue measure $\mathcal{L}^2(E)$. Let \bar{E} denote the closure of E , and ∂E stands for its boundary. We define the characteristic function χ_E of E by

$$\chi_E(x) := \begin{cases} 1, & \text{for } x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Let $D \subset \Omega$ be a Borel set with non-empty interior and sufficiently regular boundary and such that $|\Omega \setminus D| > 0$. Let $\vec{u}_0 = [u_{1,0}, u_{2,0}, \dots, u_{M,0}]^t \in L^2(\Omega \setminus D; \mathbb{R}^M)$ be an image of interest, where each coordinate represents the intensity of the corresponding spectral channel. We associate with \vec{u}_0 the panchromatic image u (the so-called total spectral energy of \vec{u})

$$u(x) = \alpha_1 u_1(x) + \dots + \alpha_M u_M(x). \quad (2.1)$$

In particular, if we take into account in (2.1) just the weight coefficients for the RGB-channels with

$$\alpha_R = 0.299, \quad \alpha_G = 0.587, \quad \alpha_B = 0.114,$$

then u can be interpreted as the luma component of \vec{u} and it represents the perceptual brightness of the multispectral image $\vec{u} : \Omega \setminus D \rightarrow \mathbb{R}^M$.

We assume that the multi-band image \vec{u}_0 is corrupted by clouds, and D is the zone of missing information. We call D the damage region. Since we have no information about the original image u_0 inside D , we assume that a SAR image $u_{SAR} : \Omega \rightarrow \mathbb{R}$ of the same region is given, and this image is well co-registered with $u_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ in $\Omega \setminus D$. This means that there exists an affine transformation $\mathcal{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of the form

$$\mathcal{F}(x) = Bx + a, \quad \forall x \in \mathbb{R}^2, \quad (2.2)$$

where

$$a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

such that the SAR image after the affine transformation $u_{SAR}(\mathcal{F}^{-1}(\cdot))$ and panchromatic image $u(\cdot)$ could be successfully matched in $\Omega \setminus D$ (see [21] for the details).

We assume that the SAR image $u_{SAR} : \Omega \rightarrow \mathbb{R}$ is a function of bounded variation, that is, $u_{SAR} \in BV(\Omega)$. Then, $u_{SAR} \in L^2(\Omega)$ and almost all level sets $\{x \in \Omega : u_{SAR}(x) \geq \lambda\}$ are sets of finite perimeter. Hence, at almost all points of almost all level sets of $u_{SAR} \in BV(\Omega)$ we can define a normal vector $\theta(x)$. This vector field of normals $\theta(x)$ can be also defined as the Radon-Nikodym derivative of the measure ∇u_{SAR} with respect to $|\nabla u_{SAR}|$, that is, it formally satisfies the following relations

$$(\theta, \nabla u_{SAR}) = |\nabla u_{SAR}| \quad \text{and} \quad |\theta| \leq 1 \text{ a.e. in } \Omega.$$

In the sequel, we will refer to the vector field θ as the vector field of unit normals to the topographic map of the image u_{SAR} .

Remark 2.1 *In practice, at the discrete level, $\theta(x, y)$ can be defined by the rule $\theta(x_i, y_j) = \frac{\nabla u_{SAR}(x_i, y_j)}{|\nabla u_{SAR}(x_i, y_j)|}$ when $\nabla u_{SAR}(x_i, y_j) \neq 0$, and $\theta = 0$ when $\nabla u_{SAR}(x_i, y_j) = 0$. However,*

as was mentioned in [5], a better choice for $\theta(x, y)$ would be to compute it as $\theta = \xi(t)$ for some small value of $t > 0$, where $\xi(t) = \frac{\nabla U(t, \cdot)}{|\nabla U(t, \cdot)|}$ and $U(t, x, y)$ is a solution of the following initial-boundary value problem with 1-Laplacian in the right hand side

$$\frac{\partial U}{\partial t} = \operatorname{div} \left(\frac{\nabla U}{|\nabla U|} \right), \quad t \in (0, +\infty), (x, y) \in \Omega, \tag{2.3}$$

$$U(0, x, y) = u_{SAR}(x, y), \quad (x, y) \in \Omega, \tag{2.4}$$

$$\frac{\partial U(0, x, y)}{\partial \nu} = 0, \quad t \in (0, +\infty), (x, y) \in \partial\Omega. \tag{2.5}$$

As a result, for any $t > 0$, there can be found a vector field

$$\xi \in L^\infty(\Omega; \mathbb{R}^2) \text{ with } \|\xi(t)\|_{L^\infty(\Omega; \mathbb{R}^2)} \leq 1$$

such that

$$(\xi(t), \nabla U(t, \cdot)) = |\nabla U(t, \cdot)| \text{ in } \Omega, \quad \xi(t) \cdot \nu = 0 \text{ on } \partial\Omega, \tag{2.6}$$

and $U_t(t, x, y) = \operatorname{div} \xi(t, x, y)$ in the sense of distributions on Ω for a.a. $t > 0$.

We notice that following this procedure, for small value of $t > 0$, we do not distort the geometry of the function $u_{SAR}(x, y)$ in an essential way. Moreover, it can be shown that this regularisation of the vector field θ satisfies condition $\operatorname{div} \theta \in L^2(\Omega)$.

Since the main problem we are going to consider in this article is to reconstruct the original multi-band image $\vec{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ in the damage region D using the knowledge of geometrical texture of the SAR image $u_{SAR} \in BV(\Omega)$ on the subset D together with the exact information about this image in $\Omega \setminus D$ (the undamaged region), we associate with each spectral channel of the restored optical image $\vec{u} = [u_1, u_2, \dots, u_M]^t : \Omega \rightarrow \mathbb{R}^M$ the so-called texture index $p_i : \Omega \rightarrow \mathbb{R}$ following the rule

$$p_i(x) := \mathfrak{F}(u_i(x)) = 1 + g(|(\nabla G_\sigma * u_i)(x)|), \quad \forall x \in \Omega, \forall i = 1, \dots, M, \tag{2.7}$$

where $g: [0, \infty) \rightarrow (0, \infty)$ is an edge-stopping function which we take in the form of the Cauchy law $g(t) = \frac{1}{1+(t/a)^2}$ with an appropriate $a > 0$, $(\nabla G_\sigma * u_i)(x)$ denotes the convolution of function u_i with the two-dimensional Gaussian filter kernel G_σ ,

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad x \in \mathbb{R}^2, \tag{2.8}$$

$$(\nabla G_\sigma * u_i)(x) := \int_\Omega \nabla G_\sigma(x - y) u_i(y) dy, \quad \forall x \in \Omega. \tag{2.9}$$

Here, the parameter $\sigma > 0$ determines the spatial size of the image details which are removed by this 2D filter. By default, we assume that the functions u_i are extended by zero outside of domain Ω .

As follows from (2.7), the magnitude $g(|(\nabla G_\sigma * u_i)(x)|)$ is close to one at those points, where the spectral intensity u_i is slowly varying, and this value is close to zero at the edges of u_i . In view of this, the function $p_i(x)$ can be interpreted as a characteristic of texture of the image \vec{u} in its i -th spectral channel. The following result plays a crucial role in the sequel.

Lemma 2.1 Let $\{v_k\}_{k \in \mathbb{N}} \subset L^\infty(\Omega)$ be a sequence of measurable non-negative functions such that $v_k \rightarrow v$ weakly- $*$ in $L^\infty(\Omega)$ for some $v \in L^\infty(\Omega)$, and each element of this sequence is extended by zero outside of Ω . Let $\{p_k = 1 + g(|(\nabla G_\sigma * v_k)|)\}_{k \in \mathbb{N}}$ be the corresponding sequence of texture indices. Then,

$$p_k(\cdot) \rightarrow p(\cdot) = 1 + g(|(\nabla G_\sigma * v)(\cdot)|) \quad \text{uniformly in } \bar{\Omega} \text{ as } k \rightarrow \infty, \\ \alpha := 1 + \delta \leq p_k(x) \leq \beta := 2, \quad \forall x \in \Omega, \forall k \in \mathbb{N}, \tag{2.10}$$

$$\text{with } \delta = a^2 \left[a^2 + \|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)}^2 \sup_{k \in \mathbb{N}} \|v_k\|_{L^1(\Omega)}^2 \right]^{-1}. \tag{2.11}$$

Proof. In view of the initial assumptions, the sequence $\{v_k\}_{k \in \mathbb{N}}$ is uniformly bounded in $L^1(\Omega)$. Hence, by smoothness of the Gaussian filter kernel G_σ , it follows that

$$|(\nabla G_\sigma * v_k)(x)| \leq \int_\Omega |\nabla G_\sigma(x-y)| v_k(y) dy \leq \|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)} \|v_k\|_{L^1(\Omega)}, \\ p_k(x) = 1 + \frac{a^2}{a^2 + |(\nabla G_\sigma * v_k)(x)|^2} \\ \geq 1 + \frac{a^2}{a^2 + \|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)}^2 \|v_k\|_{L^1(\Omega)}^2}, \quad \forall x \in \Omega,$$

where $\|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)} = \max_{\substack{z=x-y \\ x \in \Omega, y \in \Omega}} [|G_\sigma(z)| + |\nabla G_\sigma(z)|]$. Then, L^1 -boundedness of $\{v_k\}_{k \in \mathbb{N}}$ guarantees the existence of a positive value $\delta \in (0, 1)$ (see (2.11)) such that estimate (2.10) holds true for all $k \in \mathbb{N}$.

Moreover, as follows from the relations

$$|p_k(x) - p_k(y)| \leq a^2 \left| \frac{|(\nabla G_\sigma * v_k)(x)|^2 - |(\nabla G_\sigma * v_k)(y)|^2}{(a^2 + |(\nabla G_\sigma * v_k)(x)|^2)(a^2 + |(\nabla G_\sigma * v_k)(y)|^2)} \right| \\ \leq \frac{2\|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)} \|v_k\|_{L^1(\Omega)}}{a^2} \left| |(\nabla G_\sigma * v_k)(x)| - |(\nabla G_\sigma * v_k)(y)| \right| \tag{2.12} \\ \leq \frac{2\|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)} \gamma_1^2 |\Omega|}{a^2} \int_\Omega |\nabla G_\sigma(x-z) - \nabla G_\sigma(y-z)| dz, \\ \forall x, y \in \Omega \quad \text{with } \gamma_1 = \sup_{k \in \mathbb{N}} \|v_k\|_{L^\infty(\Omega)},$$

and smoothness of the function $\nabla G_\sigma(\cdot)$, there exists a positive constant $C_G > 0$ independent of k such that

$$|p_k(x) - p_k(y)| \leq \frac{2\|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)} \gamma_1^2 |\Omega| C_G}{a^2} |x - y|, \quad \forall x, y \in \Omega.$$

Setting

$$C := \frac{2\|G_\sigma\|_{C^1(\bar{\Omega}-\Omega)} \gamma_1^2 |\Omega| C_G}{a^2}, \tag{2.13}$$

we see that

$$\{p_k(\cdot)\} \subset \mathfrak{S} = \left\{ h \in C^{0,1}(\Omega) \mid \begin{array}{l} |h(x) - h(y)| \leq C|x - y|, \forall x, y, \in \Omega, \\ 1 < \alpha \leq h(\cdot) \leq \beta \text{ in } \bar{\Omega}. \end{array} \right\}$$

Since $\max_{x \in \bar{\Omega}} |p_k(x)| \leq \beta$ and each element of the sequence $\{p_k\}_{k \in \mathbb{N}}$ has the same modulus of continuity, it follows that this sequence is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli Theorem the sequence $\{p_k\}_{k \in \mathbb{N}}$ is relatively compact with respect to the strong topology of $C(\bar{\Omega})$. Taking into account the estimate (2.12) and the fact that the set \mathfrak{S} is closed with respect to the uniform convergence and

$$(\nabla G_\sigma * v_k)(x) \rightarrow (\nabla G_\sigma * v)(x) \text{ as } k \rightarrow \infty, \forall x \in \Omega$$

by definition of the weak-* convergence in $L^\infty(\Omega)$, we deduce: $p_k(\cdot) \rightarrow p(\cdot)$ uniformly in $\bar{\Omega}$ as $k \rightarrow \infty$, where $p(x) = 1 + g(|(\nabla G_\sigma * v)(x)|)$ in Ω . The proof is complete. \square

3 Problem statement

The problem of restoration of cloud contaminated optical images is to reconstruct the original multi-band image \vec{u}_0 in the damage region D using the exact information about this image in $\Omega \setminus D$ (the undamaged region) and the texture (geometry) of the corresponding SAR image on the subset D . We begin with the following key assumptions.

