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A CATEGORICAL APPROACH TO THE BAUM–CONNES CONJECTURE FOR ÉTALE GROUPOIDS

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Abstract We consider the equivariant Kasparov category associated to an étale groupoid, and by leveraging its triangulated structure we study its localization at the ‘weakly contractible’ objects, extending previous work by R. Meyer and R. Nest. We prove the subcategory of weakly contractible objects is complementary to the localizing subcategory of projective objects, which are defined in terms of ‘compactly induced’ algebras with respect to certain proper subgroupoids related to isotropy. The resulting ‘strong’ Baum–Connes conjecture implies the classical one, and its formulation clarifies several permanence properties and other functorial statements. We present multiple applications, including consequences for the Universal Coefficient Theorem, a generalized ‘going-down’ principle, injectivity results for groupoids that are amenable at infinity, the Baum–Connes conjecture for group bundles, and a result about the invariance of K-groups of twisted groupoid C^*-algebras under homotopy of twists.

Contents

Key words and phrases: groupoid; localization; triangulated category; Baum-Connes conjecture; Going-Down principle

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Introduction and main results

Over the last decades étale groupoids and their homological and K-theoretical invariants have played an increasingly important role in the fields of operator algebras, noncommutative geometry and topological dynamics. Kumjian and Renault showed that C^*-algebras associated with groupoids provide versatile models for large classes of C^*-algebras [27, 54]. More recently, Li showed that every classifiable C^*-algebra admits a (twisted) groupoid model [33]. One of the biggest open questions in the field concerns the universal coefficient theorem (UCT) and work of Barlak and Li [3] showed that the UCT problem can be translated to the question whether every nuclear C^*-algebra admits a groupoid model.

In another direction, Matui's works [35, 36] have kickstarted a fruitful line of research in topological dynamics using étale groupoids at its heart (see also [32]). In this area, it turns out that many invariants for topological dynamical systems can most naturally been defined in the framework of groupoid homology or the K-theory of groupoid C^*-algebras. Consequently, there is a great deal of interest around the homology and K-theory of étale groupoids and their interaction. Examples of recent research in this direction are the HK conjecture of Matui [35]a, or the relation between the homology theory of Smale spaces and the K-theory of their corresponding C^*-algebras [52]. In this latter example, a special case of the methods developed here (i.e., when the groupoid is torsion-free and ample) has already been applied with great success and lead to many interesting results in topological dynamics, as is demonstrated by the papers [9, 49, 51, 50].

Motivated by these developments we set out to develop the category-theory based approach to the Baum–Connes conjecture for the class of étale groupoids in full generality. This approach is very suitable for formulating and proving general statements about the Baum–Connes conjecture and for obtaining functorial properties of the assembly map and K-theoretic duality type results [21, 46]. As already observed by Meyer and Nest [40], many permanence results of the Baum–Connes conjecture become quite accessible in this setup. Besides this, several results obtained by the first named author [10, 11] and C. Dell'Aiera [12] are generalized to all étale groupoids.

The following statement summarizes a selection of applications that we are able to obtain through this approach. Some statements are deliberately vague to spare the reader the technical details at this stage, we refer to the final section of this article (Section 4) for the definitions and more precise statements.

Theorem A. *Let G be an étale groupoid which is second countable, locally compact and Hausdorff.*

1. *Suppose Σ is a twist over G. If G satisfies the strong Baum–Connes conjecture, then $C_r^*(G,\Sigma)$ satisfies the UCT.*
2. *The K-theory of $C_r^*(G,\Sigma)$ only depends on the homotopy class of Σ.*
3. *If G is strongly amenable at infinity, then there is a dual Dirac morphism for G. In particular, the Baum–Connes assembly map is split-injective.*
4. *The (strong) Baum–Connes conjecture enjoys many permanence properties both with respect to the involved groupoid (it passes to subgroupoids, direct products, increasing unions) and the coefficient algebra (inductive limits, tensor products).*

The results in Theorem A should be compared to another line of research, which uses quantitative K-theory methods to obtain many interesting related results on the UCT, the Baum–Connes conjecture and its permanence properties [23, 47, 63].

In [40], R. Meyer and R. Nest established the category theoretic framework we are after in the setting of locally compact groups and more generally for transformation groups. To this end, they leverage the triangulated structure of the equivariant bivariant Kasparov category and in particular the notion of complementary subcategories and localization. This paper extends these methods to include étale groupoids.

A related approach is described in [18], where the authors give a unified approach to various isomorphism conjectures, including the Baum–Connes conjecture by means of the orbit category and the homotopy theory of spectra. In both approaches, the role of *weakly* contractible objects, defined in terms of a certain family of subgroups of a given group G, is in a certain sense fundamental. For the Baum–Connes conjecture associated to a discrete group, this family is given by the finite subgroups of G. Analogously, when G is locally compact, the family is given by the compact subgroups.

Thus, the first task when attempting to generalise this approach is the identification of a suitable class of subgroupoids of a given étale groupoid G. Associated to this class is a homological ideal in the Kasparov category KK^G, which is the starting point for several notions of *relative* homological algebra, for example, the notion of projective object. In the words of Meyer and Nest [40, page 215], 'it is not so clear what should correspond to compact subgroups' in the case of the Baum–Connes conjecture for groupoids.

A partial solution to this question was offered in [21], where the authors show a relation of complementarity between the subcategory of proper objects and the objects $A \in \mathrm{KK}^G$ such that $p^*(A)$ is contractible in $\mathrm{KK}^{G\ltimes \underline{E}G}$. Here, p^* is the pullback functor associated to the projection $p\colon G \ltimes \underline{E}G \to G$, where $\underline{E}G$ denotes the universal example for proper actions (which is well defined for groupoids; see, for example, [59]). This approach is based on the fact that p^* is, effectively speaking, the localization functor which we seek (see Theorem 3.12). However, this is not completely satisfactory because (**a**) it relies

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on the existence of a Kasparov dual [21, Theorem 4.37], and (**b**) it does not present the projective objects in terms of a simpler class of 'building blocks' constructed via induction on a suitable family of subgroupoids.

This paper remedies these shortcomings by using a 'slice theorem' (see Proposition 3.2 below and compare with [60, Proposition 2.42]) for étale groupoids acting properly on a space, which allows us to identify a family of subgroupoids that we call 'compact actions', as they are isomorphic to action groupoids for finite groups sitting inside the isotropy of G. On a first approximation, we can say that the family of compact subgroups is replaced in our case by the family of *proper* subgroupoids of G (see Lemma 3.16 for more details on this statement).

Having this, most of the machinery from [40] can be reproduced in the groupoid context in a straightforward fashion, as it is mostly formal and inherited from the more general theory of triangulated categories. We say 'most' because we encountered another technical difficulty along the way, which we now briefly explain. Having defined projective objects as retracts of (direct sums of) 'compactly induced' objects, we were facing the issue of identifying the localizing subcategory of proper objects with the one induced by projectives. Indeed, a result of this kind is highly desirable because not only it would match up nicely with the statement in [21], but more importantly it allows to rephrase the main result of [58], on the Baum–Connes conjecture for a groupoid G satisfying the Haagerup property, as a proof that the category KK^G is generated by projective objects as defined by us.

