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PARTITIONED GROUPS AND THE ADDITIVE STRUCTURE OF CENTRALIZER NEAR-RINGS

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If G is a finite group and A is a group of automorphisms of G, the "centralizer" near-ring $C(A,G)$ consists of the identity-preserving maps from G to itself which commute with the action of A. The main concern of this paper will be with the additive structure of $C(A,G)$ in the case that this near-ring is semisimple.

As was shown by C. J. Maxson and K. C. Smith [**6**], $C(A,G)$ is semisimple precisely when the double centralizers $C_G(C_A(x))$, $x \in G^{\#}$, partition G. Since it is these double centralizers that determine the additive structure of the simple components of $C(A,G)$, our problem is a purely group theoretic one.

Partitioned groups have been studied in considerable depth by several authors and, to a large extent, are classified. However, not all partitions of a group arise from a group of automorphisms in the manner described above (the dihedral groups of order $4n$, for example, have a central involution fixed by every automorphism), and the classification does not seem to point to any obvious distinguishing features of those that do. Indeed, a definite shortcoming of the present paper is the fact that we have found no effective way of exploiting the special nature of these partitions so as to avoid invoking the deepest part of the classification, namely Suzuki's result on partitioned groups with trivial Fitting subgroup [**8**]. On the positive side however, although our interest is not so much with the structure of G as that of the double centralizers, Suzuki's theorem provides a complete description of the pairs (A,G) for which $F(G)=1$ and $C(A,G)$ is semisimple (Theorem 2.2).

The case that G has non-trivial centre requires no such deep results. Here we show that if the partition is non-trivial, then either G has prime exponent or it contains an abelian normal subgroup of prime index. (Contrast this with the obvious fact that any p-group all of whose elements of order not equal to p lie in a proper subgroup admits a non-trivial partition.) In the latter situation, the structure of G and the action of A can be pinned down fairly precisely.

The final case, when G has trivial centre but non-trivial Fitting subgroup, is less satisfactorily treated here but at least it can be asserted (using results of Baer and Kegel) that G is Frobenius with a Frobenius kernel which is either abelian or of prime exponent (Lemma 4.2).

As for the semisimple near-ring $C(A,G)$, we conclude from these results that if $C(A,G)$ is not simple, then as additive groups, each of its simple components is either abelian, of prime exponent, or is a direct product of copies of a subgroup H of G (where H is either a Frobenius group or a Frobenius complement). (Theorem 5.1.)

1. Preliminaries

All groups will be assumed to be finite and notation corresponds to that of Gorenstein's book [3].

First we summarize the results of Maxson and Smith which motivated this investigation.

Lemma 1.1 [6]. *Let G be a finite group and $A \leqq \mathrm{Aut}\, G$.*

(a) *$C(A, G)$ is simple if and only if all centralizers $C_A(x)$, $x \in G^{\#}$, are conjugate in A.*

(b) *$C(A, G)$ is semisimple if and only if for all $x, y \in G^{\#}$, either*

$$C_G(C_A(x)) \cap C_G(C_A(y)) = 1 \qquad \text{or} \qquad C_A(x) = C_A(y).$$

(c) *The action of A on $G^{\#}$ induces an action on the set of double centralizers $C_G(C_A(x))$, $x \in G$. If $x_1, x_2, \ldots, x_k \in G$ such that $\{C_G(C_A(x_i)) : 1 \leqq i \leqq k\}$ is a complete set of orbit representatives, then*

$$C(A, G) \simeq \bigoplus_{i=1}^{k} C(A_i, G_i)$$

where $G_i = C_G(C_A(x_i))$ and $A_i = N_A(C_A(x_i))/C_A(x_i) = N_A(G_i)/C_A(G_i)$. Moreover, each A_i acts semiregularly on $G_i^{\#}$ so that the $C(A_i, G_i)$ are simple.

Remark. It is immediate from statement (c) above and a theorem of Thompson (Theorem 10.2.1 of [3]) that if $A_i \neq 1$, then G_i (and hence, the additive group of the component $C(A_i, G_i)$) is nilpotent. One of the main objectives here will be to obtain the more precise statement that if $A_i \neq 1$ (and $k > 1$), then the additive group of $C(A_i, G_i)$ is either abelian or of prime exponent.

Since statement (b) of the lemma characterizes semisimplicity of $C(A, G)$ purely in terms of the action of A, we shall omit reference to $C(A, G)$ throughout most of the argument and instead simply refer to the action of A on $G^{\#}$ as "partitive" when the double centralizers partition G. We shall call (A, G) a partitive pair.

