ON SURFACE WAVES

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1. Introduction. The linearized theory of surface waves leads to several mixed boundary value problems which have been investigated by various methods. As the physical background of the theory has been repeatedly discussed, it will suffice to deal here mainly with the mathematical aspect of the question.

Let D be a finite or infinite domain in the (x, y)-plane and let $\phi(x, y)$ denote a function in D satisfying one of the following differential equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{1.1}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - k^2\phi = 0 \tag{1.2}$$

where k^2 denotes a positive constant. Let n denote the external normal to the boundary C of D. The boundary condition on the part of C corresponding to the free surface of the fluid is given by the equation

$$\frac{\partial\phi}{\partial n} = p\phi, \tag{1.3}$$

The *positive* constant p is in some cases an unknown parameter.

The boundary condition on the part of C corresponding to the rigid part of the boundary is given by the equation

$$\frac{\partial\phi}{\partial n} = 0. \tag{1.4}$$

In the classical theory ϕ denotes (up to a factor depending on the time t) the velocity potential in the physical (x, y)-plane. However, in Levi-Civita's theory of plane waves, the independent variables are the velocity potential and the stream function. The unknown function ϕ denotes in this case the angle which the velocity makes with the horizontal direction. The condition (1.3) remains unchanged in form, but (1.4) has to be replaced by the condition

$$\phi = 0. \tag{1.5}$$

In most cases in application the domain D extends to infinity and is bounded by straight lines. However in Levi-Civita's theory of periodic waves, D can be mapped conformally on a finite domain such as a circle or circular ring without changing the form of the boundary conditions.

It should be emphasized that (1.3) differs essentially from the boundary condition discussed by Fourier in his classical theory of heat conduction. In Fourier's case p is essentially negative, a fact which implies that, for a finite

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domain D, the corresponding boundary value problem admits only the trivial solution $\phi = 0$. The situation is, however, different in the case of surface waves where the corresponding boundary value problem admits one or even several non-trivial solutions for certain positive values of the constant p. The important question of the uniqueness of the solution has been overlooked by standard treatises on hydrodynamics. Besides its intrinsic mathematical interest, a survey of all solutions is of great importance for the following reasons: first, the solutions of the linearized problem give a first approximation to the exact non-linear theory of a surface wave; second, the superposition of two standing waves obtained from two different solutions of the linearized problem leads to a travelling wave as required by the theory.

There are at present three methods of approach to the various boundary problems encountered in the theory of surface waves:

(i) The eigenvalue method.
(ii) The method of reduction.
(iii) The method of singular integral equations.

Some of the problem can at present be discussed only by the first or by the third method. However this is not the case for problems discussed up to now by the second method alone. It is the purpose of the present paper to show that a combination of the method of reduction with the eigenvalue method leads to more complete results than the application of the reduction method alone.

2. The eigenvalue method. This method has been developed by A. Weinstein [3] in connection with a problem in Levi-Civita's theory. For modification of this method see the papers by G. Hoheisel [4], S. Bochner [5], J. L. B. Cooper [6] and A. E. Heins [7]. As an illustration we shall use this method for the complete solution of *Airy's Problem* which corresponds to the hydrodynamical problem of plane waves in water of constant depth: *To find all harmonic functions ϕ in the infinite strip S, $-\infty < x < +\infty, 0 \leqq y \leqq 1$ satisfying the boundary conditions*

$$\frac{\partial \phi}{\partial y} = p y, \qquad \text{for } y = 1 \tag{2.1}$$

and

$$\frac{\partial \phi}{\partial y} = 0, \qquad \text{for } y = 0. \tag{2.2}$$

Airy's work contains only a particular solution of this problem which is periodic in x and which has been reproduced in all textbooks.

In order to solve this problem let us consider first the eigenvalue problem given by the ordinary differential equation

$$Y'' + \lambda Y = 0, \qquad Y = Y(y) \tag{2.3}$$

with the boundary conditions

$$Y' = pY, \qquad \text{for } y = 1, \tag{2.4}$$

$$Y' = 0, \qquad \text{for } y = 0. \tag{2.5}$$

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A complete set of eigenfunctions and of corresponding eigenvalues is given by the formulas

$$Y_0 = \cosh \alpha_0 y, \qquad \lambda_0 = -\alpha_0^2, \tag{2.6}$$

$$Y_n = \cos \alpha_n y, \qquad \lambda_n = \alpha_n^2, \qquad (n = 1, 2, \ldots) \tag{2.7}$$

where α_0 is the unique (positive) root of the equation

$$\alpha_0 \tanh \alpha_0 = p \tag{2.8}$$

and $\alpha_1, \alpha_2, \ldots, \alpha_n$, denote the (positive) roots of the equation