A blueprint for this result ought to be found in [40], and indeed [40, Theorem 7.1] and its applications correspond to the statement we need. Nevertheless, we were not able to simply generalize the proof therein, essentially because (**a**) our compact actions are *open* subgroupoids, and (**b**) the excisive properties of $\mathrm{RKK}^G(-;A,B)$ are not entirely clear (at least to us) in general, even in simple cases such as homotopy pushouts. Nevertheless, by briefly passing to E-theory (which has long exact sequences without extra hypotheses) and using the fact that localizing subcategories are closed under direct summands, we are able to find an alternative proof of the identification of localizing subcategories of (respectively) compactly induced and proper objects.

Before passing to the organization of the paper, we present two of the core results which should serve as a brief summary of this work. For more details on definitions and applications, the reader should consult Sections 1 and 4.

Theorem B. *Let $\mathcal{N} \subseteq \mathrm{KK}^G$ be the subcategory of G-C^*-algebras A such that $\mathrm{Res}_H^G(A) \cong 0$ for any proper open subgroupoid $H \subseteq G$. Let $\mathcal{P} \subseteq \mathrm{KK}^G$ be the smallest localizing triangulated subcategory containing proper G-C^*-algebras. Then $(\mathcal{P},\mathcal{N})$ is a pair of complementary subcategories and $\mathcal{P}$ is generated by 'compactly induced' objects (see Theorem 3.4 for details).*

The previous result implies that, for any $A \in \mathrm{KK}^G$, there is an exact triangle, functorial in A and unique up to isomorphism such that $P(A) \in \mathcal{P}$ and $N(A) \in \mathcal{N}$,

$$\Sigma N(A) \longrightarrow P(A) \longrightarrow A \longrightarrow N(A).$$

Following [39], the object $P(A)$ is called the *cellular approximation* of A. We should point out that if $P(C_0(G^0))$ is a *proper* G-C^*-algebra, then any $A \in \mathcal{P}$ is KK^G-equivalent to a proper C^*-algebra (see Remark 3.11).

The next result gives a more familiar presentation of the localization $\mathrm{KK}^G/\mathcal{N}$, and expresses the ordinary Baum–Connes conjecture in terms of the natural morphism $D_A\colon P(A) \to A$ introduced above. We can view this theorem as a bridge between the somewhat abstract notions arising via the triangulated category approach and more classical objects, such as the RKK-group and the 'topological' K-theory group appearing at the left-hand side of the Baum–Connes conjecture.

Theorem C. *Let $p\colon \underline{E}G \to G^0$ be the structure map of the G-action. The pullback functor descends to an isomorphism of categories $p^*\colon \mathrm{KK}^G/\mathcal{N} \to \mathrm{RKK}(\underline{E}G)$. The induced map $(D_A \rtimes_r G)_*\colon K_*(P(A) \rtimes_r G) \to K_*(A \rtimes_r G)$ corresponds to the assembly map under the natural identification $K_*^{\mathrm{top}}(G;A) \cong K_*(P(A) \rtimes_r G)$.*

The paper is organized as follows. In Section 1, we lay out the fundamental definitions and conventions which we use throughout the paper. We define groupoid crossed products, pass on to discussing the triangulated structure of the equivariant KK- and E-categories and finish with some basic results on complementary subcategories and homotopy direct limits. Section 2 is entirely dedicated to the main technical result of the paper, that is, an adjunction between the functors $\mathrm{Ind}_H^G\colon \mathrm{KK}^H \rightleftarrows \mathrm{KK}^G\colon \mathrm{Res}_G^H$.

This adjoint situation is the technical foundation for the main results of the paper. Its proof is fairly complicated in terms of bookkeeping of variables, but it does not require particularly new conceptual ideas. In fact, the definition for unit and counit are very intuitive in terms of the open inclusion $H \subseteq G$. The model for the induction functor is perhaps a minor point of novelty, as it is based on the crossed product construction rather than on (generalized) fixed-point algebras. This is especially useful as an open subgroupoid $H \subseteq G$ need not act on G properly (see Remark 2.1).

Section 3 is entirely dedicated to proving Theorems B and C above, along with some other auxiliary results. The excisive properties of E-theory are used in this section.

Section 4 discusses several applications of the main results of the paper. In particular, we give the precise statements and proofs of the results mentioned in Theorem A.

1. Preliminaries

Let G be a second countable, locally compact, Hausdorff groupoid with unit space G^0. We let $s,r\colon G \to G^0$ denote, respectively, the source and range maps. In addition, we use the notation $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, and for a subset $A \subset G^0$, we write $G_A = \bigcup_{x\in A} G_x$, $G^A = \bigcup_{x\in A} G^x$, and $G|_A = G^A \cap G_A$. Throughout this paper, we assume the existence of a (left) Haar system $\{\lambda^x\}_{x\in G^0}$ on G [53].

Let X be second countable, locally compact, Hausdorff space. A $C_0(X)$-*algebra* is a C^*-algebra A endowed with a nondegenerate $*$-homomorphism from $C_0(X)$ to the center of the multiplier algebra $\mathcal{M}(A)$. For an open set $U \subset X$, we define $A_U = C_0(U)A$. For a locally closed subset $Y \subset X$ (i.e., $Y = U \setminus V$ for some open sets $U,V \subset X$), we set

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$A_Y = A_U/A_{U\cap V}$, and we put $A_x = A_{\{x\}} = A/AC_0(X\setminus\{x\})$ for $x\in X$. More on $C_0(X)$-algebras can be found in [7].

Let us fix our preliminary conventions on tensor products. A more in-depth discussion is provided after Definition 1.6. If A and B are $C_0(X)$-algebras, their *maximal* tensor product $A\otimes B$ is naturally equipped with a $C_0(X\times X)$-structure, and we define the (maximal) balanced tensor product $A\otimes_X B$ as the $C_0(X)$-algebra $(A\otimes B)_{\Delta_X}$, where $\Delta_X\subseteq X\times X$ is the diagonal subspace.

Note that if $f\colon Y\to X$ is a continuous map, then $C_0(Y)$ is a $C_0(X)$-algebra. It is a continuous field if and only if f is open [8]. In particular, this applies to the situation $Y=G$ and $f=s$ because the source and range maps are open when a Haar system exists [53, Proposition 2.4]. The map f defines a 'forgetful' functor, sending a $C_0(Y)$-algebra A to a $C_0(X)$-algebra $f_*(A)$, by way of the composition $C_0(X)\to\mathcal{M}(C_0(Y))\to Z\mathcal{M}(A)$. In addition, for a $C_0(X)$-algebra B, a continuous function like f above also induces a *pullback* functor $f^*B = C_0(Y)\otimes_X B$ from the category of $C_0(X)$-algebras to that of $C_0(Y)$-algebras.