Assumption 1. *The cloud contaminated optical image $\vec{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ and the corresponding SAR image $u_{SAR} \in BV(\Omega)$ are rigidly co-registered.*

Assumption 2. *The intensities u_i of all spectral channels for the retrieved image \vec{u} are subjected to the constraints $\gamma_{0,i} \leq u_i(x) \leq \gamma_{1,i}$ a.a. in Ω , where*

$$\gamma_{0,i} = \inf_{x \in \Omega \setminus D} u_{i,0}, \quad \gamma_{1,i} = \sup_{x \in \Omega \setminus D} u_{i,0}. \tag{3.1}$$

Assumption 3. *The topographic maps for all spectral channels $u_i, i \in \{1, \dots, M\}$ of the retrieved image $\vec{u} = [u_1, u_2, \dots, u_M]^t : \Omega \rightarrow \mathbb{R}^M$ have a similar geometrical structure to the topographic map of the SAR image $u_{SAR} \in BV(\Omega)$ albeit these images can have very different intensities.*

As follows from Assumption 3, all spectral channels of the retrieved image should share the geometry of the panchromatic SAR image u_{SAR} in D . It means that, due to the property $u_{SAR} \in BV(\Omega)$, for almost all points of almost all level sets of u_{SAR} we can define a vector field $\theta \in L^\infty(\Omega, \mathbb{R}^2)$ such that $\theta(x)$ has the direction of the normal to the level lines of u_{SAR} (see Remark 2.1 for the details). Therefore, the counterclockwise rotation of angle $\pi/2$, denoted by θ^\perp , represents the tangent vector to the level lines of u_{SAR} . In this case, if the spectral channels of \vec{u} share the geometry of the panchromatic image u_{SAR} , we have

$$(\theta^\perp, \nabla u_i) = 0 \text{ a.e. in } D, \forall i \in \{1, \dots, M\}. \tag{3.2}$$

We say that a function $\vec{u} = [u_1, u_2, \dots, u_M]^t : \Omega \rightarrow \mathbb{R}^M$ is the result of restoration of a cloud-corrupted image $\vec{u}_0 : D \rightarrow \mathbb{R}^M$ if for given regularisation parameters $\mu > 0, \alpha > 1$, and $\lambda > 0$, each

spectral component u_i is the solution of the following constrained minimisation problem with the nonstandard growth energy functional

$$\begin{aligned}
 (\mathcal{P}_i) \quad J_i(v, p) := & \int_{\Omega} \frac{1}{p(x)} |R_{\eta} \nabla v(x)|^{p(x)} dx + \frac{\mu}{\alpha} \int_{\Omega \setminus D} |v(x) - u_{0,i}(x)|^{\alpha} dx \\
 & + \lambda \int_D |(\theta^{\perp}, \nabla v)|^{\alpha} dx \longrightarrow \inf_{(v,p) \in \Xi_i}, \tag{3.3}
 \end{aligned}$$

where

- Ξ_i stands for the set of feasible solutions to the problem (3.3) which we define as follows

$$\Xi_i = \left\{ (v, p) \left| \begin{array}{l} v \in W^{1,p(\cdot)}(\Omega), p \in C(\overline{\Omega}), \\ 1 \leq \gamma_{0,i} \leq v(x) \leq \gamma_{1,i} \text{ a.a. in } \Omega, \\ p(x) = \mathfrak{F}(v(x)) \text{ in } \Omega. \end{array} \right. \right\}$$

Here, $W^{1,p(\cdot)}(\Omega)$ is the Sobolev-Orlicz space (for the details we refer to Appendix B),

$$\mathfrak{F}(v(x)) = 1 + g(|(\nabla G_{\sigma} * v)(x)|),$$

and $g: [0, \infty) \rightarrow (0, \infty)$ is the edge-stopping function which we take it in the form $g(t) = \frac{1}{1+(t/a)^2}$;

- $\theta \in L^{\infty}(D, \mathbb{R}^2)$ is a vector field such that

$$|\theta(x)| \leq 1 \text{ and } (\theta(x), \nabla u_{SAR}(x))_{\mathbb{R}^2} = |\nabla u_{SAR}(x)| \text{ a.e. in } D;$$

- $R_{\eta} \nabla v := \nabla v - \eta^2 \chi_D(\theta, \nabla v) \theta$ stands for the so-called directional gradient (see [8, 9]), and $\eta \in (0, 1)$ is a given threshold;
- $(G_{\sigma} * v)(x)$ denotes the convolution of function v with the two-dimensional Gaussian filter kernel G_{σ} .

Let us give a short motivation to the choice of the energy functional in the form (3.3). Since

$$R_{\eta} \nabla v = \nabla v \text{ in } \Omega \setminus D \text{ and } (1 - \eta^2)|\nabla v| \leq |R_{\eta} \nabla v| \leq |\nabla v| \text{ in } D \tag{3.4}$$

with a given $\eta \in (0, 1)$, it follows that this term can be considered as the regularisation in the Sobolev-Orlicz space $W^{1,p(\cdot)}(\Omega)$. On the other side, this term plays the role of a spacial data fidelity. Indeed, the main goal, we are going to follow in the restoration problem, is to preserve the following property: the geometry of each spectral channels in the damage zone of the retrieved image should be as close as possible to the geometry of the SAR image. Formally, it means that relations (3.2) have to be satisfied. Hence, the magnitude $\int_D |(\theta^{\perp}, \nabla v)|^{\alpha} dx$ must be small enough for each spectral channel, where θ stands for the vector field of unit normals to the topographic map of the SAR image u_{SAR} . In fact, we impose it in the energy functional (3.3) in the form of the last term. However, in order to enforce this property, we observe that the expression $R_{\eta} \nabla v$ can be reduced to $(1 - \eta^2)\nabla v$ in those places of the damage region D where v is collinear to the unit normal θ and to ∇v if ∇v is orthogonal to θ . Thus, gradients of the spectral intensities that are aligned/collinear θ are favoured as long as $|\theta| > 0$. Moreover, this property

is enforced by the exponent $p(x)$. In fact, the third term basically has a similar effect as the first one, so, in some practical implementations it can be omitted.

Since $p(x) \approx 1$ in places in Ω where edges or discontinuities are present in the spectral channel v , and $p(x) \approx 2$ in places where $v(x)$ is smooth or contains homogeneous features, the main benefit of the model (3.3) is the manner in which it accommodates the local image information. For the places where the spectral gradient is sufficiently large (i.e. likely edges), we deal with the so-called directional TV-based diffusion [8, 9], whereas the places where the gradient is close to zero (i.e. homogeneous regions), the model becomes isotropic. Specifically, the type of anisotropy at these ambiguous regions varies according to the strength of the gradient. This enables the model to have a much lower dependence on the approximation schemes for the variable exponent $p(x)$ and other thresholds. Apparently, the idea to involve a fractional norm of $W^{1,p(\cdot)}(\Omega)$ with a variable exponent $p(x)$ was firstly proposed in [7] in order to reduce the staircasing effect in the TV image restoration problem.

As for the second term in (3.3), it is the so-called data fidelity and it forces the minimiser v in domain $\Omega \setminus D$ to stay as close as possible to the spectral intensity of the cloud corrupted image $\tilde{u}_0 : D \rightarrow \mathbb{R}^M$.

Thus, the proposed model (3.3) provides a completely new approach to restoration of non-smooth multi-band optical images \tilde{u}_0 with the gap in damage region. The main characteristic feature of this model is that we involve into consideration the energy functional with nonstandard growth where the edge information for restoration of $\tilde{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ in D is accumulated in the variable exponent $p(x)$ and in the directional gradients $R_\eta \nabla$ which we derive from the SAR data. However, in contrast to [4, 11], we do not apply any selection procedure for approximation of the variable exponent $p(x)$ in (3.3).

We are now in a position to define what we mean by the solution of the restoration problem. Taking into account Assumptions (1)–(3) that we imposed above and the structure of the energy functionals $\mathcal{J}_i : \Xi_i \rightarrow \mathbb{R}$, we say that a multi-band image $\tilde{u}^0 = [u_1^0, u_2^0, \dots, u_M^0]^t : \Omega \rightarrow \mathbb{R}^M$ is the result of restoration of the cloud contaminated optical image $\tilde{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ if, for each index $i \in \{1, \dots, M\}$, the pair (u_i^0, p_i^0) , where $p_i^0 = 1 + g(|(\nabla G_\sigma * u_i^0)|)$ in Ω , is a solution of the constrained minimisation problem (3.3), that is,

$$(u_i^0, p_i^0) \in \Xi_i \text{ and } \mathcal{J}_i(u_i^0, p_i^0) = \inf_{(v,p) \in \Xi_i} \mathcal{J}_i(v, p).$$

4 Existence results

Our main intention in this section is to show that constrained minimisation problem (3.3) is consistent and admits at least one solution. Because of the specific form of the energy functional $J_i(v, p)$, the minimisation problem (3.3) is rather challenging (we can refer to [6, 7, 8, 11] for some specific details).

Following in many aspects the recent studies [15, 24], we give the following existence result.

Theorem 4.1 *For each $i = 1, \dots, M$ and given $\mu > 0, \lambda > 0, \eta \in (0, 1), \theta \in L^\infty(D, \mathbb{R}^2)$, and $\tilde{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$, the minimisation problem (3.3) admits at least one solution $(u_i^{rec}, p_i^{rec}) \in \Xi_i$.*

Proof. Since $\Xi_i \neq \emptyset$ and $0 \leq J_i(v, p) < +\infty$ for all $(v, p) \in \Xi_i$, it follows that there exists a non-negative value $\zeta \geq 0$ such that $\zeta = \inf_{(v,p) \in \Xi_i} J_i(v, p)$. Let $\{(v_k, p_k)\}_{k \in \mathbb{N}} \subset \Xi_i$ be a minimising

sequence to the problem (3.3), that is,

$$(v_k, p_k) \in \Xi_i, p_k(x) = 1 + g(|(\nabla G_\sigma * v_k)(x)|) \text{ in } \Omega \forall k \in \mathbb{N}, \text{ and } \lim_{k \rightarrow \infty} J_i(v_k, p_k) = \zeta.$$

So, without loss of generality, we can suppose that $J_i(v_k, p_k) \leq \zeta + 1$ for all $k \in \mathbb{N}$. From this and the initial assumptions, we deduce

$$\begin{aligned} \int_{\Omega} |v_k(x)|^\alpha dx &\leq \int_{\Omega} \gamma_{1,i}^\alpha dx \leq \gamma_{1,i}^\alpha |\Omega|, \quad \forall k \in \mathbb{N}, \\ \int_{\Omega} |R_\eta \nabla v_k(x)|^{p_k(x)} dx &\leq 2 \int_{\Omega} \frac{1}{p_k(x)} |R_\eta \nabla v_k(x)|^{p_k(x)} dx < 2(\zeta + 1), \quad \forall k \in \mathbb{N}, \end{aligned} \tag{4.1}$$

where $\sup_{k \in \mathbb{N}} [\sup_{x \in \Omega} p_k(x)] \leq 2$.