The two major characterizations of partitioned groups which we shall use are:

Lemma 1.2 [8]. *If G is a finite partitioned group and $F(G) = 1$, then $G \simeq PGL_2(p^n)$, $PSL_2(p^n)$ (where p is prime and $p^n \geqq 4$) or $Sz(2^{2n+1})$.*

Lemma 1.3 [5]. *Suppose G is a finite partitioned group with $F(G) \neq 1$ but $Z(G) = 1$. Then either G is Frobenius or G is isomorphic to the symmetric group S_4.*

The next fact is a trivial consequence of partitivity but it will be used continually throughout the paper.

Lemma 1.4. *If (A, G) is a partitive pair and A is not semiregular on $G^{\#}$, then*

(a) *$C_A(x) \neq 1$ for all $x \in G^{\#}$.*

(b) *If $x, y \in G^{\#}$ with $C_A(x) \leqq C_A(y)$, then $C_A(x) = C_A(y)$. In particular, if $x^n \neq 1$, then $C_A(x) = C_A(x^n)$.*

As the first application of Lemma 1.4, we eliminate the singular case $G=S_4$ from Lemma 1.3.

Lemma 1.5. *Suppose (A,G) is a partitive pair with $F(G)\neq 1$ but $Z(G)=1$. Then G is a Frobenius group.*

Proof. If not, then by Lemma 1.3, $G\simeq S_4$ so $\mathrm{Aut}\,G=\mathrm{Inn}\,G\simeq G$ and we may identify A as a subgroup of G. Now $C_G(x)=\langle x\rangle$ if x is any 3-cycle so since A is certainly not semiregular on $G^{\#}$, $\langle x\rangle=C_A(x)$ by Lemma 1.4 (a). Hence $A_4\leqq A$. On the other hand, if $x=(1234)$, then $(14)(23)\in C_A(x^2)$ but $(14)(23)$ does not centralize x, contradicting Lemma 1.4 (b).

Lemma 1.6. *Suppose (A,G) is a partitive pair and H and K are non-trivial A-invariant subgroups of G with $G=HK$ and $H\cap K=1$. Then A is semiregular on $G^{\#}$.*

Proof. If $h\in H^{\#}$ and $k\in K^{\#}$, then for $\alpha\in C_A(hk)$, we have $h^{-1}h^{\alpha}=kk^{-\alpha}\in H\cap K=1$ so $C_A(hk)=C_A(h)\cap C_A(k)$. By Lemma 1.4(b), $C_A(h)=C_A(k)$ so since h and k are arbitrary, the result follows.

If p is a prime, let $H_p(G)$ be the Hughes subgroup generated by all elements of G of order other than p. The following simple lemma is an adaptation of a basic tool in the characterization of partitioned groups:

Lemma 1.7. *Let (A,G) be a partitive pair and assume $x\in G^{\#}$.*

(a) *If x has composite order, $C_G(x)\leqq C_G(C_A(x))$.*

(b) *If x has prime order p, $H_p(C_G(x))\leqq C_G(C_A(x))$.*

Proof. Let $y\in C_G(x)$ so $(xy)^{|y|}=x^{|y|}$ and $(xy)^{|x|}=y^{|x|}$. If $|x|\neq|y|$ then Lemma 1.4(b) yields that $C_A(xy)=C_A(x)$ or $C_A(y)$ so $C_A(x)=C_A(y)$ and $y\in C_G(C_A(x))$. If $|x|=|y|$ and $p\mid |x|$, we can replace x by x^p or y by y^p in this argument and obtain the same conclusion unless $|x|=p=|y|$. The result follows.

2. The case $F(G)=1$

In this section, we obtain precise information about both A and G in the case that A is partitive on $G^{\#}$ and G has no non-trivial abelian normal subgroups. As mentioned previously, the conclusion is a corollary of Suzuki's theorem (Lemma 1.2).

Lemma 2.1. *Suppose G is a finite non-solvable group with a group of automorphisms A acting partitively on $G^{\#}$. If A contains $\mathrm{Inn}(G)$, then $G\simeq SL_2(2^n)$ for some $n\geqq 2$.*

Proof. Let x be an involution in G. If $y\in C_G(x)$ with $|y|\neq 2$, then by Lemma 1.7, y and x belong to the same component of the induced partition. Therefore $y\in C_G(C_A(x))\leqq Z(C_G(x))$ (since $\mathrm{Inn}(G)\leqq A$). If z is any involution in $C_G(x)$, then $|yz|\neq 2$ so yz (and hence z) lies in $Z(C_G(x))$. Thus, $C_G(x)$ is abelian. On the other hand, if no such y exists, $C_G(x)$ is an elementary abelian 2-group.

Now Suzuki's classification [7] together with the Feit–Thompson theorem yield that $G=H\times Z$ where $H\simeq SL_2(2^n)$, $n\geqq 2$ and Z is abelian of odd order. Since $H=G'$ and $Z=Z(G)$, Lemma 1.6 implies $Z=1$ (else A is semiregular on $G^{\#}$, contradicting $\mathrm{Inn}(G)\leqq A$).