We are ready to define the notion of groupoid action on C^*-algebras.

Definition 1.1. Let G be a second countable locally compact Hausdorff groupoid, and put $G^0=X$. A *continuous action* of G on a $C_0(X)$-algebra A (with structure map ρ) is given by an isomorphism of $C_0(G)$-algebras

$$\alpha\colon C_0(G)_s{\otimes_\rho}_X A\to C_0(G)_r{\otimes_\rho}_X A$$

such that the induced homomorphisms $\alpha_g\colon A_{s(g)}\to A_{r(g)}$ for $g\in G$ satisfy $\alpha_{gh}=\alpha_g\alpha_h$. In this case, we say that A is a *G-C^*-algebra.*

If A is a commutative C^*-algebra, say $A\cong C_0(Z)$, then we view the moment map as a continuous function $\rho\colon Z\to X$. In this case, the action α can be given as a continuous map making the following diagram commute,

$$\begin{array}{ccc} G_s{\times}_\rho Z & \xrightarrow{\alpha} & Z \\ & \searrow^{r} & \downarrow{\scriptstyle\rho} \\ & & X \end{array}$$

(above, we are slightly abusing notation by writing r for the map $(g,z)\mapsto r(g)$). The action groupoid obtained this way will be denoted $G\ltimes Z$, it has unit space Z and its generic arrow is determined by a pair $(g,z)\in G\times Z$ with range z and source $\alpha(g^{-1},z)$.

Details on the construction of groupoid crossed product C^*-algebras can be found in [25, 44]. We are going to only briefly recap the definitions here. Given a G-algebra A, define the auxiliary algebra $A_0 = C_c(G)\cdot r^*A$ and the $*$-algebra structure

$$(f\star g)(\gamma)=\int f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))\,d\lambda^{r(\gamma)}(\eta)$$
$$f^*(\gamma)=\alpha_\gamma(f(\gamma^{-1})^*)$$

for $f,g\in A_0$. For $f\in A_0$, we also define $\|f\|_1$ to be the supremum, over $x\in X$, of the quantity $\max\{\int\|f(\gamma)\|\,d\lambda_x(\gamma),\int\|f(\gamma)\|\,d\lambda^x(\gamma)\}$, where $\lambda_x(\gamma)=\lambda^x(\gamma^{-1})$. The enveloping

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C^*-algebra of the Banach $*$-algebra obtained by completing A_0 with respect to $\|\cdot\|_1$ is called the *full* crossed product of A by G.

In this paper, unless otherwise stated, we are going to consider the *reduced* crossed product C^*-algebra of A by G, denoted $A \rtimes_r G$ (at times we might drop the subscript 'r'), which is obtained as a quotient of the full crossed product as follows. For $x \in X$, consider the A_x-Hilbert module $L^2(G^x, \lambda^x) \otimes A_x$. The formula $\Lambda_x(f)g = f \star g$ defines an adjointable operator and extends to a $*$-representation of the full crossed product.

Definition 1.2. The *reduced* crossed product $A \rtimes_r G$ is defined as the quotient of the full crossed product by the joint kernel of the family $(\Lambda_x)_{x\in X}$ of representations.

Let us consider the G-equivariant Kasparov category KK^G whose objects are separable and trivially graded C^*-algebras equipped with an action of G and whose set of morphisms $A \to B$ is Le Gall's groupoid equivariant Kasparov group $\mathrm{KK}^G(A,B)$ (see [30]); the composition in this category is the Kasparov product. We can view KK^G as a functor from the category of (separable) G-C^*-algebras sending equivariant $*$-homomorphisms $A \to B$ to their respective class in the abelian group $\mathrm{KK}^G(A,B)$. When viewed in this way, the functor KK^G enjoys an important property: It is the universal split-exact, C^*-stable and homotopy invariant functor (see [37, 49, 56] for more details).

Given a G-action on a space Z with moment map $p_Z \colon Z \to G^0$, we have introduced above the pullback functor p_Z^* sending G-C^*-algebras to $G \ltimes Z$-C^*-algebras. Thanks to the universal property discussed above, we can promote this functor to a functor between equivariant Kasparov categories $p_Z^* \colon \mathrm{KK}^G \to \mathrm{KK}^{G\ltimes Z}$. This will be particularly useful when we take Z to be a model for the classifying space for proper actions of G (and in this case we may use the notation $Z = \underline{E}G$) [59, Proposition 6.15].

Moreover, given a map $f \colon G \to G \ltimes Z$, the universal property ensures f_* yields well-defined functor between the corresponding KK-categories. Furthermore, when $f \colon X \to Z$ is *proper*, we have a standard adjunction (see [40])

$$\mathrm{KK}^{G\ltimes X}(f^*A, B) \cong \mathrm{KK}^{G\ltimes Z}(A, f_*B). \tag{1}$$

Finally, let us define the category $\mathrm{RKK}(Z)$ as follows.

Definition 1.3. The category $\mathrm{RKK}^G(Z)$ has the same objects as KK^G, and its Hom-sets $\mathrm{Hom}(A,B)$ are given by the abelian groups $\mathrm{KK}^{G\ltimes Z}(p_Z^*A, p_Z^*B)$.

For a map f as above (not necessarily proper), the functor $f^* \colon \mathrm{KK}^{G\ltimes Z} \to \mathrm{KK}^{G\ltimes X}$ induces natural maps (slightly abusing notation)

$$f^* \colon \mathrm{RKK}^G(Z; A, B) \to \mathrm{RKK}^G(X; A, B)$$

whenever the factorization $p_Z \circ f = p_Y$ holds. In this sense, for fixed A and B, RKK^G is a contravariant functor. It is also homotopy invariant, that is, $f_1^* = f_2^*$ if the maps f_1, f_2 are G-homotopic. In order to see this, note that we have an isomorphism

$$\mathrm{RKK}^G(Y \times [0,1]; A, B) \simeq \mathrm{RKK}^G(Y; A, B[0,1]) \tag{2}$$

induced by Equation (1), hence the claim follows from the homotopy invariance of $\mathrm{KK}^G(A,B)$ in the second variable B.

1.1. Triangulated structure and comparison with E-theory

Let us start by fixing some standard conventions. For a C^*-algebra A, we have a suspension functor ΣA defined as $\Sigma A = C_0(\mathbb{R}) \otimes A$. For an equivariant $*$-homomorphism of G-C*-algebras $f\colon A \to B$, we define its associated mapping cone by

$$\mathrm{Cone}(f) = \{(a,b_*) \in A \oplus C_0((0,1],B) \mid f(a) = b_1\}.$$

This inherits a structure of G-C*-algebra from A and B.

An *exact triangle* in KK^G is the data of a diagram of the form

$$A \to B \to C \to \Sigma A,$$

and a $*$-homomorphism $f\colon A' \to B'$ of G-C*-algebras, together with a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \Sigma A \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Sigma B' & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & A' & \longrightarrow & B', \end{array}$$

where the vertical arrows are equivalences in KK^G, and the rightmost downward arrow is equal to the leftmost downward arrow, up to applying Σ and the Bott periodicity isomorphism $\Sigma^2 B' \simeq B'$ in KK^G.