Utilising the fact that $v_k(x) \leq \gamma_{1,i}$ for almost all $x \in \Omega$, we infer the following estimate

$$\|v_k\|_{L^1(\Omega)} \leq \gamma_{1,i} |\Omega|, \quad \forall k \in \mathbb{N}.$$

Then arguing as in Lemma 2.1, it can be shown that $p_k \in C^{0,1}(\overline{\Omega})$ and

$$\alpha := 1 + \delta \leq p_k(x) \leq \beta := 2, \quad \forall x \in \Omega, \quad \forall k \in \mathbb{N}, \tag{4.2}$$

$$\text{with } \delta = \frac{a^2}{a^2 + \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}^2 \gamma_1^2 |\Omega|^2}. \tag{4.3}$$

Taking this fact into account, we deduce from (4.1), (4.2) and (B5) that

$$\begin{aligned} \|v_k\|_{W^{1,\alpha}(\Omega)} &= \left(\int_{\Omega} [|v_k(x)|^\alpha + |\nabla v_k(x)|^\alpha] dx \right)^{1/\alpha} \\ &\leq (1 + |\Omega|)^{1/\alpha} \left(\int_{\Omega} [|v_k(x)|^{p_k(x)} + |\nabla v_k(x)|^{p_k(x)}] dx + 2 \right)^{1/\alpha} \\ &\stackrel{\text{by (3.4)}}{\leq} \left(\frac{1 + |\Omega|}{(1 - \eta^2)^2} \right)^{1/\alpha} \left(\int_{\Omega} [|v_k(x)|^{p_k(x)} + |R_\eta \nabla v_k(x)|^{p_k(x)}] dx + 2 \right)^{1/\alpha} \\ &\stackrel{\text{by (4.1)}}{\leq} \left(\frac{1 + |\Omega|}{(1 - \eta^2)^2} \right)^{1/\alpha} (\gamma_1^2 |\Omega| + 2\zeta + 4)^{1/\alpha} \end{aligned}$$

uniformly with respect to $k \in \mathbb{N}$. Therefore, there exists a subsequence of $\{v_k\}_{k \in \mathbb{N}}$, still denoted by the same index, and a function $u_i^{rec} \in W^{1,\alpha}(\Omega)$ such that

$$\begin{aligned} v_k &\rightarrow u_i^{rec} \text{ strongly in } L^q(\Omega) \text{ for all } q \in [1, \alpha^*), \\ v_k &\rightharpoonup u_i^{rec} \text{ weakly in } W^{1,\alpha}(\Omega) \text{ as } k \rightarrow \infty, \end{aligned} \tag{4.4}$$

where, by Sobolev embedding theorem, $\alpha^* = \frac{2\alpha}{2-\alpha} = \frac{2+2\delta}{1-\delta} > 2$.

Moreover, passing to a subsequence if necessary, we have (see Proposition B.2 and Lemma 2.1):

$$v_k(x) \rightarrow u_i^{rec}(x) \text{ a.e. in } \Omega. \tag{4.5}$$

$$v_k \rightharpoonup u_i^{rec} \text{ weakly in } L^{p_k(\cdot)}(\Omega),$$

$$\nabla v_k \rightharpoonup \nabla u_i^{rec} \text{ weakly in } L^{p_k(\cdot)}(\Omega; \mathbb{R}^2),$$

$$p_k(\cdot) \rightarrow p_i^{rec}(\cdot) = 1 + g(|(\nabla G_\sigma * u_i^{rec})(\cdot)|) \text{ uniformly in } \overline{\Omega} \text{ as } k \rightarrow \infty,$$

where $u_i^{rec} \in W^{1,p_i^{rec}(\cdot)}(\Omega)$ with $p_i^{rec}(x) = 1 + g(|(\nabla G_\sigma * u_i^{rec})(x)|)$ in Ω .

Since $\gamma_{0,i} \leq v_k(x) \leq \gamma_{1,i}$ a.a. in Ω for all $k \in \mathbb{N}$, it follows from (4.5) that the limit function u_i^{rec} is also subjected to the same restriction. Thus, (u_i^{rec}, p_i^{rec}) is a feasible solution to minimisation problem (3.3).

Let us show that (u_i^{rec}, p_i^{rec}) is a minimiser of this problem. With that in mind, we note that in view of the obvious inequality

$$|v_k(x) - u_{0,i}(x)|^\alpha \leq 2^{\alpha-1} (|v_k(x)|^\alpha + |u_{0,i}(x)|^\alpha)$$

and the fact that $u_{0,i} \in L^2(\Omega \setminus D)$, we have: the sequence $\{v_k(x) - u_{0,i}(x)\}_{k \in \mathbb{N}}$ is bounded in $L^\alpha(\Omega \setminus D)$ and converges weakly in $L^\alpha(\Omega \setminus D)$ to $u_i^{rec} - u_{0,i}$. Hence, by Proposition B.2 (see (B17)), $u_i^{rec} - u_{0,i} \in L^{p^{rec}(\cdot)}(\Omega \setminus D)$ and

$$\liminf_{k \rightarrow \infty} \int_{\Omega \setminus D} |v_k(x) - u_{0,i}(x)|^\alpha dx \geq \int_{\Omega \setminus D} |u_i^{rec}(x) - u_{0,i}(x)|^\alpha dx. \tag{4.6}$$

As for the last terms in (3.3), in view of the weak convergence $v_k \rightarrow u_i^{rec}$ in $W^{1,\alpha}(\Omega)$ (see (4.4)), we have

$$\liminf_{k \rightarrow \infty} \int_D |(\theta^\perp, \nabla v_k)|^\alpha dx \geq \int_D |(\theta^\perp, \nabla u_i^{rec})|^\alpha dx dx. \tag{4.7}$$

It remains to notice that due to the properties (4.1), (4.4) and the fact that $\theta \in L^\infty(D, \mathbb{R}^2)$, the sequence $\{|R_\eta \nabla v_k| \in L^{p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$ is bounded and weakly convergence to $|R_\eta \nabla u_i^{rec}|$ in $L^\alpha(\Omega)$. Hence, by Proposition B.2, the following lower semicontinuous property

$$\liminf_{k \rightarrow \infty} \int_\Omega \frac{1}{p_k(x)} |R_\eta \nabla v_k(x)|^{p_k(x)} dx \geq \int_\Omega \frac{1}{p(x)} |R_\eta \nabla u_i^{rec}(x)|^{p(x)} dx \tag{4.8}$$

holds true.

As a result, utilising relations (4.6), (4.7) and (4.8), we finally obtain

$$\zeta = \inf_{(v,p) \in \Xi_i} J_i(v,p) = \lim_{k \rightarrow \infty} J_i(v_k,p_k) = \liminf_{k \rightarrow \infty} J_i(v_k,p_k) \geq J_i(u_i^{rec}, p_i^{rec}).$$

Thus, (u_i^{rec}, p_i^{rec}) is a minimiser to the problem (3.3), whereas its uniqueness remains as an open question. □

5 Iterative algorithm based on the variational convergence of extremal problems

It is clear that because of the nonstandard energy functional and its non-convexity, constrained minimisation problem (3.3) is not trivial in its practical implementation. The main difficulty in its study comes from the state constraints

$$1 \leq \gamma_{0,i} \leq v(x) \leq \gamma_{1,i} \text{ a.a. in } \Omega, \quad p(x) = 1 + g(|(\nabla G_\sigma * v)(x)|)$$

that we impose on the set of feasible solutions Ξ_i . This motivates us to pass to some relaxation (we refer to [16, 28, 33] where the similar approach has been utilised). In view of this, we propose the following iteration procedure which is based on the concept of relaxation of extremal problems and their variational convergence [25, 26, 27, 30].

At the first step, we set up

$$p_0(x) = \begin{cases} 1 + g(|(\nabla G_\sigma * u_{0,i})(x)|), & \text{if } x \in \Omega \setminus D, \\ 1 + g(|(\nabla G_\sigma * u_{SAR})(x)|), & \text{if } x \in D, \end{cases} \tag{5.1}$$

$$u^0 = \underset{v \in \mathcal{B}_{i,p_0(\cdot)}}{\text{Argmin}} J_i(v, p_0(\cdot)). \tag{5.2}$$

Here, $\mathcal{B}_{i,p(\cdot)} = \{v \in W^{1,p(\cdot)}(\Omega) : 1 \leq \gamma_{0,i} \leq v(x) \leq \gamma_{1,i} \text{ a.a. in } \Omega\}$.

Then, for each $k \geq 1$, we set

$$p_k(x) = 1 + g(|(\nabla G_\sigma * u^{k-1})(x)|), \quad \forall x \in \Omega, \quad u^k = \underset{v \in \mathcal{B}_{i,p_k(\cdot)}}{\text{Argmin}} J_i(v, p_k(\cdot)). \tag{5.3}$$

Before proceeding further, we set

$$\mathfrak{S} = \left\{ h \in C(\overline{\Omega}) \mid \begin{array}{l} |h(x) - h(y)| \leq C|x - y|, \quad \forall x, y \in \Omega, \\ \alpha := 1 + \delta \leq h(x) \leq \beta := 2, \quad \forall x \in \Omega, \end{array} \right\} \tag{5.4}$$

where $C > 0$ and $\delta > 0$ are defined by (2.13) and (4.3), respectively.

Arguing as in the proof of Theorem 4.1 and using the convexity arguments, it can be shown that, for each $p(\cdot) \in \mathfrak{S}$, there exists a unique element $u_i^{0,p(\cdot)} \in \mathcal{B}_{i,p(\cdot)}$ such that $u_i^{0,p(\cdot)} = \underset{v \in \mathcal{B}_{i,p(\cdot)}}{\text{Argmin}} J_i(v, p(\cdot))$. Moreover, it can be shown that, for given $i = 1, \dots, M$, $\mu > 0$, $\lambda > 0$, $\eta \in (0, 1)$, $u_{SAR} \in BV(\Omega)$, and $\tilde{u}_0 \in L^2(\Omega \setminus D, \mathbb{R}^M)$, the sequence $\{u^k \in W^{1,p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$ is compact with respect to the weak topology of $W^{1,\alpha}(\Omega)$, whereas the exponents $\{p_k\}_{k \in \mathbb{N}}$ are compact with respect to the strong topology of $C(\overline{\Omega})$.