Note that since the centralizers of non-identity elements in $SL_2(2^n)$ are abelian, the inner automorphism group of $SL_2(2^n)$ is actually partitive on $SL_2(2^n)^\#$ so no stronger conclusion about G is possible. On the other hand, without the assumption that $\mathrm{Inn}(G) \leqq A$, Lemma 2.1 is false. (For a counterexample, let G be a non-solvable Frobenius group with kernel K and complement H, and let A be the group of inner automorphisms induced by $KZ(H)$.)

Theorem 2.2. *Let G be a finite group and A be a non-trivial group of automorphisms of G acting partitively on $G^\#$. If $F(G)=1$, then $G \simeq SL_2(2^n)$ for some $n \geqq 2$ and* $\mathrm{Inn}(G) \leqq A$. *In fact, if $n \geqq 3$, then* $\mathrm{Inn}(G)=A$.

Proof. Since G has trivial centre, we may identify G with $\mathrm{Inn}(G)$ (so both A and G may be thought of as subgroups of $\mathrm{Aut}(G)$). In view of Lemma 1.2, G is isomorphic to one of $PGL_2(q)$, $PSL_2(q)$, or $Sz(q)$ for an appropriate prime power q. Hence, G admits a group of "field" automorphisms F_0 induced by the action of the Galois group of $GF(q)$.

If G is $PGL_2(q)$ or $Sz(q)$, the semidirect product F_0G is all of $\mathrm{Aut}(G)$. If G is $PSL_2(q)$, we may identify G as the commutator subgroup of $PGL_2(q)$ and then, every automorphism of G is the restriction of an automorphism of $PGL_2(q)$. If q is a power of the prime p, G has an F_0-invariant Sylow p-subgroup P_0 whose centre may be identified with $GF(q)$ in such a way that the action of F_0 on $Z(P_0)$ is isomorphic to the action of the Galois group on $GF(q)$. In particular, F_0 is faithful on $Z(P_0)$. $N_{\mathrm{Aut}(G)}(P_0)$ is transitive on $Z(P_0)^\# = \Omega_1(P_0)^\#$. Finally, $C_G(x)=P_0$ for every $x \in Z(P_0)^\#$ so P_0 is a T.I. set. (These facts and several others to be used in the argument may be found in [**1**], [**2**] and [**9**].)

We claim first that if P is any Sylow p-subgroup of G, then $A \cap P \neq 1$.

Suppose $A \cap P = 1$. P is conjugate in G to P_0 so some conjugate F of F_0 normalizes P. Let $x \in C_{Z(P)}(F)^\#$. From the structure of $\mathrm{Aut}(G)$ and the fact that $C_G(x)=P$, we obtain $C_{\mathrm{Aut}(G)}(x)=FP$. Since $C_A(x) \cap P = A \cap P = 1$, it follows that $C_A(x)=\langle \sigma y \rangle$ for some $\sigma \in F$ and $y \in P$.

If $G \simeq Sz(q)$, $P=H_2(P)$ so by Lemma 1.7, $C_A(P)=C_A(x)=\langle \sigma y \rangle$ so $\sigma \in C_F(Z(P))=1$ and $C_A(x)=\langle y \rangle \leqq A \cap P = 1$. But then partitivity implies A is semiregular on $G^\#$, an impossibility. Thus, we may assume G is isomorphic to $PGL_2(q)$ or $PSL_2(q)$.

If $|\sigma|=m$, then

$$(\sigma y)^m = \sigma^m y^{\sigma^{m-1}} \dots y^\sigma y = y^{\sigma^{m-1}} \dots y^\sigma y \in A \cap P = 1.$$

Since the action of F on $P=Z(P)$ is isomorphic to the action of the Galois group on $GF(q)$, Hilbert's Theorem 90 yields that $y = z^{-\sigma} z$ for some $z \in P$. Therefore, $\sigma y = \sigma^z$ so $C_G(C_A(x))$ is conjugate in G to $C_G(\sigma)$ and hence is isomorphic to $PGL_2(p^k)$ or $PSL_2(p^k)$ for some k (where $GF(p^k)$ is the fixed field of the Galois automorphism corresponding to σ). Then $C_G(C_A(x))$ contains a maximal cyclic subgroup of order $(p^k+1)/d \neq 1$ (where $d=1$ if $C_G(\sigma) \simeq PGL_2(p^k)$ and $d=2$ otherwise). But by Lemma 1.4(b), such a subgroup must be a maximal cyclic subgroup of G and so has order $q \pm 1$ or $(q \pm 1)/2$ according as $G \simeq PGL_2(q)$ or $G \simeq PSL_2(q)$. It follows that $p^k+1=q-1$ or $(q-1)/2$ and hence, that either $G \simeq PSL_2(4)$ with $C_G(C_A(x)) \simeq PGL_2(2)$ or $G \simeq PSL_2(9)$ with $C_G(C_A(x)) \simeq PGL_2(3)$.