$$\alpha_n \tan \alpha_n = p, \qquad (n = 1, 2, \ldots). \tag{2.9}$$

Turning back to our boundary value problem (1.1), (2.1), (2.2) we develop $\phi(x, y)$ for a fixed value of x, into the series

$$\phi(x, y) = \sum_{n=0}^{\infty} c_n(x) Y_n(y). \tag{2.10}$$

This development is possible as ϕ satisfies, for any fixed value of x, the same boundary conditions as Y_n. The Fourier coefficients c_n are given by the formulas

$$c_n(x) = C_n \int_0^1 \phi(x, y) Y_n(y) dy \tag{2.11}$$

where

$$C_n = \left(\int_0^1 Y_n^2 dy \right)^{-\frac{1}{2}}. \tag{2.12}$$

The constant C_n is the normalization factor.

From (2.11) and (1.1) it follows by differentiation that

$$c''_n(x) = -C_n \int_0^1 \phi_{yy} Y_n dy.$$

Integrating twice by parts we obtain the formula

$$c''_n(x) = -C_n[\phi_y Y_n - \phi Y'_n]_0^1 - C_n \int_0^1 \phi Y''_n dy.$$

The square bracket vanishes in view of the boundary conditions (2.1), (2.2), (2.4) and (2.5). By (2.3) we have therefore for $c_n(x)$ the differential equation

$$c''_n(x) - \lambda_n c_n(x) = 0 \tag{2.13}$$

which has the following solutions:

$$c_0(x) = a_0 \cos \alpha_0 x + b_0 \sin \alpha_0 x \tag{2.14}$$

$$c_n(x) = a_n e^{\alpha_n x} + b_n e^{-\alpha_n x}, \qquad (n = 1, 2, \ldots). \tag{2.15}$$

On the other hand $c_n(x)$ is given by the formula (2.11).

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Let us assume now that $\phi(x, y)$ satisfies the inequality

$$\int_0^1 \phi^2(x, y)dy < e^{2A|x|}, \qquad A > 0 \tag{2.16}$$

for $|x| \to \infty$. An application of Schwarz' inequality to the formulas

$$a_0 \cos \alpha_0 x + b_0 \sin \alpha_0 x = C_0 \int_0^1 \phi(x, y) Y_0(y)dy \tag{2.17}$$

$$a_n e^{\alpha_n x} + b_n e^{-\alpha_n x} = C_n \int_0^1 \phi(x, y) Y_n(y)dy, \qquad (n = 1, 2, \ldots) \tag{2.18}$$

shows immediately that $a_n = b_n = 0$ for all values of n for which α_n is greater than A, $n = 0, 1, 2, \ldots$. We have therefore the following result. All solutions of our boundary value problem satisfying the inequality (2.16) are given by the formulas

$$\phi(x, y) = (a_0 \cos \alpha_0 x + b_0 \sin \alpha_0 x) \cosh \alpha_0 y + \sum_{n=1}^{h} (a_n e^{\alpha_n x} + b_n e^{-\alpha_n x}) \cos \alpha_n y \tag{2.19}$$

where a and b are arbitrary constants. The exponents α_n satisfy the inequality

$$0 < \alpha_1 < \alpha_2 < \ldots < \alpha_h < A < \alpha_{h+1} < \ldots. \tag{2.20}$$

In particular the only bounded solution of the problem is given by the formula

$$\phi(x, y) = (a_0 \cos \alpha_0 x + b \sin \alpha_0 x) \cos \alpha_0 y. \tag{2.21}$$

This solution is periodic in x and coincides with the particular solution given by Airy. The problem considered in this paragraph cannot be solved by the reduction method which will be discussed in the following section.

3. The reduction method. In this method the unknown function ϕ is replaced by a new unknown $\Phi(x, y)$ satisfying the same differential equation as ϕ but vanishing on the boundary of D. The mixed boundary problem is reduced to a problem with the classical boundary condition $\Phi = 0$.

T. Boggio [1] was the first to determine all harmonic functions ϕ satisfying the condition (1.3) on the boundary of a circle of radius one. Let us put

$$f(z) = \phi + i\psi, \qquad z = x + iy = re^{i\theta} \tag{3.1}$$

where ψ denotes the conjugate function to ϕ. Since the real part of $z \dfrac{df}{dz}$ equals $r \dfrac{\partial \phi}{\partial r}$ and since the external normal to the circle has the direction of the radius r, the harmonic function

$$\Phi = r \frac{\partial \phi}{\partial r} - p\phi \tag{3.2}$$

vanishes by (1.3) on the boundary $r = 1$. Assuming that ϕ is regular for

$0 \leqq r \leqq 1$, we see that ϕ vanishes identically and that f satisfies therefore the ordinary linear differential equation

$$z \frac{df}{dz} - pf = ia \tag{3.3}$$

where a is a real constant. The integration of this equation shows that a regular solution ϕ exists only for $p = 1, 2, 3, \ldots$, in which case ϕ is given by the formula

$$\phi = r^p(\alpha \cos p\theta + \beta \sin p\theta).$$

A more complete analysis of the problem could have been made by the eigenvalue method, which yields also all solutions ϕ with an isolated singularity at the origin $r = 0$. (See Sec. 4.)