We say that a pair $(\widehat{u}_i, \widehat{p}_i)$ is a weak solution to the original problem (3.3) if

$$\begin{aligned} \widehat{u}_i &= \underset{v \in \mathcal{B}_{i,\widehat{p}_i(\cdot)}}{\text{Argmin}} J_i(v, \widehat{p}_i(\cdot)), \quad \widehat{u}_i \in \mathcal{B}_{i,\widehat{p}_i(\cdot)}, \\ \widehat{p}_i(x) &= 1 + g(|(\nabla G_\sigma * \widehat{u}_i)(x)|), \quad \forall x \in \Omega. \end{aligned} \tag{5.5}$$

Remark 5.1 *The relation between a weak solution and a solution to the problem (3.3) is very intricate. Since the uniqueness of solutions to (3.3) is a rather questionable option, it follows that, in principle, these definitions can lead to the different pairs in Ξ_i . As immediately follows from (5.5), a weak solution is a merely feasible one to the original problem. However, if the problem (3.3) admits a unique solution $(u_i^0, p_i^0) \in \Xi_i$, then (5.5) implies that this pair is a weak solution. On the other hand, if $(u_i^*, p_i^*) \in \Xi_i$ is some solution to (3.3), then setting in (5.1) $p_0(x) = p_i^*(x)$ for all $x \in \Omega$, we immediately arrive at the conclusion: $(u^k, p_k) = (u_i^*, p_i^*)$ for all $k \in \mathbb{N}$ and, therefore, (u_i^*, p_i^*) is a weak solution to the above problem.*

Our main goal in this section is to establish the existence of a weak solution to the original problem (3.3) and show that it can be attained by some iterative algorithm.

We begin with some technical results.

Lemma 5.1 *For given $\mu > 0$, $\lambda > 0$, $\eta \in (0, 1)$, $u_{SAR} \in BV(\Omega)$ and $\tilde{u}_0 \in L^2(\Omega \setminus D, \mathbb{R}^M)$, the sequence $\{u^k \in W^{1,p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$ is compact with respect to the weak topology of $W^{1,\alpha}(\Omega)$.*

Proof. To begin with, let us show that the sequence of minimisers $\{u^k\}_{k \in \mathbb{N}}$ is bounded in the sense of condition (B10). Let $\widehat{u} \in C^1(\overline{\Omega})$ be an arbitrary function such that $\gamma_{0,i} \leq \widehat{u}(x) \leq \gamma_{1,i}$ in Ω . Since

$$J_i(u^k, p_k(\cdot)) \leq J_i(\widehat{u}, p_k(\cdot)), \quad \forall k = 0, 1, 2, \dots \tag{5.6}$$

and

$$\begin{aligned} \int_{\Omega} \frac{1}{p_k(x)} |R_{\eta} \nabla \widehat{u}(x)|^{p_k(x)} dx &\leq \int_{\Omega} \frac{1}{p_k(x)} |\nabla \widehat{u}(x)|^{p_k(x)} dx \\ &\leq \int_{\Omega} \frac{1}{p_k(x)} (1 + \|\widehat{u}\|_{C^1(\overline{\Omega})})^{p_k(x)} dx \leq \frac{|\Omega|}{\alpha} (1 + \|\widehat{u}\|_{C^1(\overline{\Omega})})^2, \\ \int_{\Omega \setminus D} \frac{1}{p_k(x)} |\widehat{u}(x) - u_{0,i}(x)|^{\alpha} dx &\leq 2 \int_{\Omega \setminus D} [(1 + \|\widehat{u}\|_{C^1(\overline{\Omega})})^{\alpha} + |u_{0,i}(x)|^{\alpha}] dx \\ &\leq 2|\Omega \setminus D| (1 + \|\widehat{u}\|_{C^1(\overline{\Omega})})^{\alpha} + 2|\Omega|^{\frac{2-\alpha}{2}} \|u_{0,i}\|_{L^2(\Omega)}^{\alpha}, \end{aligned}$$

it follows that

$$\sup_{k \in \mathbb{N}} J_i(u^k, p_k(\cdot)) \leq \sup_{k \in \mathbb{N}} J_i(\widehat{u}, p_k(\cdot)) \leq \widehat{C}$$

with some constant $\widehat{C} > 0$.

From this and estimate (3.4), we deduce

$$\int_{\Omega} |u^k(x)|^{\alpha} dx \leq \gamma_{1,i}^{\alpha} |\Omega|, \quad \forall k \in \mathbb{N}, \tag{5.7}$$

$$\int_{\Omega} |\nabla u^k(x)|^{p_k(x)} dx \leq 2 \int_{\Omega} \frac{1}{p_k(x)} |\nabla u^k(x)|^{p_k(x)} dx < \frac{2}{(1 - \eta^2)^2} \widehat{C}, \quad \forall k \in \mathbb{N}. \tag{5.8}$$

Then estimate (B5) implies that the sequence $\{u^k\}_{k \in \mathbb{N}}$ is bounded in $W^{1,\alpha}(\Omega)$. So, its weak compactness is a direct consequence of the reflexivity of $W^{1,\alpha}(\Omega)$. \square

We notice that boundedness of the sequence $\{u^k\}_{k \in \mathbb{N}}$ in $W^{1,\alpha}(\Omega)$ and compactness of the embedding $W^{1,\alpha}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, \frac{2\alpha}{2-\alpha})$ imply the existence of element $u^* \in W^{1,\alpha}(\Omega)$ such that, up to a subsequence,

$$u^k(x) \rightarrow u^*(x) \text{ a.e. in } \Omega, \tag{5.9}$$

$$u^k \rightharpoonup u^* \text{ in } L^{\alpha}(\Omega), \text{ and } \nabla u^k \rightharpoonup \nabla u^* \text{ in } L^{\alpha}(\Omega; \mathbb{R}^2). \tag{5.10}$$

Then using (5.9) and passing to the limit in two-side inequality $\gamma_{0,i} \leq u^k(x) \leq \gamma_{1,i}$, we obtain

$$\gamma_{0,i} \leq u^*(x) \leq \gamma_{1,i} \text{ for a.a. } x \in \Omega. \tag{5.11}$$

Utilising this fact together with the pointwise convergence (5.9), by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathfrak{F}(u^k(x)) &= 1 + \frac{a^2}{a^2 + (\lim_{k \rightarrow \infty} |(\nabla G_{\sigma} * u^k)(x)|)^2} \\ &= 1 + \frac{a^2}{a^2 + \left(\left| (\nabla G_{\sigma} * \lim_{k \rightarrow \infty} u^k)(x) \right| \right)^2} = \mathfrak{F}(u^*(x)), \quad \forall x \in \Omega. \end{aligned} \tag{5.12}$$

Since, by Arzelà–Ascoli theorem, the set $\{p_k = 1 + g(|(\nabla G_\sigma * u^{k-1})(x)|)\}_{k \in \mathbb{N}}$ is compact with respect to the strong topology of $C(\overline{\Omega})$, it follows from (5.12) (see also the proof of Lemma 2.1) that

$$p_k \rightarrow p^* = \mathfrak{F}(u^*(x)) \text{ strongly in } C(\overline{\Omega}) \text{ as } k \rightarrow \infty, \text{ and } p^* \in \mathfrak{G}. \tag{5.13}$$

Then properties (5.9)–(5.13) and Proposition B.2 imply:

$$u^* \in \mathcal{B}_{i,p^*(\cdot)} = \left\{ u \in W^{1,p^*(\cdot)}(\Omega) : 1 \leq \gamma_{0,i} \leq u(x) \leq \gamma_{1,i} \text{ a.a. in } \Omega \right\}. \tag{5.14}$$

Thus, $(u^*, p^*) \in \mathcal{B}_{i,p^*(\cdot)} \times \mathfrak{G}$ is a cluster point of iterative procedure (5.3) with respect to the convergence (5.9)–(5.10), (5.13).

The main result of this section can be stated as follows:

Theorem 5.1 *Let $\mu > 0, \lambda > 0, \eta \in (0, 1), u_{SAR} \in BV(\Omega)$ and $\tilde{u}_0 \in L^2(\Omega \setminus D, \mathbb{R}^M)$ be given data. Then, for each $i \in \{1, \dots, M\}$, the sequence of approximated solutions $\{(u^k, p_k)\}_{k \in \mathbb{N}}$ possesses the following asymptotic properties:*

$$u^k(x) \rightarrow \tilde{u}_i(x) \text{ a.e. in } \Omega, \tag{5.15}$$

$$u^k \rightharpoonup \tilde{u}_i \text{ in } L^\alpha(\Omega), \text{ and } \nabla u^k \rightharpoonup \nabla \tilde{u}_i \text{ in } L^\alpha(\Omega; \mathbb{R}^2), \tag{5.16}$$

$$p_k \rightarrow \tilde{p}_i = \mathfrak{F}(\tilde{u}_i) \text{ strongly in } C(\overline{\Omega}) \text{ as } k \rightarrow \infty, \tag{5.17}$$

where $(\tilde{u}_i, \tilde{p}_i)$ is a weak solution to the original problem (3.3), that is,

$$\tilde{u}_i \in \mathcal{B}_{i,\tilde{p}_i(\cdot)}, \quad \tilde{u}_i = \underset{v \in \mathcal{B}_{i,\tilde{p}_i(\cdot)}}{\text{Argmin}} J_i(v, \tilde{p}_i(\cdot)),$$

and, in addition, the following variational property holds true

$$\begin{aligned} \lim_{k \rightarrow \infty} J_i(u^k, p_k(\cdot)) &= \lim_{k \rightarrow \infty} \left[\inf_{v \in \mathcal{B}_{i,p_k(\cdot)}} J_i(v, p_k(\cdot)) \right] \\ &= \inf_{v \in \mathcal{B}_{i,\tilde{p}_i(\cdot)}} J_i(v, \tilde{p}_i(\cdot)) = J_i(\tilde{u}_i, \tilde{p}_i(\cdot)). \end{aligned} \tag{5.18}$$

Proof. Let $(\tilde{u}_i, \tilde{p}_i)$ be a cluster point of the sequence $\{(u^k, p_k)\}_{k \in \mathbb{N}}$ with respect to the convergence (5.15)–(5.17). The existence of such pair immediately follows from Lemma 5.1 and reasoning given above. Let’s assume the converse – namely, there is a function $z \in \mathcal{B}_{i,\tilde{p}_i(\cdot)}$ such that

$$J_i(z, \tilde{p}_i(\cdot)) = \inf_{v \in \mathcal{B}_{i,\tilde{p}_i(\cdot)}} J_i(v, \tilde{p}_i(\cdot)) < J_i(\tilde{u}_i, \tilde{p}_i(\cdot)). \tag{5.19}$$

Using the procedure of the direct smoothing, we set

$$u_\varepsilon(x) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^2} K\left(\frac{x-s}{\varepsilon}\right) \tilde{z}(s) ds,$$

where $\varepsilon > 0$ is a small parameter, K is a positive compactly supported smooth function with properties

$$K \in C_0^\infty(\mathbb{R}^2), \quad \int_{\mathbb{R}^2} K(x) dx = 1, \quad \text{and } K(x) = K(-x),$$

and \tilde{z} is zero extension of z outside of Ω .