Now $N_A(C_A(x))/C_A(x)$ acts semiregularly on $C_G(C_A(x))$ which is not nilpotent, so by Theorem 10.2.1 of [**3**], $N_A(C_A(x))=C_A(x)$. Since $|C_A(x)|=|\sigma|=2$, it follows that $C_A(x)$ is a

Sylow 2-subgroup of A. Let y be an element of order 5 in G so $C_G(y)=\langle y\rangle$. If $|C_A(y)|$ were even, it would contain a conjugate of $C_A(x)$ by Sylow's theorem and so, by partitivity, would, in fact, be conjugate to $C_A(x)$ in A. But then $y\in C_G(C_A(y))\simeq C_G(C_A(x))$, a contradiction since neither $PGL_2(2)$ nor $PGL_2(3)$ has order divisible by 5. Thus, $|C_A(y)|$ is odd so, since $|\mathrm{Aut}\,G:G|=4$, $C_A(y)\leqq G$. Then $1\neq C_A(y)\leqq C_G(y)=\langle y\rangle$ so $y\in A$. Since G is simple, it is generated by its elements of order 5 and so, $G\leqq A$. This contradicts the assumption that $A\cap P=1$ and hence, the claim that $A\cap P\neq 1$ for every Sylow p-subgroup P of G is proved.

Let $H=A\cap G$. $(A/C_A(H),H)$ is a partitive pair and $A/C_A(H)$, identified as a subgroup of $\mathrm{Aut}(H)$, contains $\mathrm{Inn}(H)$. By Lemma 2.1, either H is solvable or $H\simeq SL_2(2^m)$ for some $m\geqq 2$. Also, we have shown that H intersects every Sylow p-subgroup of G non-trivially so, since the Sylow p-subgroups of G are T.I. sets, H contains precisely the same number of Sylow p-subgroups as G.

If $G\simeq Sz(q)$, G (and hence H) has q^2+1 Sylow 2-subgroups. Since $SL_2(2^m)$ has 2^m+1 Sylow 2-subgroups and

$$|SL_2(q^2)|=q^2(q^2-1)(q^2+1)>q^2(q-1)(q^2+1)=|Sz(q)|,$$

H must be solvable. Since q^2+1 is a Hall divisor of $|Sz(q)|$, H contains a subgroup of order q^2+1. But $Sz(q)$ contains no such subgroup, a contradiction.

G is, therefore, isomorphic to $PGL_2(q)$ or $PSL_2(q)$, $q=p^n\geqq 4$, and in particular, G and H each have $q+1$ Sylow p-subgroups.

If $p\neq 2$, then no solvable subgroup of $PSL_2(q)$ can intersect non-trivially every Sylow p-subgroup of $PSL_2(q)$ so if H is solvable, $p=2$ and $G\simeq SL_2(2^n)$. In this case, H must be dihedral of order $2(2^n+1)$. But if x is any element of order 2^n+1 in $G=SL_2(2^n)$, $C_{\mathrm{Aut}\,G}(x)=\langle x\rangle$ so $H\cap\langle x\rangle=A\cap\langle x\rangle=C_A(x)$. Since partivity implies $C_A(x)\neq 1$, H must intersect non-trivially every cyclic subgroup of order 2^n+1 in G. Because H has a unique such subgroup, this is absurd.

We are now reduced to the case that $H\simeq SL_2(2^m)$ for some $m\geqq 2$. If $p\neq 2$, any involution x in H (and in fact, in G) is contained in a self-centralizing cyclic p'-subgroup $\langle y\rangle$ of G and, by partitivity, $C_A(x)=C_A(y)$. Then $C_H(x)\leqq C_G(y)=\langle y\rangle$ and so $C_H(x)$ is cyclic. This contradicts $m\geqq 2$. Thus, $p=2$ so, since H and G have the same number of Sylow 2-subgroups, $m=n$ and $G=H$ as required.

Finally, we argue that if $n\geqq 3$, $G=A$. Since $G\simeq SL_2(2^n)$, $\mathrm{Aut}(G)=F_0G$ where F_0 is the group of field automorphisms. Since $G\leqq A$, $A=BG$ where $B=A\cap F_0$. If $B\neq 1$, $C_G(B)\simeq SL_2(2^m)$ for some proper divisor m of n. Let $x\in C_G(B)$ of order 2^m+1 (so $x\neq 1$). Now $x\in\langle y\rangle$ where $y\in G$ has order $2^n\pm 1$ so by Lemma 1.4, $C_A(x)=C_A(y)$. Since $B\leqq C_A(x)$, $y\in C_G(B)\simeq SL_2(2^m)$ so $\langle x\rangle=\langle y\rangle$. Thus $2^m+1=2^n-1$ so $n=2$ and $m=1$. Therefore, if $n\geqq 3$, $B=1$ so $A=G$ as required. This completes the proof of Theorem 2.2.