As we see from above, the most natural triangulated structure lives on the opposite category $(\mathrm{KK}^G)^{\mathrm{op}}$. The opposite category of a triangulated category inherits a canonical triangulated category structure, which has 'the same' exact triangles. The passage to opposite categories exchanges suspensions and desuspensions and modifies some sign conventions. Thus, the functor Σ becomes in principle a *desuspension* functor in KK^G, but due to Bott periodicity Σ and Σ^{-1} agree so that we can safely overlook this fact. Moreover, depending on the definition of triangulated category, one may want the suspension to be an equivalence or an isomorphism of categories. In the latter case, KK^G should be replaced by an equivalent category (see [40, Section 2.1]). This is not terribly important and will be ignored in the sequel.

The triangulated category axioms are discussed in greater detail in [45, 62]. Most of them amount to formal properties of mapping cones and mapping cylinders, which can be shown in analogy with classical topology. The fundamental axiom requires that any morphism $A \to B$ should be part of an exact triangle. In our setting, this can be proved as a consequence of the generalization of [37] to groupoid-equivariant KK-theory (see also [29, Lemma A.3.2]). Having done that, the rest of the proof follows the same outline of [40, Appendix A], where the triangulated structure is established in the case of action groupoids.

There is an alternative, perhaps more conceptual path which consists in *defining* the Kasparov category as a certain localization of the Spanier–Whitehead category associated to the standard tensor category of G-C^*-algebras and $*$-homomorphisms [20]. The triangulated structure of the Spanier–Whithead category is proved in [20, Theorem A.5.3]. The argument given there can be directly used to show that KK^G is triangulated, because it makes use of only two facts, which we prove below.

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Proposition 1.4. *Let $\mathcal{C}$ be the standard tensor category of separable G-C^*-algebras (with $\otimes_X$) and $*$-homomorphisms. Denote by F the canonical functor from $\mathcal{C}$ to KK^G. The following hold:*

- *up to an isomorphism of morphisms in KK^G, each morphism of KK^G is in the image of F;*
- *up to an isomorphism of diagrams $Q \to K \to D$ in KK^G, each composable pair of morphisms of KK^G is in the image of F.*

Proof. In order to show the lifting properties above we make use of 'extension triangles'. Let $f \in \mathrm{KK}_0^G(Q,K)$ be a morphism and denote by $\tilde{f}$ the corresponding element $\tilde{f} \in \mathrm{KK}_1^G(\Sigma Q,K)$. By applying [29, Lemma A.3.4], we can represent $\tilde{f}$ by a Kasparov module where the operator T is G-equivariant. Then the proof of [29, Lemma A.3.2] gives that $\tilde{f}$ is represented by an equivariant (semi-)split extension which fits a diagram as follows (see [40, Section 2.3]):

$$\begin{array}{ccccccc}
\Sigma^2 Q & \xrightarrow{f\beta_Q^{-1}} & K & \longrightarrow & E & \xrightarrow{p_f} & \Sigma Q \\
\| & & \downarrow{\scriptstyle \epsilon_K} & & \| & & \| \\
\Sigma^2 Q & \xrightarrow{\iota_f} & \mathrm{Cone}(p_f) & \longrightarrow & E & \xrightarrow{p_f} & \Sigma Q,
\end{array}$$

where β_Q is the Bott isomorphism and ϵ_K is an equivalence. Hence, we have that $F(\iota_f) \cong f$. Notice how this argument automatically shows that f is contained in an exact triangle (up to equivalence).

Now, given $g \in \mathrm{KK}_0^G(K,D)$, set $h = g \circ \epsilon_K^{-1}, C_f = \mathrm{Cone}(p_f)$ and consider the diagram

$$\begin{array}{ccccc}
Q & \xrightarrow{f} & K & \xrightarrow{g} & D \\
{\scriptstyle \beta_Q}\downarrow & & \downarrow{\scriptstyle \epsilon_K} & & \| \\
\Sigma^2 Q & \xrightarrow{\iota_f} & C_f & \xrightarrow{h} & D \\
{\scriptstyle \beta_{\Sigma^2 Q}}\downarrow & & \downarrow{\scriptstyle \beta_{C_f}} & & \downarrow{\scriptstyle \epsilon_D} \\
\Sigma^4 Q & \xrightarrow{\Sigma^2\iota_f} & \Sigma^2 C_f & \xrightarrow{\iota_h} & C_h.
\end{array}$$

This shows that the pair (f,g) can be lifted to a composable pair $(\Sigma^2\iota_f,\iota_h)$. $\square$

Remark 1.5. The proof above depends on the fact that extensions with an equivariant, contractive, completely positive section can be shown to be isomorphic to mapping cone triangles. From an abstract standpoint, we may express this by saying that KK^G is the result of the Verdier quotient [26, 45] of the Spanier–Whitehead category of G-C^*-algebras [20] by the thick tensor ideal of objects $\mathrm{Cone}(\epsilon_K)$, for all canonical comparison maps ϵ_K associated to equivariant semisplit extensions (to be precise, we need to take into account yet another class of morphisms, to ensure that KK^G is a stable functor; see [20, Section A6.1] and Definition 1.6 below).

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Definition 1.6. Let $\mathrm{SW}(C)$ be the Spanier–Whitehead category of the standard category of G-C^*-algebras, and let $\mathcal{I} \subseteq \mathrm{SW}(C)$ be the thick tensor ideal generated by the mapping cones of morphisms:

- ϵ_K for any extension $K \hookrightarrow E \twoheadrightarrow Q$ in C;
- $\mathcal{K}(H_1) \to \mathcal{K}(H_1 \oplus H_2)$ for any two nonzero G-Hilbert spaces H_1, H_2, where the map is induced by the canonical inclusion in the first factor.

The equivariant E-theory category is defined as the Verdier quotient $E_G = \mathrm{SW}(C)/\mathcal{I}$.

It should be clear from the definition above that E_G, viewed as functor from the category of separable G-C^*-algebras is the universal *half-exact*, C^*-stable, and homotopy invariant functor. In this sense, we can understand E-theory as the universal 'correction' of KK-theory in terms of excision properties. The universal property implies in particular that any functor between 'concrete' categories of C^*-algebras such as f_* and f^* extends to E-theory the same way it does for KK-theory.

By the same token, for a separable G-C^*-algebra B we can define a functor σ_B which is given by $\sigma_B(A) = A \otimes_X B$ on objects and $\sigma_B(\phi) = \phi \otimes 1_B$ on morphisms. It is important to discuss whether or not σ_B is a triangulated functor on our K-theory categories KK^G and E_G. By this, we mean whether or not σ_A preserves exact triangles. Since we are adopting the convention of using the maximal tensor product, the preservation of exact triangles is a simple consequence of the fact that $- \otimes B$ is an exact functor, and clearly it preserves semisplit extensions.