Since $z \in W^{1,\tilde{p}(\cdot)}(\Omega)$ and $\tilde{p}(x) \geq \alpha = 1 + \delta$ in Ω , it follows that $z \in W^{1,\alpha}(\Omega)$. Then,

$$u_\varepsilon \in C_0^\infty(\mathbb{R}^2) \text{ for each } \varepsilon > 0, \\ u_\varepsilon \rightarrow z \text{ in } L^\alpha(\Omega), \quad \nabla u_\varepsilon \rightarrow \nabla z \text{ in } L^\alpha(\Omega; \mathbb{R}^2) \tag{5.20}$$

by the classical properties of smoothing operators (see [18]). From this, we deduce that

$$u_\varepsilon(x) \rightarrow z(x) \text{ a.e. in } \Omega. \tag{5.21}$$

Moreover, taking into account the estimates

$$u_\varepsilon(x) = \int_{\mathbb{R}^2} K(y) \tilde{z}(x - \varepsilon y) dy \leq \gamma_{1,i} \int_{\mathbb{R}^2} K(y) dy = \gamma_{1,i}, \\ u_\varepsilon(x) \geq \int_{y \in \varepsilon^{-1}(x-\Omega)} K(y) \tilde{z}(x - \varepsilon y) dy \geq \gamma_{0,i} \int_{y \in \varepsilon^{-1}(x-\Omega)} K(y) dy \geq \gamma_{0,i},$$

we see that each element u_ε is subjected to the pointwise constraints

$$\gamma_{0,i} \leq u_\varepsilon(x) \leq \gamma_{1,i} \text{ a.a. in } \Omega, \quad \forall \varepsilon > 0.$$

Since, for each $\varepsilon > 0$, $u_\varepsilon \in W^{1,p_k(\cdot)}(\Omega)$ for all $k \in \mathbb{N}$, it follows that $u_\varepsilon \in \mathcal{B}_{i,p_k(\cdot)}$, that is, each element of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is a feasible solution to all approximating problems $(\inf_{v \in \mathcal{B}_{i,p_k(\cdot)}} J_i(v, p_k(\cdot)))$. Hence,

$$J_i(u^k, p_k(\cdot)) \leq J_i(u_\varepsilon, p_k(\cdot)), \quad \forall \varepsilon > 0, \forall k = 0, 1, \dots \tag{5.22}$$

Further we notice that

$$\liminf_{k \rightarrow \infty} J_i(u^k, p_k(\cdot)) \geq J_i(\tilde{u}_i, \tilde{p}_i(\cdot)) \tag{5.23}$$

by Proposition B.2 and Fatou’s lemma, and

$$\lim_{k \rightarrow \infty} J_i(u_\varepsilon, p_k(\cdot)) = \lim_{k \rightarrow \infty} \int_{\Omega} \frac{1}{p_k(x)} |R_\eta \nabla u_\varepsilon(x)|^{p_k(x)} dx \\ + \frac{\mu}{\alpha} \int_{\Omega \setminus D} |u_\varepsilon(x) - u_{0,i}(x)|^\alpha dx + \lambda \int_D |(\theta^\perp, \nabla u_\varepsilon)|^\alpha dx. \tag{5.24}$$

Since

$$\frac{1}{p_k(x)} |R_\eta \nabla u_\varepsilon(x)|^{p_k(x)} \rightarrow \frac{1}{\tilde{p}_i(x)} |R_\eta \nabla u_\varepsilon(x)|^{\tilde{p}_i(x)} \text{ uniformly in } \Omega \text{ as } k \rightarrow \infty,$$

it follows from the Lebesgue dominated convergence theorem and (5.24) that

$$\lim_{k \rightarrow \infty} J_i(u_\varepsilon, p_k(\cdot)) = J_i(u_\varepsilon, \tilde{p}_i(\cdot)), \quad \forall \varepsilon > 0. \tag{5.25}$$

As a result, passing to the limit in (5.22) and utilising properties (5.23)–(5.25), we obtain

$$J_i(\tilde{u}_i, \tilde{p}_i(\cdot)) \leq J_i(u_\varepsilon, \tilde{p}_i(\cdot)) = \int_{\Omega} \frac{1}{\tilde{p}_i(x)} |R_\eta \nabla u_\varepsilon(x)|^{\tilde{p}_i(x)} dx \\ + \frac{\mu}{\alpha} \int_{\Omega \setminus D} |u_\varepsilon(x) - u_{0,i}(x)|^\alpha dx + \lambda_2 \int_D |(\theta^\perp, \nabla u_\varepsilon)|^\alpha dx, \tag{5.26}$$

for all $\varepsilon > 0$. Taking into account the pointwise convergence (see (5.21) and property (5.20))

$$\begin{aligned} |R_\eta \nabla u_\varepsilon(x)|^{\tilde{p}_i(x)} &\rightarrow |R_\eta \nabla z(x)|^{\tilde{p}_i(x)}, \\ |u_\varepsilon(x) - u_{0,i}(x)|^\alpha &\rightarrow |z(x) - u_{0,i}(x)|^\alpha, \\ |(\theta^\perp, \nabla u_\varepsilon)|^\alpha &\rightarrow |(\theta^\perp, \nabla z)|^\alpha \end{aligned}$$

as $\varepsilon \rightarrow 0$, and the fact that, for ε small enough,

$$\begin{aligned} |R_\eta \nabla u_\varepsilon(x)|^{\tilde{p}_i(x)} &\leq (1 + |\nabla z(\cdot)|)^{\tilde{p}_i(\cdot)} \in L^1(\Omega) \quad \text{a.e. in } \Omega, \\ |u_\varepsilon(\cdot) - u_{0,i}(\cdot)|^\alpha &\leq \left[2(1 + |z(\cdot)|)^\alpha + 2(1 + |u_{0,i}(\cdot)|)^\alpha \right] \in L^1(\Omega) \quad \text{a.e. in } \Omega \setminus D, \\ |(\theta^\perp, \nabla u_\varepsilon)|^\alpha &\leq \|\theta\|_{L^\infty(D, \mathbb{R}^2)} (1 + |\nabla z(\cdot)|)^{\tilde{p}_i(\cdot)} \in L^1(\Omega) \quad \text{a.e. in } \Omega, \end{aligned}$$

we can pass to the limit in (5.26) as $\varepsilon \rightarrow 0$ by the Lebesgue dominated convergence theorem. This yields

$$J_i(\tilde{u}_i, \tilde{p}_i(\cdot)) \leq \lim_{\varepsilon \rightarrow 0} J_i(u_\varepsilon, \tilde{p}_i(\cdot)) = J_i(z, \tilde{p}_i(\cdot)).$$

Combining this inequality with (5.26) and (5.19), we finally get

$$J_i(z, \tilde{p}_i(\cdot)) = \inf_{v \in \mathcal{B}_{i, \tilde{p}_i(\cdot)}} J_i(v, \tilde{p}_i(\cdot)) < J_i(\tilde{u}_i, \tilde{p}_i(\cdot)) \leq J_i(z, \tilde{p}_i(\cdot)),$$

that leads us into conflict with the initial assumption. Thus,

$$J_i(\tilde{u}_i, \tilde{p}_i(\cdot)) = \inf_{v \in \mathcal{B}_{i, \tilde{p}_i(\cdot)}} J_i(v, \tilde{p}_i(\cdot)) \tag{5.27}$$

and, therefore, $(\tilde{u}_i, \tilde{p}_i)$ is a weak solution to the original problem (3.3). As for the variational property (5.18), it is a direct consequence of (5.27) and (5.25). \square

6 Optimality conditions

To characterise the solution $u^{0,p(\cdot)} \in \mathcal{B}_{i,p(\cdot)}$ of the approximating optimisation problem $\langle \inf_{v \in \mathcal{B}_{i,p(\cdot)}} J_i(v, p(\cdot)) \rangle$, we check that the functional $J_i(v, p(\cdot))$ is Gâteaux differentiable with respect to v , that is,

$$\begin{aligned} &\lim_{t \rightarrow 0} \frac{J_i(u^{0,p(\cdot)} + tv, p(\cdot)) - J_i(u^{0,p(\cdot)}, p(\cdot))}{t} \\ &= \int_\Omega (|R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-2} R_\eta \nabla u^{0,p(\cdot)}(x), R_\eta \nabla v(x)) \, dx \\ &\quad + \mu \int_{\Omega \setminus D} |u^{0,p(\cdot)}(x) - u_{0,i}(x)|^{\alpha-2} u^{0,p(\cdot)}(x) v(x) \, dx \\ &\quad + \alpha \lambda \int_\Omega |(\theta^\perp, \nabla u^{0,p(\cdot)})|^{\alpha-1} (\theta^\perp, \nabla v) \, dx, \quad \forall v \in W^{1,p(\cdot)}(\Omega). \end{aligned} \tag{6.1}$$

To this end, we note that

$$\frac{|R_\eta \nabla u^{0,p(\cdot)}(x) + tR_\eta \nabla v(x)|^{p(x)} - |R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t} \rightarrow (|R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-2} R_\eta \nabla u^{0,p(\cdot)}(x), R_\eta \nabla v(x)) \text{ as } t \rightarrow 0$$

almost everywhere in Ω . Since, by convexity,

$$|\xi|^p - |\eta|^p \leq 2p (|\xi|^{p-1} + |\eta|^{p-1}) |\xi - \eta|,$$

it follows that

$$\begin{aligned} & \left| \frac{|R_\eta \nabla u^{0,p(\cdot)}(x) + tR_\eta \nabla v(x)|^{p(x)} - |R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t} \right| \\ & \leq 2 (|R_\eta \nabla u^{0,p(\cdot)}(x) + tR_\eta \nabla v(x)|^{p(x)-1} + |R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-1}) |R_\eta \nabla v(x)| \\ & \leq \text{const} (|R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-1} + |R_\eta \nabla v(x)|^{p(x)-1}) |R_\eta \nabla v(x)|. \end{aligned} \tag{6.2}$$

Taking into account that

$$\begin{aligned} & \int_\Omega |R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-1} |R_\eta \nabla v(x)| \, dx \\ & \stackrel{\text{by (B.18)}}{\leq} 2 \| |R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-1} \|_{L^{p'(\cdot)}(\Omega)} \|R_\eta \nabla v(x)\|_{L^{p(\cdot)}(\Omega)} \\ & \leq 2 \| \nabla u^{0,p(\cdot)}(x) \|_{L^{p'(\cdot)}(\Omega, \mathbb{R}^2)} \| \nabla v(x) \|_{L^{p(\cdot)}(\Omega, \mathbb{R}^2)}, \end{aligned}$$

and $\int_\Omega |\nabla v(x)|^{p(x)} \, dx \stackrel{\text{by (B5)}}{\leq} \| \nabla v \|_{L^{p(\cdot)}(\Omega, \mathbb{R}^2)}^2 + 1$, we see that the right hand side of inequality (6.2) is an $L^1(\Omega)$ function. Therefore,

$$\begin{aligned} & \int_\Omega \frac{|R_\eta \nabla u^{0,p(\cdot)}(x) + tR_\eta \nabla v(x)|^{p(x)} - |R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t} \, dx \\ & \rightarrow \int_\Omega (|R_\eta \nabla u^{0,p(\cdot)}(x)|^{p(x)-2} R_\eta \nabla u^{0,p(\cdot)}(x), R_\eta \nabla v(x)) \, dx \text{ as } t \rightarrow 0 \end{aligned}$$

by the Lebesgue dominated convergence theorem.

Utilising similar arguments to the rest terms in (3.3), we deduce that the representation (6.1) for the Gâteaux differential of $J_i(\cdot, p(\cdot))$ at the point $u^{0,p(\cdot)} \in \mathcal{B}_{i,p(\cdot)}$ is valid.

Thus, in order to derive some optimality conditions for the minimising element $u^{0,p(\cdot)} \in \mathcal{B}_{i,p(\cdot)}$ to the problem $\inf_{v \in \mathcal{B}_{i,p(\cdot)}} J_i(v, p(\cdot))$, we note that $\mathcal{B}_{i,p(\cdot)}$ is a non-empty convex subset of $W^{1,p(\cdot)}(\Omega)$ and the objective functional $J_i(\cdot, p(\cdot)) : \mathcal{B}_{i,p(\cdot)} \rightarrow \mathbb{R}$ is strictly convex. Hence, the well-known classical result (see [32, Theorem 1.1.3]) and representation (6.1) lead us to the following conclusion.