Of course, if the near-ring $C(A,G)$ is semisimple and if $F(G)=1$, we can now specify precisely the simple components of $C(A,G)$. However, since we eventually want to suppress any assumptions about the structure of G, we content ourselves with the following observation:

Corollary 2.3. *If $C(A,G)$ is simple and $F(G)=1$, then $C(A,G)$ has precisely three simple components, all of which are abelian as additive groups.*

Proof. Theorem 2.2 states that $G \leqq A$ in this case so $C_G(C_A(x)) \leqq Z(C_G(x))$ for all $x \in G^{\#}$. By Lemma 1.1(c), every simple component is abelian. In fact, since $G \cong SL_2(2^n)$ and $G \leqq A$, the subgroups $C_G(C_A(x))$, $x \in G^{\#}$ are each either Sylow 2-subgroups of G or else cyclic of order $2^n - 1$ or $2^n + 1$, and there is one conjugacy class of each type in G. Thus, Lemma 1.1(c) implies that there are exactly three simple components.

3. The case $Z(G) \neq 1$

Lemma 3.1. *Let (A, G) be a partitive pair and assume A is not semiregular on $G^{\#}$. Suppose N is an A-invariant normal subgroup of G and $x \in G^{\#}$ such that $N \leqq C_G(C_A(x))$. If $C_G(N) \leqq N$ then N is abelian and, for any $g \in G \backslash C_G(C_A(N))$, $C_A(N)$ is transitive on the coset gN.*

Proof. Let $g \in G \backslash C_G(C_A(N))$ with g of prime order p. (Such an element exists for, if $h \in G \backslash C_G(C_A(N))$, then $\langle h \rangle \cap C_G(C_A(N)) = 1$ by Lemma 1.4(b).) Let $G_0 = \langle N, g \rangle$ and $A_0 = C_A(G_0/N)$.

Since $C_G(N) = Z(N)$, the Three Subgroups lemma [3, Theorem 2.2.3] implies $C_A(N) \leqq C_A(G/Z(N))$ so if $n \in N^{\#}$, $C_A(n) = C_A(N) \leqq C_A(G/Z(N)) \leqq A_0$. On the other hand, if $h \in G_0 \backslash N$ then $G_0 = \langle N, h \rangle$ so $C_A(h) \leqq C_A(G_0/N) = A_0$. It follows that if $\bar{A}_0 = A_0/C_A(G_0)$ then $(\bar{A}_0, G_0)$ is a partitive pair and $\bar{A}_0$ is not semiregular on $G_0^{\#}$. In order to prove N is abelian, therefore, we may assume $G = G_0$ and $A = \bar{A}_0$.

Now since $C_A(N) \leqq C_A(G/Z(N))$, the map $\alpha \mapsto [g, \alpha] = g^{-1}g^{\alpha}$ defines a homomorphism from $C_A(N)$ into $Z(N)$ and, in fact, it is a monomorphism since $G = \langle N, g \rangle$. Since $A/C_A(N)$ is semiregular on $N^{\#}$, $(|A : C_A(N)|, |N|) = 1$ and hence $(|A : C_A(N)|, |C_A(N)|) = 1$. By the Schur–Zassenhaus theorem, $A = C_A(N)B$ for some subgroup B with $C_A(N) \cap B = 1$.

Since $C_A(g) \cap C_A(N) = C_A(G) = 1$, $|C_A(g)|$ divides $|B|$ so $C_A(g^{\alpha}) = C_A(g)^{\alpha} \leqq B$ for some $\alpha \in A$. Since $A = C_A(G/N)$, the coset $g^{\alpha}N$ is A-invariant so, because N acts transitively on $g^{\alpha}N$ by right multiplication and $(|N|, |B|) = 1$, a result of Glauberman [4, (13.8)] yields that $B \leqq C_A(g^{\alpha}n)$ for some $n \in N^{\#}$. Thus $C_A(g^{\alpha}) = B = C_A(g^{\alpha}n)$, proving that $C_A(g)$ is conjugate in A to B. The same argument applies to every element of $G \backslash N$ (and, hence, to every element of gN).

Suppose $C_A(g) = C_A(gn)$ for some $n \in N$. Then $C_A(g) \leqq C_A(n)$ so, since $g \notin C_G(C_A(N))$, we have $n = 1$. Hence, each element of gN has as its centralizer in A a distinct conjugate of B. In particular, $|N| = |gN|$ is bounded by $|A : N_A(B)| \leqq |C_A(N)|$.