Recently some other interesting mixed boundary value problems corresponding to waves on sloping beaches have been discussed by Miche [8], H. Lewy [9], and J. J. Stoker [10], and others. The method of Lewy and Stoker introduces a different reduction procedure. In the following we shall discuss as an example one of the problems treated by Stoker and show that a combination of the reduction method and of the eigenvalue method yields a complete solution of the problem.

4. A mixed boundary value problem in three-dimensional wave motion. Let us consider (see Stoker, loc. cit. [10] paragraph 9) the problem of waves in an ocean of infinite depth bounded on one side by a vertical cliff when the wave crests are not assumed to be parallel to the shore line. The corresponding boundary value problem is the following:

To find all solutions $\phi(x, y)$ of the differential equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - k^2\phi = 0 \tag{4.1}$$

in the domain D, $x \geqq 0$, $y \leqq 0$, satisfying the boundary conditions

$$\frac{\partial\phi}{\partial y} = \phi, \qquad \text{for } y = 0, \quad x > 0, \tag{4.2}$$

$$\frac{\partial\phi}{\partial x} = 0, \qquad \text{for } x = 0, \quad y < 0. \tag{4.3}$$

Here k^2 denotes an arbitrary positive constant. According to Stoker we reduce the boundary conditions (4.2), (4.3) to the boundary condition $\phi = 0$ by the introduction of the function

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} - 1\right)\phi = \Phi(x, y) \tag{4.4}$$

which obviously satisfies the differential equation (4.1). It may be written in polar coordinates as follows:

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} - k^2\Phi = 0. \tag{4.5}$$

The boundary condition is

$$\Phi = 0, \text{ for } x = 0, y < 0 \text{ and } y = 0, x > 0. \tag{4.6}$$

Instead of prescribing specifically the singularities of Φ (see Stoker, *loc. cit.*, n. 28, p. 39) we shall determine Φ by the eigenvalue method. A subsequent integration of the differential equation (4.4) will give us then all possible solutions of ϕ.

As in Sec. **2**, we consider first the eigenvalue differential problem given by the equation

$$\frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0 \tag{4.7}$$

with the boundary conditions

$$\Theta = 0, \text{ for } \theta = 0 \text{ and } \theta = -\tfrac{1}{2}\pi. \tag{4.8}$$

A complete set of eigenfunctions and of the corresponding eigenvalues is given by the formulas

$$\Theta_n(\theta) = \sin 2n\theta,\ \lambda = 4n^2, \qquad (n = 1, 2, \ldots). \tag{4.9}$$

For a fixed value of r we have for the unknown function Φ the expansion

$$\Phi = \sum_{n=1}^{\infty} c_n(r) \sin 2n\theta \tag{4.10}$$

where

$$c_n(r) = C_n \int_{-\frac{1}{2}\pi}^{0} \Phi(r, \theta) \sin 2n\theta d\theta, \qquad (n = 1, 2, \ldots) \tag{4.11}$$

where C_n is the normalizing factor. From this formula we find by differentiation with respect to r and by use of (4.5)

$$c''_n(r) + \frac{1}{r}c'_n(r) - \left(k^2 + \frac{4n^2}{r^2}\right)c_n(r) = \frac{C_n}{r^2}\int_0^{-\frac{1}{2}\pi}\left(\frac{\partial^2\Phi}{\partial\theta^2} + 4n^2\Phi\right)\sin 2n\theta d\theta.$$

The right-hand side in this equation is equal to zero as can easily be seen by two successive integrations by parts and by the use of the boundary condition (4.6). We have therefore the following differential equation for $c_n(r)$

$$c''_n + \frac{1}{r}c'_n - \left(k^2 + \frac{4n^2}{r^2}\right)c_n = 0. \tag{4.12}$$