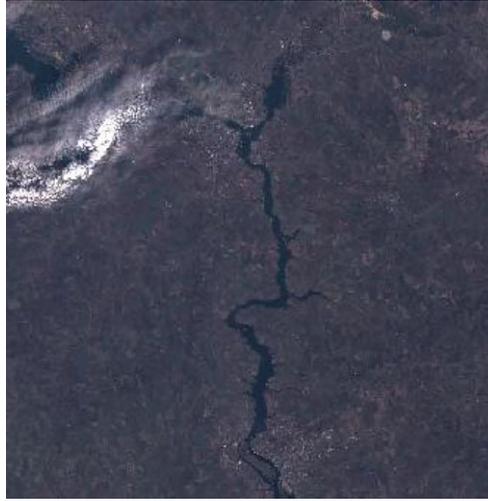


FIGURE 1. Optical image from 2021/040.

Theorem 6.1. *Let $p_k(\cdot) \in \mathfrak{S}$ be an exponent given by the iterative rule (5.3). Then, the unique minimiser $u^k \in \mathcal{B}_{i,p_k(\cdot)}$ to the approximating problem $\inf_{v \in \mathcal{B}_{i,p_k(\cdot)}} J_i(v, p_k(\cdot))$ is characterised by*

$$\begin{aligned} & \int_{\Omega} \left(\left| R_{\eta} \nabla u^k(x) \right|^{p_k(x)-2} R_{\eta} \nabla u^k(x), R_{\eta} \nabla v(x) - R_{\eta} \nabla u^k(x) \right) dx \\ & \quad + \mu \int_{\Omega \setminus D} \left| u^k(x) - u_{0,i}(x) \right|^{\alpha-2} u^k(x) (v(x) - u^k(x)) dx \\ & \quad + \alpha \lambda \int_{\Omega} \left| (\theta^{\perp}, \nabla u^{0,p(\cdot)}) \right|^{\alpha-1} (\theta^{\perp}, \nabla v - \nabla u^k) dx \geq 0, \quad \forall v \in \mathcal{B}_{i,p_k(\cdot)}. \end{aligned} \quad (6.3)$$

7 Numerical experiments

In order to illustrate the proposed algorithm for the restoration of the cloud-corrupted satellite multispectral optical images, we have provided some numerical experiments. As input data, we have used a series of optical images and a radar image that were delivered from two twin satellites, Sentinel-2A and Sentinel-2B. Each of the optical image is corrupted by clouds with a different level of degradation (see Figures 1, 2, 3, 4). The corresponding SAR image is shown in Figure 5.

The results of restoration after 5 iterations of the proposed algorithm are shown in Figures 6, 7, 8, 9 for $\eta = 0.8$, $\mu = 10$, and $\lambda = 20$. As for the exponent α , we define it following the rule (2.10) with $\delta = a^2 \left[a^2 + \|G_{\sigma}\|_{C^1(\overline{\Omega-\Omega})}^2 255^2 |\Omega|^2 \right]^{-1}$ and $a = 0.01$.

Comparing the restored images and the contaminated ones, we could see that the texture of original images is well preserved. However, some details in the damage zones are blurred. This problem will be addressed in the following research.



FIGURE 2. Optical image from 2021/032.



FIGURE 3. Optical image from 2021/041.

8 Conclusion

This paper proposes a novel restoration model for the satellite optical images that is based on their co-registration with SAR images of the same region and the solution of a special variational problem with nonstandard growth objective functional. The characteristic feature of the restoration problem is the fact that the optical images are supposed to be corrupted and there is a subdomain (the so-called damage region) where we do not have any information about the brightness of such images. Instead, we propose to make use of the structure of the SAR images of the same regions. The novelty of our approach is that we involve into consideration the objective functional with nonstandard growth $p(x)$, where the variable exponent is unknown a priori



FIGURE 4. Optical image from 2021/0406.

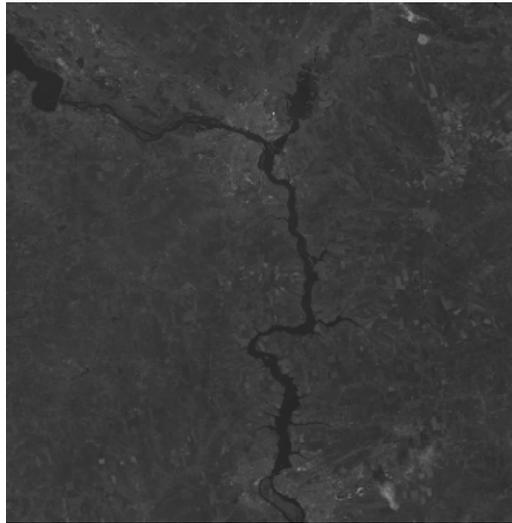


FIGURE 5. The SAR image of the same territory.

and it directly depends on the texture of an image that has to be restored. In order to study the consistency of the proposed non-convex minimisation problem, we develop a special technique and supply this approach by the rigorous mathematical substantiation.

However, this problem is not trivial from its practical implementation and has limited performance from numerical point of view. With that in mind, we propose an alternating minimisation procedure which is based on the concept of relaxation of extremal problems with the state constraints and their variational convergence. We show that at each step of the iteration procedure, we obtain a strictly convex optimisation problem which admits a unique solution. It is established that the so-defined sequence of approximations is precompact and each convergent subsequence



FIGURE 6. Result of its restoration.

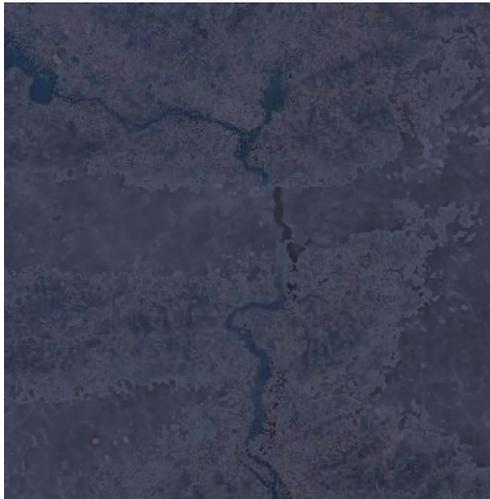


FIGURE 7. Result of its restoration.

leads to the so-called weak solutions of the original problem. So, following this way some weak solutions of the restoration problem can be attained in the suitable topology by a special iterative algorithm.

As for the numerical scheme for practical implementation of the proposed procedure, we use the straightforward finite difference method (see [12] for some details) and it will be described in the forthcoming paper. Besides of that, in the second part of this paper we mainly focus on the solvability issues of variational inequality (6.3), its qualitative analysis and the study of its asymptotic behaviour as k tends to ∞ .

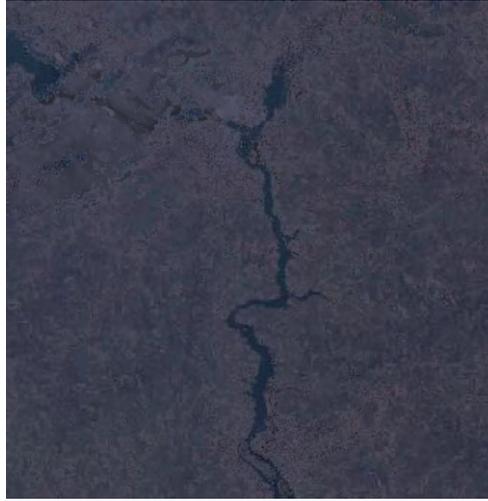


FIGURE 8. Result of its restoration.

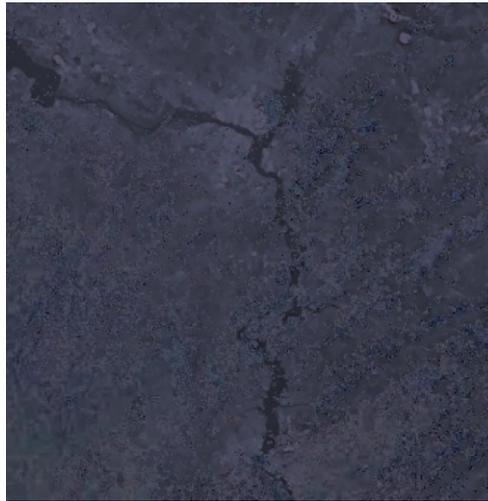


FIGURE 9. Result of its restoration.

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Conflicts of interest

The authors declare that all of them have contributed to the submission. Moreover, they declare that they have no known competing financial interest or personal relationships that could have appeared to influence the work reported in this paper.

References

- [1] AMBROSIO, L., FUSCO, N. & PALLARA, D. (2000) *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, New York.
- [2] ATTOUCH, H., BUTTAZZO, G. & MICHAÏLLE, G. (2006) *Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization*, SIAM, Philadelphia.
- [3] ANTIL, H. & BARTELS, S. (2017) Spectral approximation of fractional PDEs in image processing and phase field modeling. *Comput. Methods Appl. Math.* **17**(4), 661–678.
- [4] ANTIL, H. & RAUTENBERG, C. N. (2019) Sobolev spaces with non-Muckenhoupt weights, fractional elliptic operators, and applications. *SIAM J. Math. Anal.* **51**(3), 2479–2503.
- [5] BALLESTER, C., CASELLES, V., IGUAL, L., VERDERA, J. & ROUGÉ, B. (2006) A variational model for P+XS image fusion. *Int. J. Comput. Vis.* **69**, 43–58.
- [6] BLOMGREN, P. (1998) *Total Variation Methods for Restoration of Vector Valued Images*. PhD thesis, pp. 384–387.
- [7] BLOMGREN, P., CHAN, T. F., MULET, P. & WONG, C. (1997) Total variation image restoration: numerical methods and extensions. In: *Proceedings of the 1997 IEEE International Conference on Image Processing*, III, pp. 384–387.
- [8] BUNGER, L., COOMES, D. A., EHRHARDT, M. J., RASCH, J., REISENHOFER, R. & SCHÖNLIEB, C.-B. (2018) Blind image fusion for hyperspectral imaging with the directional total variation. *Inverse Problems* **34**(4), Article 044003.
- [9] BUNGER, L. & EHRHARDT, M. J. (2020) Robust image reconstruction with misaligned structural information. *IEEE Access* **8**, 222944–222955.
- [10] CAFFARELLI, L. & SILVESTRE, L. (2016) An extension problem related to the fractional Laplacian. *Commun. Partial Differ. Equations* **32**(7–9), 767–907.
- [11] CHEN, Y., LEVINE, S. & RAO, M. (2006) Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.* **66**(4), 1383–1406.
- [12] CHEN, Y., LEVINE, S. & STANICH, J. (2004) Image Restoration via Nonstandard Diffusion. Technical Report, Duquesne University, 13 p.
- [13] CRUZ-URIBE, D. V. & FIORENZA, A. (2013) *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Birkhäuser, New York.
- [14] D’APICE, C., DE MAIO, U. & KOGUT, P. I. (2018) Thermistor problem: Multi-dimensional modelling, optimization, and approximation. In: *Proceedings of the 32nd European conference on modelling and simulation*, May 22nd–May 25th, 2018, Wilhelmshaven, Germany.
- [15] D’APICE, C., DE MAIO, U. & KOGUT, P. I. (2020) An indirect approach to the existence of quasi-optimal controls in coefficients for multi-dimensional thermistor problem. In: V. A. Sadovnichiy and M. Zgurovsky (editors), *Contemporary Approaches and Methods in Fundamental Mathematics and Mechanics*, Chapter 24, Springer, pp. 489–522.
- [16] DE MAIO, U., KOGUT, P. I. & MANZO, R. (2020) Asymptotic analysis of an optimal boundary control problem for ill-posed elliptic equation in domains with Rugous boundary. *Asymp. Anal.* **118**(3), 209–234. doi: [10.3233/ASY-191570](https://doi.org/10.3233/ASY-191570).
- [17] DIENING, L., HARJULEHTO, P., HÄSTÖ, P. & RUZICKA, M. (2011) *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, New York.
- [18] EVANS, L. C. & GARIÉPY, R. F. (1992) *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton.
- [19] GARKUSHA, I. N., HNATUSHENKO, V. V. & VASYLIEV, V. V. (2017) Research of influence of atmosphere and humidity on the data of radar imaging by Sentinel-1. In: *IEEE 37th International Conference on Electronics and Nanotechnology (ELNANO)*. doi: [10.1109/elnano.2017.7939787](https://doi.org/10.1109/elnano.2017.7939787).