But now $|N| \leqq |C_A(N)| \leqq |Z(N)|$ so N is abelian. Moreover we have shown that if $g \in G \backslash C_G(C_A(N))$ is of prime order, the homomorphism $C_A(N) \to Z(N) = N$ is actually an epimorphism from which it follows that $C_A(N)$ is transitive on gN. If g is any element of $G \backslash C_G(C_A(N))$, then for some k, g^k has prime order and, since $\langle g \rangle \cap C_G(C_A(N)) = 1$, $g^k \in G \backslash C_G(C_A(N))$. Thus, the map $\alpha \mapsto [g^k, \alpha]$ from $C_A(N)$ to N is surjective. But the map $\alpha \mapsto [g, \alpha]$ has kernel

$$C_A(N) \cap C_A(g) \leqq C_A(N) \cap C_A(g^k)$$

so it must also be surjective. Therefore, $C_A(N)$ is transitive on gN regardless of the order of g and the proof of the lemma is complete.

Lemma 3.2. *Let V be an abelian group and H be a group of operators on V. If $X \leqq H$, let $\hat{X} = \sum_{x \in X} x$ (considered as an endomorphism of V). Assume H admits a partition π (so π is a collection of subgroups with each non-identity element of H contained in a unique member of π) and that $\hat{X} = 0$ for each $X \in \pi$. If $\hat{H} = 0$, then $|\pi| \equiv 1 \bmod (\exp V)$.*

Proof.

$$0 = \hat{H} = I + \sum_{X \in \pi} (\hat{X} - I) = I - |\pi| I = (1 - |\pi|)\, I \text{ in End } V$$

so the result follows.

Theorem 3.3. *Suppose (A, G) is a partitive pair with A not semiregular on $G^{\#}$. If $Z(G) \neq 1$ then either G has exponent p for some prime p or the following hold:*

(a) *$Z(G)$ has exponent p for some prime p,*

(b) *$H_p(G)$ is abelian of index p in G,*

(c) *$C_{Aut(G)}(H_p(G)) \leqq A \leqq C_{\mathrm{Aut}(G)}(G/H_p(G))$,*

(d) *$C_A(H_p(G))$ is a normal Hall subgroup of A isomorphic to $H_p(G)$,*

(e) *$C_A(H_p(G))$ acts transitively on the set of subgroups of G of order p outside $H_p(G)$.*

Proof. If $Z(G)$ contained an element x of composite order, Lemma 1.7 would imply $G = C_G(C_A(x))$ so $C_A(x) = 1$, contradicting the assumption that A is not semiregular on $G^{\#}$. Thus, $Z(G)$ has prime exponent p for some p. For any $x \in Z(G)^{\#}$, $H_p(G) \leqq C_G(C_A(x))$ by Lemma 1.7(b) so by Lemma 3.1, if G is not of exponent p, $H_p(G)$ is abelian and $C_A(H_p(G))$ is transitive on each coset $gH_p(G)$ where $g \in G \backslash C_G(C_A(H_p(G)))$.

Let $\bar{G} = G/H_p(G)$ so $\bar{G}$ has exponent p and acts by conjugation on $H_p(G)$. Suppose $|G : H_p(G)| > p$ so $\bar{G}$ contains a subgroup $\bar{H}$ of order p^2. If $\bar{x} \in \bar{G}$ then for any $u \in H_p(G)$, $u u^{\bar{x}} u^{\bar{x}^2} \dots u^{\bar{x}^{p-1}} = 1$ so, in the notation of Lemma 3.2, $\hat{\bar{x}} = 0$. Moreover, if $\hat{\bar{H}} = \langle \hat{\bar{x}}, \hat{\bar{y}} \rangle$, then $\hat{\bar{H}} = \hat{\bar{x}} \hat{\bar{y}} = 0$. Since $\bar{H}$ is partitioned by its $p+1$ subgroups of order p, Lemma 3.2 yields that $\exp(H_p(G)) = p$, contradicting the assumption that G does not have exponent p. Thus, $|G : H_p(G)| = p$. Moreover, it follows that $C_G(C_A(H_p(G))) = H_p(G)$ so $C_A(H_p(G))$ is transitive on $gH_p(G)$ for every $g \in G \backslash H_p(G)$ and hence, transitive on the set of subgroups of order p in G outside $H_p(G)$.