When B is $C_0(X)$-nuclear, that is, a continuous field over X with nuclear fibers [4], we have an isomorphism $A \otimes_X B \cong (A \otimes^{\min} B)_{\Delta_X}$ [7]. Note that this applies in particular to the pullback functor f^* associated to an open map $f\colon Y \to X$, such as the range and source maps $r, s\colon G \to G^0 = X$. Thus, if B is exact or $C_0(X)$-nuclear the functor σ_B is triangulated, regardless of the specific choice of tensor product.

The property of being $C_0(X)$-nuclear, or rather its K-theoretic counterpart called KK^X-*nuclearity*, is important to establish a useful identification between KK- and E-theory groups as follows. More information on KK^X-nuclearity can be found in [4]; here, we limit ourselves to record the following simple fact, which is proved in [58, Proposition 5.1 & Corollary 5.2] (see Definition 3.1 for proper groupoids).

Proposition 1.7. *Suppose G is proper. If A is KK^{G^0}-nuclear, for example, A is a continuous field over the unit space of G with nuclear fibers, then the functor $B \mapsto \mathrm{KK}^G(A,B)$ is half-exact.*

Having this, the following is a simple consequence of the universal properties.

Corollary 1.8 [48]. *If G is proper and A is a KK^G-nuclear C^*-algebra, there is a natural isomorphism $\mathrm{KK}^G(A,B) \cong E_G(A,B)$ for any separable G-C^*-algebra B.*

Proof. Denote by F the standard KK-functor from the category of separable C^*-algebras. The universal property of KK-theory gives us a map $\Phi_{C,B}\colon \mathrm{KK}^G(C,B) \to E_G(C,B)$. Let F' be the functor (from separable C^*-algebras) given by $F'(B) = \mathrm{KK}^G(A,B)$ and

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$F'(f\colon C\to B)$ induced by Kasparov product with $F(f)$. Since $\mathrm{KK}^G(A,-)$ is half-exact, the universal property of E-theory yields a map $\Psi_{C,B}\colon \mathrm{KK}^G(A,C)\times E_G(C,B)\to \mathrm{KK}^G(A,B)$. It is clear that $\Psi(-,\Phi\circ F)=F'$. In particular, for $f\colon A\to B$, we have

$$\Psi_{A,B}(1_A,\Phi_{A,B}F(f)) = F'(f)(1_A) = F(f),$$

which implies that $\Psi_{A,B}(1_A,-)$ is a left inverse for $\Phi_{A,B}$. The argument for showing it is a right inverse is analogous. □

1.2. Complementary subcategories and cellular approximation

In this subsection, we recall some facts about complementary subcategories, homotopy colimits in triangulated categories and the fundamental notion of *cellular approximation*. The material in this section is summarized from [38, 39, 40, 41].

Let $F\colon \mathcal{T}\to\mathcal{S}$ be an exact functor between triangulated categories. This means that F intertwines suspensions and preserves exact triangles. The kernel of F (on morphisms), denoted $\mathcal{I}=\ker F$, will be called a *homological ideal* (see [41, Remark 19]). We say that $\mathcal{I}$ is *compatible with direct sums* if F commutes with countable direct sums (see [39, Proposition 3.14]). Note that triangulated categories involving KK-theory have no more than countable direct sums because separability assumptions are needed for certain analytical results in the background.

An object $P\in\mathcal{T}$ is called *$\mathcal{I}$-projective* if $\mathcal{I}(P,A)=0$ for all objects $A\in\mathcal{T}$. An object $N\in\mathcal{T}$ is called *$\mathcal{I}$-contractible* if id_N belongs to $\mathcal{I}(N,N)$. Reference to $\mathcal{I}$ is often omitted in the sequel. Let $P_{\mathcal{I}},N_{\mathcal{I}}\subseteq\mathcal{T}$ be the full subcategories of projective and contractible objects, respectively.

We denote by $\langle P_{\mathcal{I}}\rangle$ the *localizing* subcategory generated by the projective objects, that is, the smallest triangulated subcategory that is closed under countable direct sums and contains $P_{\mathcal{I}}$. In particular, $\langle P_{\mathcal{I}}\rangle$ is closed under isomorphisms, suspensions, and if

$$A\longrightarrow B\longrightarrow C\longrightarrow \Sigma A$$

is an exact triangle in $\mathcal{T}$ where any two of the objects A,B,C are in $\langle P_{\mathcal{I}}\rangle$, so is the third. Note that $N_{\mathcal{I}}$ is localizing, and any localizing subcategory is *thick*, that is, closed under direct summands (see [45]).

Definition 1.9. Given an object $A\in\mathcal{T}$ and a chain complex

$$\cdots\xrightarrow{\delta_{n+1}} P_n\xrightarrow{\delta_n}\cdots\xrightarrow{\delta_1} P_0\xrightarrow{\delta_0} A\,, \tag{3}$$

we say that Equation (3) is a *projective resolution* of A if

- all the P_n's are projective;
- the chain complex below is split exact

$$F(P_\bullet)\xrightarrow{F(\delta_0)} F(A)\longrightarrow 0.$$

We say that $\mathcal{T}$ has *enough projectives* if any object admits a projective resolution.

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Proposition 1.10 [41, Proposition 44]. *The construction of projective resolutions yields a functor $\mathcal{T} \to \mathrm{Ho}(\mathcal{T})$. In particular, two projective resolutions of the same object are chain homotopy equivalent.*

Definition 1.11. We call two thick triangulated subcategories $\mathcal{P}$,$\mathcal{N}$ of $\mathcal{T}$ *complementary* if $\mathcal{T}(P,N) = 0$ for all $P \in \mathcal{P}, N \in \mathcal{N}$ and, for any $A \in \mathcal{T}$, there is an exact triangle

$$P \longrightarrow A \longrightarrow N \longrightarrow \Sigma P\ ,$$

where $P \in \mathcal{P}$ and $N \in \mathcal{N}$.

Proposition 1.12 [40, Proposition 2.9]. *Let $(\mathcal{P},\mathcal{N})$ be a pair of complementary subcategories of $\mathcal{T}$.*

- *We have $N \in \mathcal{N}$ if and only if $\mathcal{T}(P,N) = 0$ for all $P \in \mathcal{P}$. Analogously, we have $P \in \mathcal{P}$ if and only if $\mathcal{T}(P,N) = 0$ for all $N \in \mathcal{N}$.*
- *The exact triangle $P \to A \to N \to \Sigma P$ with $P \in \mathcal{P}$ and $N \in \mathcal{N}$ is uniquely determined up to isomorphism and depends functorially on A. In particular, its entries define functors*

$$P\colon \mathcal{T} \to \mathcal{P} \qquad\qquad N\colon \mathcal{T} \to \mathcal{N}$$
$$A \mapsto P \qquad\qquad A \mapsto N.$$

- *The functors P and N are respectively left adjoint to the embedding functor $\mathcal{P} \to \mathcal{T}$ and right adjoint to the embedding functor $\mathcal{N} \to \mathcal{T}$.*
- *The localizations $\mathcal{T}/\mathcal{N}$ and $\mathcal{T}/\mathcal{P}$ exist and the compositions*

$$\mathcal{P} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{N}$$
$$\mathcal{N} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{P}$$

  are equivalences of triangulated categories (see [26] for localization).
- *If $K\colon \mathcal{T} \to \mathcal{C}$ is a covariant functor, then its localization with respect to $\mathcal{N}$ is defined by $\mathbb{L}K = K \circ P$ and the natural maps $P(A) \to A$ provide a natural transformation $\mathbb{L}K \Rightarrow K$.*

The following result will be very important for us.