The general solution of (4.10) is given in terms of Bessel functions by the formula

$$c_n(r) = A_{2n}I_{2n}(kr) + B_{2n}i^{2n+1}H_{2n}^{(1)}(ikr) \tag{4.13}$$

with arbitrary real constants A_{2n} and B_{2n}. The function I_{2n} vanishes for $r = 0$ like r^{2n} but tends to infinity like $e^r r^{-\frac{1}{2}}$ for $r = \infty$. The functions $i^{2n+1} H_{2n}^{(1)}$ behave like r^{-2n} for r tending to zero and tend to zero like $e^{-r} r^{-\frac{1}{2}}$ at infinity. By the same procedure as in Sec. **2** we obtain the following results. The solutions Φ of (4.5) and (4.6) given by (4.10), can be classified according to the behaviour of the integral

$$(4.14) \qquad \int_{-\frac{1}{2}\pi}^{0} \Phi^2(r, \theta) d\theta.$$

The coefficients A_{2n} in (4.13) are all equal to zero for any solution Φ for which the integral (4.14) is $o(e^{2r} r^{-1})$ at infinity. The coefficients B_{2n} vanish for $n > h$ for all solutions Φ for which the integral (4.14) is $o(r^{-2h})$ at the origin. The only solution Φ which is regular everywhere is $\Phi = 0$.

By taking $\Phi = 0$ and $\Phi = iH_2^{(1)}(ikr) \sin 2\theta$ and by integrating the corresponding differential equations (4.4) for ϕ, Stoker obtains two standing waves which can be combined into a travelling wave. One of these standing waves has a logarithmic singularity at the origin. From the results of the present paper we see the presence of a singularity is an unavoidable consequence of the linearized theory of surface waves. The contradiction of the original assumption of small amplitudes is somewhat mitigated by taking the solution with the weakest singularity at the origin. From the mathematical viewpoint, however, there is no reason to introduce any limitations on the behaviour of the solutions.

5. The method of singular integral equations. We conclude with a few remarks about this method which has been applied to the case when the domain D is a parallel strip, as in Sec. **2**. Let us replace in Airy's problem the condition (2.2) by the condition

$$(5.1) \qquad \phi = 0, \qquad \text{for } y = 0.$$

Under certain restrictive assumptions on the behaviour of ϕ at infinity the modified problem can be reduced to a Picard integral equation [2]. However, as has been mentioned in Sec. **2**, the eigenvalue method gives the solution of the same problem under less restrictive conditions. The situation is, however, different in the dock problem in a channel of finite depth, which is obtained by imposing the condition (2.1) for $y = 1$, $x > 0$ and the condition (2.2) on the remaining part of the boundary of the strip. This problem, which seems at present inaccessible by any other method, has been solved by A. E. Heins [11] by a reduction of the problem to a Wiener-Hopf equation. The assumptions which are required in order that this problem be formulated as a Wiener-Hopf integral equation are discussed in paragraph 9 of the paper by Heins. Uniqueness is studied in relation to the Wiener-Hopf integral equation to be solved. This integral equation is equivalent to the original boundary value problem subject to the conditions mentioned above. The general uniqueness theorem under less restrictive conditions, has not been discussed yet for the dock problem.

References

[1] T. Boggio, *Rendiconti della R. Accademia di Torino*, vol. 47 (1912), 22; see also E. Goursat, *Cours d'Analyse Mathématique*, vol. 3 (Paris, 1927), 240.

[2] A. Weinstein, *Rendiconti della R. Accademia dei Lincei*, vol. 5, Series 6^a (1927), 259.

[3] A. Weinstein, *C. R. Acad. Sci.*, Paris, vol. 184 (1927), 497.

[4] G. Hoheisel, *Jber. Deutschen Math. Verein.*, vol. 39 (1930), 54.

[5] S. Bochner, *Fouriersche Integrale* (Berlin, 1932), chap. VII.

[6] J. L. B. Cooper, *J. London Math. Soc.*, vol. 14 (1939), 124.

[7] A. E. Heins, *Bull. Amer. Math. Soc.*, vol. 49 (1943), 130.

[8] A. Miche, *Annales des ponts et chaussées*, vol. 114 (1944).

[9] H. Lewy, *Bull. Amer. Math. Soc.*, vol. 52 (1946), 737.

[10] J. J. Stoker, *Quarterly of Applied Math.*, vol. 5 (1947), 1.

[11] A. E. Heins, *Amer. J. Math.*, vol. 70 (1948) 730.

[12] K. O. Friedrichs, *Communications on Appl. Math.*, vol. 1 (1948), 109.

[13] K. O. Friedrichs and H. Lewy, *Communications on Appl. Math.*, vol. 1 (1948), 135.

[14] F. John, *Communications on Appl. Math.*, vol. 1 (1948), 149.

[15] E. Isaacson, *Communications on Appl. Math.*, vol. 1 (1948), 201.

[16] G. Kreisel, *Quarterly of Applied Math.*, vol. 7 (1949), 21.

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