As in the proof of Lemma 3.1, if $g \in G \backslash H_p(G)$, the map $C_A(H_p(G)) \to H_p(G)$ (defined by $\alpha \to [g, \alpha]$) is bijective. It is still injective when extended to $C_{\mathrm{Aut}\, G}(H_p(G))$ (since $G = \langle H_p(G), g \rangle$) and so $C_{\mathrm{Aut}\, G}(H_p(G)) \leqq A$. $A/C_A(H_p(G))$ is semiregular on $H_p(G)^{\#}$ so since $C_A(H_p(G)) \simeq H_p(G)$, $C_A(H_p(G))$ is a normal Hall subgroup of A. If B is a complement for $C_A(H_p(G))$ in A, B centralizes some element of $gH_p(G)$ by Glauberman's lemma, so B (and hence A) is contained in $C_{\mathrm{Aut}\, G}(G/H_p(G))$. This completes the proof.

The theorem suggests a recipe for constructing examples. Let H be an abelian group which admits an automorphism σ of order m, say, such that $h^{1+\sigma \dots + \sigma^{m-1}} = 1$ for every $h \in H$. Assume H also admits a fixed-point-free automorphism α of prime order such that $\alpha\sigma = \sigma\alpha$. If $G = H\langle\sigma\rangle$, then α extends to an automorphism of G (which fixes σ). For each $h \in H$, define β_h to be the automorphism which fixes H and maps σ to $h\sigma$. If $B = \{\beta_h : h \in H\}$, then B is a subgroup of $\mathrm{Aut}\, G$ which is normalized by α, so we may let

$A=B\langle\alpha\rangle$. It is then easily checked that (A,G) is a partitive pair. If H is a 2-group of exponent at least 4 which admits a fixed-point-free automorphism of prime order (such as $Z_4\times Z_4$) and if σ is the automorphism which inverts H, then (A,G) is one of the exceptional cases described in Theorem 3.3. On the other hand, if p is an odd prime, H is a 2-dimensional space over $GF(p)$,

$$\sigma=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \text{ and } \alpha=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

with respect to some basis, then we obtain a partitive pair in which G is non-abelian of exponent p. If $\alpha=\sigma$, we obtain a partitive pair in which G is a Frobenius group, a situation to be discussed in the next section.

Corollary 3.4. *Suppose G is a finite group, $A\leqq \operatorname{Aut} G$ and $C(A,G)$ is semisimple but not simple. If $Z(G)\neq 1$, then either*

(a) *$(C(A,G),+)$ has prime exponent, or*

(b) *$C(A,G)\simeq C(\bar{A},H)\oplus C(1,K)$ where $H=H_p(G)$, $\bar{A}=A/C_A(H)$, and K is a subgroup of order p in G outside H. In this case, $C(A,G)$ is abelian.*

Proof. This is a direct consequence of Theorem 3.3. and Lemma 1.1. There are only two components in the second case since A is transitive on subgroups of order p in G outside H. Moreover, $|H|=|A{:}C_A(K)|$ by transitivity so, since $C_A(H)\simeq H$, $A=C_A(H)C_A(K)$. Since $N_A(K)\cap C_A(H)\leqq C_A(K)$, it follows that $N_A(K)/C_A(K)=1$.

4. The Case $F(G)\neq 1=Z(G)$

In this case, G is Frobenius by Lemma 1.5 so we may write $G=FK$ where $F=F(G)$, the Frobenius kernel (and the Fitting subgroup of G), and K is a complement for F (which acts semiregularly on $F^{\#}$).

Lemma 4.1 *$K\leqq C_G(C_A(x))$ for every $x\in K^{\#}$. Moreover, if $x\in K^{\#}$ and $N_A(C_A(x))/C_A(x)\neq 1$, then $K=C_G(C_A(x))$ and K is cyclic of odd order.*

Proof. Let $x\in K^{\#}$. If $\alpha\in C_A(x)$ then $x\in K\cap K^{\alpha}$ so since K is a T.I. set and K^{α} is conjugate in G to K, $\alpha\in N_A(K)$. Thus, $C_A(x)\leqq N_A(K)$ so $N_A(K)/C_A(K)$ acts partitively on $K^{\#}$. K, being a Frobenius complement has non-trivial centre, so Theorem 3.3 applies to the pair $(N_A(K)/C_A(K),K)$. Since K cannot itself contain a Frobenius group (as may be easily seen from [3, Theorem 3.4.4]), we conclude that either $N_A(K)/C_A(K)=1$ or K is nilpotent. Since $C_A(x)\leqq N_A(K)$, the first case immediately yields $K\leqq C_G(C_A(x))$. In the second case, each Sylow subgroup of K contains a unique subgroup of prime order [3, Theorem 10.3.1] and hence, Lemma 1.4 implies $K\leqq C_G(C_A(x))$.