Theorem 1.13 [39, Theorem 3.16]. *Let $\mathcal{T}$ be a triangulated category with countable direct sums, and let $\mathcal{I}$ be a homological ideal with enough projective objects. Suppose that $\mathcal{I}$ is compatible with countable direct sums. Then the pair of localizing subcategories $(\langle P_{\mathcal{I}}\rangle, N_{\mathcal{I}})$ in $\mathcal{T}$ is complementary.*

A pair of complementary subcategories helps clarify the degree to which a projective resolution 'computes' a homological functor into the category of abelian groups. The object $P(A)$ resulting from Proposition 1.12 is called the *$P_{\mathcal{I}}$-cellular approximation* of A (it is called *simiplicial* approximation in [40]).

Definition 1.14. In general, the homotopy direct limit of a countable inductive system (A_n, α_m^n) is defined as the object A_∞^h fitting into the exact triangle below:

$$\bigoplus A_n \xrightarrow{\mathrm{id}-S} \bigoplus A_n \longrightarrow A_\infty^h \longrightarrow \Sigma \bigoplus A_n,$$

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where $S|_{A_n}: A_n \to A_{n+1}$ is just the connecting map α_n^{n+1}. We write $\text{ho-lim}(A_n, \alpha_m^n) = A_\infty^h$, or simply $\text{ho-lim}\, A_n$ when the connecting maps are clear from context.

Remark 1.15. The object $P(A)$ can be computed as the homotopy limit of an inductive system (P_n, ϕ_n) with $P_n \in P_{\mathcal{I}}$ (in fact, P_n belongs to a subclass of objects in $P_{\mathcal{I}}$, see [39, Proposition 3.18] for more details).

We mention a few more properties of this limit that will be useful for our later arguments. First of all, the last map in the triangle above is equivalent to a sequence of maps $\alpha_n^\infty : A_n \to A_\infty^h$ with the compatibility relation $\alpha_n^\infty \circ \alpha_m^n = \alpha_m^\infty$ when $m \leq n$.

Lemma 1.16 [45]. *Suppose F is a (co)homological functor, that is, it sends exact triangles to long exact sequences of abelian groups.*

- *(homological case): If $F(\bigoplus A_n) \cong \bigoplus F(A_n)$, then the maps α_n^∞ give an isomorphism $\varinjlim F_k(A_n) \cong F_k(A_\infty^h)$.*
- *(cohomological case): If $F(\bigoplus A_n) \cong \prod F(A_n)$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^1 F^{k-1}(A_n) \longrightarrow F^k(A_\infty^h) \longrightarrow \varprojlim F^k(A_n) \longrightarrow 0,$$

where the last map is induced by $(\alpha_n^\infty)_{n \in \mathbb{N}}$.

Let us consider the ordinary inductive limit of C^*-algebras A_∞ associated to the system (A_n, α_m^n), where the maps α_m^n are equivariant $*$-homomorphisms. We keep using α_n^∞ for the canonical maps $A_n \to A_\infty$. The relation between A_∞^h and A_∞, as discussed in [40, Section 2.4], is based on the notion of an *admissible* system in KK^G. We do not need this definition here, but we recall a sufficient condition: The system (A_n, α_m^n) is *admissible* if there exist equivariant completely positive contractions $\phi_n : A_\infty \to A_n$ such that $\alpha_n^\infty \circ \phi_n : A_\infty \to A_\infty$ converges to the identity in the point norm topology [40, Lemma 2.7]. The situation is simpler in E_G-theory: By Definition 1.6, since all extensions in E_G-theory are admissible, all inductive systems are admissible too.

Proposition 1.17. *We have $A_\infty^h \cong A_\infty$ in the category E_G. If the inductive system (A_n, α_m^n) is admissible, we have $A_\infty^h \cong A_\infty$ in the category KK^G.*

1.3. Crossed products of Hilbert modules and descent

In this section, we recall the notion of crossed product of Hilbert modules and define the Kasparov descent morphism in the context of groupoids. We will focus on *reduced* crossed products. To this end, we start by recasting $C_0(X)$-algebras under the perspective of C^*-bundles. If A is a $C_0(X)$-algebra, there exists a topology on $\mathcal{A} = \bigsqcup_{x \in X} A_x$ making the natural map $\mathcal{A} \to X$ an upper-semicontinuous C^*-bundle. The associated algebra of sections vanishing at infinity, denoted $\Gamma_0(X, \mathcal{A})$, admits a $C_0(X)$-linear isomorphism onto A. The correspondence $A \mapsto \mathcal{A}$ sends $C_0(X)$-linear morphisms to C^*-bundles morphisms.

If $f: Y \to X$ is a continuous map, the pullback C^*-algebra f^*A can also be defined by first constructing the pullback bundle $f^*\mathcal{A}$, then setting $f^*A = \Gamma_0(Y, f^*\mathcal{A})$. A G-action

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on A can be given by defining a functor from G (viewed as a category) to the category of C^*-algebras, sending $x \in X$ to A_x, then imposing continuity on the resulting G-action on the topological space $\mathcal{A}$. The definition of $A \rtimes G$ can then be reframed by endowing the compactly supported sections $\Gamma_c(G, r^*\mathcal{A})$ with a $*$-algebra structure and completing in the appropriate norm as explained previously.

Given a G-algebra (A,α) and a Hilbert A-module $\mathcal{E}$, for each $x \in X$ one defines the Hilbert A_x-module $\mathfrak{E}_x$ to be the balanced tensor product $\mathcal{E} \otimes_A A_x$. The space $\mathfrak{E} := \bigsqcup_{x\in X} \mathfrak{E}_x$ may be topologized to obtain an upper-semicontinuous Hilbert $\mathcal{A}$-module bundle $p_{\mathfrak{E}} : \mathfrak{E} \longrightarrow X$. The space of sections $\Gamma_0(X;\mathfrak{E})$ is equipped with pointwise operations to furnish a Hilbert $\Gamma_0(X;\mathcal{A})$-module, to which $\mathcal{E}$ is canonically isomorphic as a Hilbert A-module. We will identify $\mathcal{E}$ with its associated section space $\Gamma_0(X;\mathfrak{E})$. We have associated bundles of C^*-algebras $\mathcal{K}(\mathfrak{E})$ and $\mathcal{L}(\mathfrak{E})$, whose fibres over $x \in X$ are $\mathcal{K}(\mathfrak{E}_x)$ and $\mathcal{L}(\mathfrak{E}_x)$, respectively (the former bundle is upper-semicontinuous). By the identification $\mathcal{E} = \Gamma_0(X;\mathfrak{E})$, we then also have $\mathcal{K}(\mathcal{E}) = \Gamma_0(X;\mathcal{K}(\mathfrak{E}))$ and $\mathcal{L}(\mathcal{E}) = \Gamma_b(X;\mathcal{L}(\mathfrak{E}))$ (strictly continuous bounded sections).