- [20] GARKUSHA, I. N., HNATUSHENKO, V. V. & VASYLIEV, V. V. (2017) Using Sentinel-1 data for monitoring of soil moisture. In: *IEEE International Geoscience and Remote Sensing Symposium (IGARSS)*. doi: [10.1109/igarss.2017.8127291](https://doi.org/10.1109/igarss.2017.8127291).
- [21] HNATUSHENKO, V. V., KOGUT, P. I. & UVAROW, M. V. (2021) On flexible co-registration of optical and SAR satellite images. In: S. Babichev, V. Lytvynenko, W. Wójcik and S. Vyshemyrskaya (editors). *Lecture Notes in Computational Intelligence and Decision Making*, Advances in Intelligent Systems and Computing, Springer, pp. 515–534.
- [22] HNATUSHENKO, V. V., KOGUT, P. I. & UVAROW, M. V. (2022) Variational approach for rigid co-registration of optical/SAR satellite images in agricultural areas. *J. Comput. Appl. Math.* **400**, Id:113742.
- [23] IRONY, R., COHEN-OR, D. & LISCHINSKI, D. (2005) Colorization by example. In: *Proceedings of the Sixteenth Eurographics Conference on Rendering Techniques, EGSR 2005*. Switzerland, Eurographics Association, pp. 201–210.
- [24] KOGUT, P. I. (2019) On optimal and quasi-optimal controls in coefficients for multi-dimensional thermistor problem with mixed Dirichlet-Neumann boundary conditions. *Control Cybern.* **48**(1), 31–68.
- [25] KOGUT, P. I. (2014) On approximation of an optimal boundary control problem for linear elliptic equation with unbounded coefficients. *Discrete Contin. Dyn. Syst. Ser. A* **34**(5), 2105–2133.
- [26] KOGUT, P. I. (1996) Variational S-convergence of minimization problems. Part I. Definitions and basic properties. *Problemy Upravleniya i Informatiki (Avtomatika)* **5**, 29–42.
- [27] KOGUT, P. I. (1997) S-convergence of the conditional optimization problems and its variational properties. *Problemy Upravleniya i Informatiki (Avtomatika)* **4**, 64–79.
- [28] KOGUT, P. I., KUPENKO, O. P. & MANZO, R. (2020) On regularization of an optimal control problem for ill-posed nonlinear elliptic equations. *Abstract Appl. Anal.* 2020, Article ID 7418707, 15 p.
- [29] KOGUT, P. I. & LEUGERING, G. (2011) *Optimal Control Problems for Partial Differential Equations on Reticulated Domains. Approximation and Asymptotic Analysis*, Systems and Control, Birkhäuser Verlag, Boston.
- [30] KOGUT, P. I. & LEUGERING, G. (2001) On S-homogenization of an optimal control problem with control and state constraints. *Zeitschrift für Analysis und ihre Anwendung* **20**(2), 395–429.
- [31] LEVIN, A., LISCHINSKI, D. & WEISS, Y. (2004) Colorization using optimization. *ACM Trans. Graph.* **23**, 689–694.
- [32] LIONS, J.-L. (1971) *Optimal Control of Systems Governed by Partial Differential Equations*, Springer, Berlin.
- [33] MANZO, R. (2020) On Tikhonov regularization of optimal Neumann boundary control problem for an ill-posed strongly nonlinear elliptic equation with an exponential type of non-linearity. *Differ. Integral Equations* **33**(3–4), 139–162.
- [34] MULLER, M. U., SHEPHERD, J. D. & DYMOND, J. R. (2015) Support vector machine classification of woody patches in New Zealand from synthetic aperture radar and optical data, with LiDAR training. *J. Appl. Remote Sens.* **9**(1), Id:095984.
- [35] RONCAL, L. & STINGA, P. R. (2016) Fractional Laplacian on the torus. *Commun. Contemp. Math.* **18**(3), Id:1550033.
- [36] SARADJIAN, M. R. & HOSSEINI, M. (2011) Comparison of optical, radar and hybrid soil moisture estimation models using experimental data. *J. Appl. Remote Sens.* **5**(1), Id:053524.
- [37] SAPIRO, G. (2005) Inpainting the colors. In: *ICIP 2005. IEEE International Conference on Image Processing*, Vol. 60, pp. 698–701.
- [38] SAPIRO, G. & YATZIV, L. (2006) Fast image and video colorization using chrominance blending. *IEEE Trans. Image Process.* **15**, 1120–1129.
- [39] STINGA, P. R. & TORREA, J. L. (2010) Extension problem and Harnack's inequality for some fractional operators. *Commun. Partial Differ. Equations* **35** (11), 2092–2122.
- [40] SÝKORA, D., BURIÁNEK, J. & ZÁRA, J. (2004) Unsupervised colorization of black-and-white cartoons. In: *Proceedings of the 3rd International Symposium on Non-photorealistic Animation and Rendering, NPAR 2004*, ACM, New York, NY, USA, pp. 121–127.

- [41] ZHIKOV, V. V. (2008) Solvability of the three-dimensional thermistor problem. *Proc. Steklov Inst. Math.* **281**, 98–111.
- [42] ZHIKOV, V. V. (2011) On variational problems and nonlinear elliptic equations with nonstandard growth conditions. *J. Math. Sci.* **173**(5), 463–570.

Appendix A. *BV*-Space

Let $C_0^\infty(\Omega)$ be the infinitely differentiable functions with a compact support in Ω . By $BV(\Omega)$ we denote the space of all functions $u \in L^1(\Omega)$ for which their distributional derivatives are representable by finite Borel measures in Ω , that is,

$$\int_{\Omega} u \frac{\partial \phi}{\partial x_i} dx = - \int_{\Omega} \phi dD_i u, \quad \forall \phi \in C_0^\infty(\Omega), \quad i = 1, 2$$

for some \mathbb{R}^2 -valued measure $Du = (D_1u, D_2u) \in \mathcal{M}^2(\Omega)$. It can be shown that $BV(\Omega)$, endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$$

is a Banach space, where

$$|Du|(\Omega) := \int_{\Omega} d|Du| = \sup \left\{ \int_{\Omega} u \operatorname{div} \varphi dx : \varphi \in C_0^1(\Omega; \mathbb{R}^2), |\varphi(x)| \leq 1 \text{ for } x \in \Omega \right\} \quad (\text{A1})$$

stands for the total variation of u in Ω . It is clear that $|Du|(\Omega) = \int_{\Omega} |\nabla u| dx$ if u is continuously differentiable in Ω .

The following embedding results for BV -function play a crucial role for qualitative analysis of variational problems that we study in this paper.

Proposition A.1 [2, p. 378] *Let Ω be an open bounded Lipschitz subset of \mathbb{R}^2 . Then, the embedding $BV(\Omega; \mathbb{R}^M) \hookrightarrow L^2(\Omega; \mathbb{R}^M)$ is continuous and the embeddings $BV(\Omega; \mathbb{R}^M) \hookrightarrow L^p(\Omega; \mathbb{R}^M)$ are compact for all p such that $1 \leq p < 2$. Moreover, there exists a constant $C_{em} > 0$ which depends only on Ω and p such that for all u in $BV(\Omega; \mathbb{R}^M)$,*

$$\left(\int_{\Omega} |u|^p dx \right)^{1/p} \leq C_{em} \|u\|_{BV(\Omega; \mathbb{R}^M)}, \quad \forall p \in [1, 2].$$

Further information on BV -functions and their properties can be found in [1, 2].

Appendix B. On Orlicz Spaces

Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 < \alpha \leq p(x) \leq \beta < \infty$ a.e. in Ω , where α and β are given constants. Let $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ be the corresponding conjugate exponent. It is clear that

$$1 \leq \underbrace{\frac{\beta}{\beta-1}}_{\beta'} \leq p'(x) \leq \underbrace{\frac{\alpha}{\alpha-1}}_{\alpha'} \text{ a.e. in } \Omega,$$

where β' and α' stand for the conjugates of constant exponents. Denote by $L^{p(\cdot)}(\Omega)$ the set of all measurable functions $f(x)$ on Ω such that $\int_{\Omega} |f(x)|^{p(x)} dx < \infty$. Then, $L^{p(\cdot)}(\Omega)$ is a reflexive

separable Banach space with respect to the Luxemburg norm (see [13, 17] for the details)

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \rho_p(\lambda^{-1}f) \leq 1 \}, \tag{B1}$$

where $\rho_p(f) := \int_{\Omega} |f(x)|^{p(x)} dx$.

It is well-known that $L^{p(\cdot)}(\Omega)$ is reflexive provided $\alpha > 1$, and its dual is $L^{p'(\cdot)}(\Omega)$, that is, any continuous functional $F = F(f)$ on $L^{p(\cdot)}(\Omega)$ has the form (see [42, Lemma 13.2])

$$F(f) = \int_{\Omega} fg \, dx, \quad \text{with } g \in L^{p'(\cdot)}(\Omega).$$

3.7 As for the infimum in (B1), we have the following result (for reader's convenience, we furnish it with the proof).

Proposition B.1 *The infimum in (B1) is attained if $\rho_p(f) > 0$. Moreover*

$$\text{if } \lambda_* := \|f\|_{L^{p(\cdot)}(\Omega)} > 0, \text{ then } \rho_p(\lambda_*^{-1}f) = 1. \tag{B2}$$

Proof. Indeed, as follows from (B1), $\|f\|_{L^{p(\cdot)}(\Omega)} = 0$ if and only if $f(x) = 0$ a.e. in Ω , i.e. $\rho_p(f) = 0$. Assume that $\rho_p(f) > 0$. We define a function $\psi : [0, \infty) \rightarrow \mathbb{R}$ as

$$\psi(s) := \rho_p(\lambda^{-1}f) = \int_{\Omega} |sf(x)|^{p(x)} dx.$$

Since $\psi(0) = 0$, $\lim_{s \rightarrow \infty} \psi(s) = +\infty$, and

$$\frac{d}{ds} \psi(s) = \int_{\Omega} p(x)s^{p(x)-1} |f(x)|^{p(x)} dx > 0,$$

it follows that $\psi = \psi(s)$ is a monotonically increasing function. Hence, there exists a positive value $\lambda_* > 0$ such that

$$\psi(\lambda_*^{-1}) = 1 \quad \text{and} \quad \psi(\lambda^{-1}) = \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \leq 1 \quad \forall \lambda \in [\lambda_*, +\infty).$$

Therefore,

$$\inf \{ \lambda > 0 : \psi(\lambda^{-1}) \leq 1 \} = \lambda_* \quad \text{and} \quad \int_{\Omega} \left| \frac{f(x)}{\lambda_*} \right|^{p(x)} dx = 1. \tag{B3}$$

As a result, we deduce from (B3) and (B1) that $\lambda_* = \|f\|_{L^{p(\cdot)}(\Omega; \mathbb{R}^2)}$.