If $N_A(C_A(x))/C_A(x)\neq 1$, then since this group acts semiregularly on $C_G(C_A(x))^{\#}$, Theorem 10.2.1 of [3] implies $C_G(C_A(x))$ is nilpotent. Since K is semiregular on $F^{\#}$, $C_F(C_A(x))=1$ so since $K\leqq C_G(C_A(x))$, we must have $K=C_G(C_A(x))$. As in the preceding paragraph, each Sylow subgroup of K has a unique subgroup of prime order so if $|K|$ were even, then taking u to be the unique involution in K, we get $N_A(C_A(x))=$

$N_A(K) \leqq C_A(u) = C_A(x)$, a contradiction. Thus, $|K|$ is odd so by [3, Theorem 10.3.1], it is cyclic.

Lemma 4.2. *If F is not of prime exponent then the following are satisfied:*

(a) *F is abelian and $F = C_G(C_A(x))$ for every $x \in F^{\#}$.*

(b) *A contains the inner automorphisms induced by F (and hence, is transitive on the set of Frobenius complements).*

(c) $N_A(K)/C_A(K) = 1$.

Proof. Assume F is not of prime exponent and let $H = H_p(F) \neq 1$ (where p divides $|Z(F)|$). If $z \in Z(F)^{\#}$, $H \leqq H_p(C_G(z))$ by Lemma 1.7.

Suppose $F \nleqq C_G(C_A(z))$. Then $A/C_A(F)$ is not semiregular on $F^{\#}$ so Theorem 3.3 implies that H is abelian of index p in F. Let $\bar{G} = G/H = [\bar{F}]\bar{K}$ where $\bar{F} = F/H \simeq Z_p$ and $\bar{K} \simeq K$. Now $\bar{G}$ is Frobenius and so is partitioned by $\bar{F}$ and the p conjugates of $\bar{K}$ in $\bar{G}$. Considering the action of $\bar{G}$ on H, we conclude from Lemma 3.2 that the exponent of H divides p, a contradiction. Thus, $F \leqq C_G(C_A(z))$.

By Lemma 3.1, F is abelian so, identifying G as a subgroup of Aut G, $F \leqq C_{\mathrm{Aut}\,G}(F)$. By the Three Subgroups lemma, $[G, C_{\mathrm{Aut}\,G}(F)] \leqq C_G(F) = F$. If $\alpha \in C_{\mathrm{Aut}\,G}(F)$, then $K^{\alpha} = K^f$ for some $f \in F$ so $\alpha f^{-1} \in N_{\mathrm{Aut}\,G}(K) \cap C_{\mathrm{Aut}\,G}(F)$. Hence, $[G, \alpha f^{-1}] \leqq F \cap K = 1$ so $\alpha = f \in F$. This proves that $C_{\mathrm{Aut}\,G}(F) \leqq F$ so $C_{\mathrm{Aut}\,G}(F) = F$. In particular, $C_A(F) = A \cap F$. Now by Lemma 3.1, $A \cap F$ is transitive on the coset xF for any $x \in G \backslash C_G(C_A(F))$. Since $C_F(x) = 1$ for any such x, it follows that $|A \cap F| = |xF| = |F|$ so, in fact, $F \leqq A$. Then $C_G(C_A(z)) \leqq C_G(F) = F$ so $C_G(C_A(z)) = F$.

Finally, if $x \in K^{\#}$, $C_A(x) = C_A(K)$ by Lemma 4.1 so $C_A(x) \cap F = 1$. Since A is transitive on xF, $|A : C_A(x)| = |F|$ so $A = FC_A(x) = FC_A(K)$. Hence, $N_A(K) = N_F(K)C_A(K) = C_A(K)$, completing the proof of (c) and the lemma.

Note that Lemma 4.2 and Corollary 3.4 imply that if $C(A, G)$ is semisimple but not simple and $F(G)$ is non-trivial but not of prime exponent, then

$$C(A, G) \simeq C(\bar{A}, F(G)) \oplus C(1, K) \quad \text{where} \quad \bar{A} = A/C_A(F(G)).$$

When $N_A(C_A(x))/C_A(x) = 1$ for any $x \in K^{\#}$, we cannot say much more about K than that it is a Frobenius complement. This is unfortunate but not surprising, at least when F is abelian since, if $G = FK$ is *any* Frobenius group with abelian kernel, the group $A = FZ(K)$ obviously acts partitively on $G^{\#}$ by conjugation.

5. Epilogue

Summarizing the results of Sections 2, 3 and 4 in a form which is free of hypothesis about the structure of A or G, we obtain the following:

Theorem 5.1. *If G is a finite group, $A \leqq \mathrm{Aut}\,G$ and $C(A, G)$ is semisimple but not simple, then the additive group of each simple component of $C(A, G)$ is either*

(a) *of prime exponent*

(b) *abelian*, or

(c) *a direct sum of copies of a subgroup H of G which is either a Frobenius group with kernel of prime exponent or else a Frobenius complement.*

The last case arises when $G=FK$, a Frobenius group with kernel F, and the component in question is of the form $C(1,H)$ where $H=C_G(C_A(x))$, $x\in K^{\#}$.

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