A G-action $\mathcal{E} = \Gamma_0(X;\mathfrak{E})$ consists of a family $\{W_\gamma\}_{\gamma\in G}$ such that:

- for each $\gamma \in G$, $W_\gamma : \mathfrak{E}_{s(\gamma)} \longrightarrow \mathfrak{E}_{r(\gamma)}$ is an isometric isomorphism of Banach spaces such that $\langle W_\gamma e, W_\gamma f\rangle_{r(\gamma)} = \alpha_\gamma(\langle e,f\rangle_{s(\gamma)})$ for all $e,f \in \mathfrak{E}_{s(\gamma)}$;
- the map $G_s\times_{p_{\mathfrak{E}}} \mathfrak{E} \longrightarrow \mathfrak{E}$, $(\gamma,e) \mapsto W_\gamma e$ defines a continuous action of G on $\mathfrak{E}$.

Conjugation by W gives rise to a strictly continuous action $\varepsilon : G_s \times_{p_{\mathfrak{E}}} \mathcal{L}(\mathfrak{E}) \longrightarrow \mathcal{L}(\mathfrak{E})$ of G on the upper-semicontinuous bundle $\mathcal{L}(\mathfrak{E})$ (the restriction of ε to the compact operators is continuous in the usual sense).

If (B,β) is a G-algebra and $\pi\colon B \to \mathcal{L}(\mathcal{E})$ a $C_0(X)$-linear representation, we define a G-representation by requiring equivariance, namely for all $\gamma \in G$ we have

$$\varepsilon_\gamma \circ \pi_{s(\gamma)} = \pi_{r(\gamma)} \circ \beta_\gamma.$$

Given a Kasparov module $(\pi,\mathcal{E},T)$ representing a class in $\mathrm{KK}^G(B,A)$, let us consider the $B \rtimes_r G$-$A \rtimes_r G$-module $(\tilde{\pi}, \mathcal{E}\widehat{\otimes}_A(A\rtimes_r G), T\widehat{\otimes}1)$, where $\tilde{\pi}$ is a representation of $B \rtimes_r G$ induced by π as follows. First of all, note that $\mathcal{E}\widehat{\otimes}_A(A\rtimes_r G)$ is isomorphic to the completion of $\Gamma_c(G, r^*\mathfrak{E})$ with respect to the $\Gamma_c(G;r^*\mathcal{A})$-valued inner product

$$\langle \xi,\xi'\rangle(\gamma) := \int_G \alpha_\eta\big(\langle \xi(\eta^{-1}), \xi'(\eta^{-1}\gamma)\rangle_{s(\eta)}\big)\, d\lambda^{r(\gamma)}(\eta),$$

for $\xi,\xi' \in \Gamma_c(G;r^*\mathfrak{E})$ and $\gamma \in G$. We denote this completion $\mathcal{E} \rtimes G$. Consider the formula below, defined for $f \in \Gamma_c(G;r^*\mathcal{A})$, $\xi \in \Gamma_c(G;r^*\mathfrak{E})$, and $\gamma \in G$,

$$(f\cdot\xi)(\gamma) := \int_G \pi_{r(\eta)}(f(\eta)) W_\eta\big(\xi(\eta^{-1}\gamma)\big)\, d\lambda^{r(\gamma)}(\eta).$$

This determines a bounded representation $\tilde{\pi} = \pi \rtimes G : A \rtimes_r G \longrightarrow \mathcal{L}(\mathcal{E} \rtimes G)$ (see, for example, [34, Prop. 7.6]).

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Definition 1.18. We define the Kasparov *descent* morphism to be the homomorphism of abelian groups

$$\jmath^G\colon \mathrm{KK}^G(B,A) \to \mathrm{KK}(B \rtimes_r G, A \rtimes_r G)$$

which sends the class of $(\pi,\mathcal{E},T)$ to the class of $(\tilde{\pi},\mathcal{E}\widehat{\otimes}_A(A \rtimes_r G),T\widehat{\otimes}1)$.

It can be checked that $\jmath^G$ is compatible with the product in KK^G, meaning that $\jmath^G(x\,\widehat{\otimes}_D\, y) = \jmath^G(x)\,\widehat{\otimes}_{D\rtimes_r G}\,\jmath^G(y)$, giving us a well-defined functor [31, Theorem 3.4].

2. Induction-restriction adjunction

Consider a subgroupoid $H \subseteq G$. The inclusion map $H \hookrightarrow G$ induces a natural restriction functor $\mathrm{Res}_G^H\colon \mathrm{KK}^G \to \mathrm{KK}^H$. In this section, we will construct a functor in the other direction, called the induction functor, and prove that these two functors are adjoint when $H \subseteq G$ is *open*. This generalizes earlier results for transformation groups [40] and ample groupoids [10].

2.1. The induction functor

Let $(B,\beta) \in \mathrm{KK}^H$ with moment map $\rho\colon C_0(H^0) \to Z(\mathcal{M}(B))$. In this subsection, it is sufficient to assume H is locally closed in G. Recall G_{H^0} is the subspace of G consisting of arrows with source in H^0. We consider the restriction of the source map $\phi = s|_{G_{H^0}}$: $G_{H^0} \to H^0$ and construct the pullback algebra

$$\phi^* B = C_0(G_{H^0})_s{\otimes}_{\rho\, H^0} B.$$

This balanced tensor product is then a $C_0(H^0)$-algebra in its own right and can be equipped with the diagonal action $\mathrm{rt}\otimes\beta$ of H, where rt denotes the action of H on $C_0(G_{H^0})$ induced by right translation. We define the induced algebra as the corresponding reduced crossed product

$$\mathrm{Ind}_H^G B := (C_0(G_{H^0})_s{\otimes}_{\rho\, H^0} B) \rtimes_{\mathrm{rt}\otimes\beta} H.$$

To define a G-action on $\mathrm{Ind}_H^G B$, notice that G also acts on the balanced tensor product $C_0(G_{H^0}) \otimes_{H^0} B$ by $\mathrm{lt}\otimes \mathrm{id}_B$, where lt denotes the action of G on $C_0(G_{H^0})$ induced by left translation. A straightforward computation reveals that the actions $\mathrm{rt}\otimes\beta$ and $\mathrm{lt}\otimes\mathrm{id}_B$ commute and therefore the left translation action of G descends to an action on the crossed product $(C_0(G_{H^0}) \otimes_{H^0} B) \rtimes_{\mathrm{rt}\otimes\beta} H$.