Taking this result and condition $1 \leq \alpha \leq p(x) \leq \beta$ into account, we see that

$$\begin{aligned} \frac{1}{\lambda_*^\beta} \int_{\Omega} |f(x)|^{p(x)} dx &\leq \int_{\Omega} \left| \frac{f(x)}{\lambda_*} \right|^{p(x)} dx \leq \frac{1}{\lambda_*^\alpha} \int_{\Omega} |f(x)|^{p(x)} dx, \\ \frac{1}{\lambda_*^\beta} \int_{\Omega} |f(x)|^{p(x)} dx &\leq 1 \leq \frac{1}{\lambda_*^\alpha} \int_{\Omega} |f(x)|^{p(x)} dx. \end{aligned}$$

Hence, (see [13, 17, 41] for the details)

$$\begin{aligned} \|f\|_{L^{p(\cdot)}(\Omega)}^\alpha &\leq \int_{\Omega} |f(x)|^{p(x)} dx \leq \|f\|_{L^{p(\cdot)}(\Omega)}^\beta, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} \geq 1, \\ \|f\|_{L^{p(\cdot)}(\Omega)}^\beta &\leq \int_{\Omega} |f(x)|^{p(x)} dx \leq \|f\|_{L^{p(\cdot)}(\Omega)}^\alpha, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1, \end{aligned} \tag{B4}$$

and, therefore,

$$\|f\|_{L^{p(\cdot)}(\Omega)}^\alpha - 1 \leq \int_{\Omega} |f(x)|^{p(x)} dx \leq \|f\|_{L^{p(\cdot)}(\Omega)}^\beta + 1, \quad \forall f \in L^{p(\cdot)}(\Omega), \tag{B5}$$

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \int_{\Omega} |f(x)|^{p(x)} dx, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} = 1. \tag{B6}$$

The following estimates are well-known (see, for instance, [13, 17, 41]): if $f \in L^{p(\cdot)}(\Omega)$ then

$$\|f\|_{L^\alpha(\Omega)} \leq (1 + |\Omega|)^{1/\alpha} \|f\|_{L^{p(\cdot)}(\Omega)}, \tag{B7}$$

$$\|f\|_{L^{p(\cdot)}(\Omega)} \leq (1 + |\Omega|)^{1/\beta'} \|f\|_{L^\beta(\Omega)}, \quad \beta' = \frac{\beta}{\beta - 1}, \quad \forall f \in L^\beta(\Omega). \tag{B8}$$

Let $\{p_k\}_{k \in \mathbb{N}} \subset C^{0,\delta}(\overline{\Omega})$, with some $\delta \in (0, 1]$, be a given sequence of exponents. Hereinafter in this subsection, we assume that

$$p, p_k \in C^{0,\delta}(\overline{\Omega}) \text{ for } k = 1, 2, \dots, \text{ and } p_k(\cdot) \rightarrow p(\cdot) \text{ uniformly in } \overline{\Omega} \text{ as } k \rightarrow \infty. \tag{B9}$$

We associate with this sequence the following collection $\{f_k \in L^{p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$. The characteristic feature of this set of functions is that each element f_k lives in the corresponding Orlicz space $L^{p_k(\cdot)}(\Omega)$. We say that the sequence $\{f_k \in L^{p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$ is bounded if (see [29, Section 6.2])

$$\limsup_{k \rightarrow \infty} \int_{\Omega} |f_k(x)|^{p_k(x)} dx < +\infty. \tag{B10}$$

Definition B.1 A bounded sequence $\{f_k \in L^{p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$ is weakly convergent in the variable Orlicz space $L^{p_k(\cdot)}(\Omega)$ to a function $f \in L^{p(\cdot)}(\Omega)$, where $p \in C^{0,\delta}(\overline{\Omega})$ is the limit of $\{p_k\}_{k \in \mathbb{N}} \subset C^{0,\delta}(\overline{\Omega})$ in the uniform topology of $C(\overline{\Omega})$, if

$$\lim_{k \rightarrow \infty} \int_{\Omega} f_k \varphi dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2). \tag{B11}$$

We make use of the following result (we refer to [42, Lemma 13.3] for comparison) concerning the lower semicontinuity property of the variable $L^{p_k(\cdot)}$ -norm with respect to the weak convergence in $L^{p_k(\cdot)}(\Omega)$.

Proposition B.2 *If a bounded sequence $\{f_k \in L^{p_k(\cdot)}(\Omega)\}_{k \in \mathbb{N}}$ converges weakly in $L^\alpha(\Omega)$ to f for some $\alpha > 1$, then $f \in L^{p(\cdot)}(\Omega)$, $f_k \rightharpoonup f$ in variable $L^{p_k(\cdot)}(\Omega)$, and*

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |f_k(x)|^{p_k(x)} dx \geq \int_{\Omega} |f(x)|^{p(x)} dx. \tag{B12}$$

Proof. In view of the fact that

$$\begin{aligned} & \left| \int_{\Omega} |f_k(x)|^{p_k(x)} dx - \int_{\Omega} \frac{p(x)}{p_k(x)} |f_k(x)|^{p_k(x)} dx \right| \\ & \leq \|p_k - p\|_{C(\bar{\Omega})} \int_{\Omega} \frac{1}{p_k(x)} |f_k(x)|^{p_k(x)} dx \\ & \leq \frac{\|p_k - p\|_{C(\bar{\Omega})}}{\alpha} \int_{\Omega} |f_k(x)|^{p_k(x)} dx \xrightarrow{\text{by (B10)}} 0 \text{ as } k \rightarrow \infty, \end{aligned}$$

we see that

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |f_k(x)|^{p_k(x)} dx = \liminf_{k \rightarrow \infty} \int_{\Omega} \frac{p(x)}{p_k(x)} |f_k(x)|^{p_k(x)} dx. \tag{B13}$$

Using the Young inequality $ab \leq |a|^p/p + |b|^{p'}/p'$, we have

$$\int_{\Omega} \frac{p(x)}{p_k(x)} |f_k(x)|^{p_k(x)} dx \geq \int_{\Omega} p(x) f_k(x) \varphi(x) dx - \int_{\Omega} \frac{p(x)}{p'_k(x)} |\varphi(x)|^{p'_k(x)} dx, \tag{B14}$$

for $p'_k(x) = p_k(x)/(p_k(x) - 1)$ and any $\varphi \in C_0^\infty(\mathbb{R}^2)$.

Then passing to the limit in (B14) as $k \rightarrow \infty$ and utilising property (B9) and the fact that

$$\lim_{k \rightarrow \infty} \int_{\Omega} f_k(x) \varphi(x) dx = \int_{\Omega} f(x) \varphi(x) dx \quad \text{for all } \varphi \in L^{a'}(\Omega), \tag{B15}$$

we obtain

$$\liminf_{k \rightarrow \infty} \int_{\Omega} |f_k(x)|^{p_k(x)} dx \geq \int_{\Omega} p(x) f(x) \varphi(x) dx - \int_{\Omega} \frac{p(x)}{p'(x)} |\varphi(x)|^{p'(x)} dx. \tag{B16}$$

Since the last inequality is valid for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ and $C_0^\infty(\mathbb{R}^2)$ is dense subset of $L^{p'(\cdot)}(\Omega)$, it follows that this relation also holds true for $\varphi \in L^{p'(\cdot)}(\Omega)$. So, taking $\varphi = |f(x)|^{p(x)-2} f(x)$, we arrive at the announced inequality (B12). So, as a consequence of (B12) and estimate (B5), we get: $f \in L^{p(\cdot)}(\Omega)$.

To end the proof, it remains to observe that relation (B15) holds true for $\varphi \in C_0^\infty(\mathbb{R}^2)$ as well. From this the weak convergence $f_k \rightharpoonup f$ in the variable Orlicz space $L^{p_k(\cdot)}(\Omega)$ follows. \square

Remark B.1 Arguing in a similar manner and using, instead of (B14), the estimate

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \frac{1}{p_k(x)} |f_k(x)|^{p_k(x)} dx \geq \int_{\Omega} p(x) f(x) \varphi(x) dx - \int_{\Omega} \frac{1}{p'_k(x)} |\varphi(x)|^{p'_k(x)} dx,$$

it can be shown that the lower semicontinuity property (B12) can be generalised as follows

$$\liminf_{k \rightarrow \infty} \int_{\Omega} \frac{1}{p_k(x)} |f_k(x)|^{p_k(x)} dx \geq \int_{\Omega} \frac{1}{p(x)} |f(x)|^{p(x)} dx. \tag{B17}$$

The following result can be viewed as an analogous of the Hölder inequality in Lebesgue spaces with variable exponents (for the details we refer to [13, 17]).

Proposition B.3 *If $f \in L^{p(\cdot)}(\Omega)^N$ and $g \in L^{p'(\cdot)}(\Omega)^N$, then $(f, g) \in L^1(\Omega)$ and*

$$\int_{\Omega} (f, g) \, dx \leq 2 \|f\|_{L^{p(\cdot)}(\Omega)^N} \|g\|_{L^{p'(\cdot)}(\Omega)^N}. \tag{B18}$$

Appendix C. Sobolev Spaces with Variable Exponent

We recall here well-known facts concerning the Sobolev spaces with variable exponent. Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 < \alpha \leq p(x) \leq \beta < \infty$ a.e. in Ω , where α and β are given constants. We associate with it the so-called Sobolev-Orlicz space

$$W^{1,p(\cdot)}(\Omega) := \left\{ u \in W^{1,1}(\Omega) : \int_{\Omega} [|u(x)|^{p(x)} + |\nabla u(x)|^{p(x)}] \, dx < +\infty \right\} \tag{C1}$$

and equip it with the norm $\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega; \mathbb{R}^2)}$.

It is well-known that, in general, unlike classical Sobolev spaces, smooth functions are not necessarily dense in $W = W^{1,p(\cdot)}(\Omega)$. Hence, with variable exponent $p = p(x)$ ($1 < \alpha \leq p \leq \beta$) it can be associated another Sobolev space,

$$H = H^{1,p(\cdot)}(\Omega) \text{ as the closure of the set } C^\infty(\overline{\Omega}) \text{ in } W^{1,p(\cdot)}(\Omega)\text{-norm.}$$

Since the identity $W = H$ is not always valid, it makes sense to say that an exponent $p(x)$ is regular if $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.

The following result reveals the important property that guarantees the regularity of exponent $p(x)$.

Proposition C.1 *Assume that there exists $\delta \in (0, 1]$ such that $p \in C^{0,\delta}(\overline{\Omega})$. Then the set $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, and, therefore, $W = H$.*

Proof. Let $p \in C^{0,\delta}(\overline{\Omega})$ be a given exponent. Since

$$\lim_{t \rightarrow 0} |t|^\delta \log(|t|) = 0 \quad \text{with } \delta \in (0, 1], \tag{C2}$$

it follows from the Hölder continuity of $p(\cdot)$ that

$$|p(x) - p(y)| \leq C|x - y|^\delta \leq \left[\sup_{x,y \in \Omega} \frac{|x - y|^\delta}{\log(|x - y|^{-1})} \right] \omega(|x - y|), \quad \forall x, y \in \Omega, \tag{C3}$$

where $\omega(t) = C/\log(|t|^{-1})$, and $C > 0$ is some positive constant.

Then, property (C2) implies that $p(\cdot)$ is a log-Hölder continuous function. So, to deduce the density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$ it is enough to refer to Theorem 13.10 in [42]. \square