Having defined Ind_H^G on objects, let us consider the case of morphisms. Consider a right Hilbert B-module $\mathcal{E}$. Considering the canonical action $B \longrightarrow \mathcal{M}(C_0(G_{H^0}) \otimes_{H^0} B)$ given by multiplication in the second factor, we can form the ϕ^*B-module

$$\phi^*\mathcal{E} = \mathcal{E} \otimes_B (C_0(G_{H^0}) \otimes_{H^0} B).$$

Note the module above corresponds to the space of section of the pullback bundle $\phi^*\mathfrak{E}$. Assume now that $\mathcal{E}$ carries an action of H (call it ϵ) along with a nondegenerate equivariant representation $\pi\colon A \to \mathcal{L}(\mathcal{E})$ of an H-algebra A. First of all, we note that $\epsilon\otimes(\mathrm{rt}\otimes\beta)$ defines an H-action on $\phi^*\mathcal{E}$. Then we define a representation of ϕ^*A on $\phi^*\mathcal{E}$

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by considering elements $f \otimes a$, with $f \in C_c(G_{H^0})$ and $a \in A$, whose linear span is dense in $\Gamma_c(G_{H^0}, \phi^*\mathcal{A}) \subseteq \phi^* A$ and setting $\phi^*\pi(f \otimes a) = \pi(a) \otimes (f \cdot)$.

Now, if $(\pi, \mathcal{E}, T)$ is an A-B-Kasparov module, then $(\phi^*\pi, \phi^*\mathcal{E}, T\widehat{\otimes}1)$ is a ϕ^*A-ϕ^*B-module equipped with an action of H, and we can define the induction functor by means of the descent morphism defined above, as follows:

$$\mathrm{Ind}_H^G(\pi, \mathcal{E}, T) = \jmath_H(\phi^*\pi, \phi^*\mathcal{E}, T\widehat{\otimes}1).$$

To complete the description of Ind_H^G, we need two more observations. The ϕ^*B-module $\mathcal{E} \otimes_B (C_0(G_{H^0}) \otimes_{H^0} B)$ admits a G-action induced by left translation on $C_0(G_{H^0})$. Notice this action is defined by fibreing over the range map. Clearly $T\widehat{\otimes}1$ is equivariant with respect to this translation. To check the equivariance of $\phi^*\pi$, by definition it is sufficient to consider $\gamma \in G$ and $f \in C_c(G_{H^0})$, and write

$$[\gamma \cdot (f \cdot (\gamma^{-1} \cdot g))](\eta) = f(\gamma\eta) g(\gamma\gamma^{-1}\eta) = (\mathrm{lt}_\gamma(f) \cdot g)(\eta)$$

with $g \in C_0(G_{H^0})$, $\eta \in G_{H^0}$ with $r(\eta) = s(\gamma)$. This ensures the G-action commutes with the H-action on $(\phi^*\pi, \phi^*\mathcal{E}, T\widehat{\otimes}1)$, hence $\jmath_H(\phi^*\pi, \phi^*\mathcal{E}, T\widehat{\otimes}1) \in \mathrm{KK}^G(A,B)$. Finally, as Ind_H^G is defined as a composition of the pullback functor ϕ^* with the descent functor $\jmath^G$, it is indeed a functor $\mathrm{Ind}_H^G \colon \mathrm{KK}^H \to \mathrm{KK}^G$.

Remark 2.1. Both the descent functor $\jmath_G : \mathrm{KK}^G \to \mathrm{KK}$ and the induction functor $\mathrm{Ind}_H^G : \mathrm{KK}^H \to \mathrm{KK}^G$ can be abstractly constructed using the universal property of equivariant KK-theory, by observing that the respective constructions on the C^*-level are compatible with split-exact sequences, stabilisations and homotopies (compare [41]). In many applications, however, it is useful to have a concrete model at hand. This is certainly the case for the adjunction result in Theorem 2.3 below but has also proven to be a useful construction in [49, 9].

The model for the induction functor in [10] is different from the one employed here. Given an H-C^*-algebra A, the construction of $\mathrm{Ind}_H^G(A)$ in [10] prescribes constructing the pullback algebra $\phi^*A = C_0(G_{H^0})_s \otimes_\rho A$ as above, but then considers the (generalized) fixed-point algebra ϕ^*A^H associated to the diagonal H-action. If H is acting properly on G, then the main result in [13] implies that ϕ^*A^H is strongly Morita equivalent to $\mathrm{Ind}_H^G(A)$. It is not hard to see that the imprimitivity bimodule witnessing this equivalence gives a G-equivariant KK-equivalence.

It should be noted that, when $H \subseteq G$ is closed (hence $G \rtimes H$ is proper), then the spectrum of $\phi^*C_0(Z)^H$ is homeomorphic to the ordinary induction space $G \times_H Z$ (see [10, Proposition 3.22]). However, if $H \subseteq G$ is open, then it need not act properly on G, and it is well known that quotients by nonproper actions can lead to pathological topological spaces (e.g., non-Hausdorff, nonlocally compact). It is for this reason that in this paper, where induction from *open* subgroupoids is considered, we have taken the approach of defining induction via crossed products.

2.2. Proof of the adjunction

Recall that if G acts freely and properly on a second countable, locally compact, Hausdorff space Y, then $G \ltimes Y$ is Morita equivalent as a groupoid to Y/G and hence the groupoid

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C^*-algebra $C_0(Y) \rtimes G \cong C^*(G \ltimes Y)$ is strongly Morita equivalent to $C_0(Y/G)$ [13]. Note that $G \ltimes Y$ is an amenable groupoid, so the reduced and full crossed products are isomorphic; see, for example, [1, Corollary 2.1.17 & Proposition 6.1.10]).

In particular, when Y equals G itself and the action is given by right translation, the associated imprimitivity bimodule X^G gives a $*$-isomorphism $C_0(G) \rtimes_{\mathrm{rt}} G \cong \mathcal{K}(L^2(G))$, where $L^2(G)$ is the standard continuous field of Hilbert spaces associated to G. The KK-class induced by X^G will be important in a moment.

If (A,G,α) is a groupoid dynamical system, then the pushforward along the source map $s_*\alpha$ is an isomorphism of C^*-dynamical systems:

$$s_*\alpha : (s_*(C_0(G)_s \otimes_{\rho\, G^0} A), G, \mathrm{rt} \otimes \alpha) \to (s_*(C_0(G)_r \otimes_{\rho\, G^0} A), G, \mathrm{rt} \otimes \mathrm{id}_A),$$

where the intertwining map is given precisely by α [30]. As a consequence, we have the following.

Lemma 2.2. *If $H \subseteq G$ is a locally closed subgroupoid and A is a G-algebra, then we have a canonical isomorphism*

$$\Phi\colon\ \mathrm{Ind}_H^G \mathrm{Res}_G^H A \cong (C_0(G_{H^0}) \rtimes_{\mathrm{rt}} H) \otimes_{G^0} A.$$

After Φ, the G-action on the right-hand side is given by $lt \otimes \alpha$, that is, left translation on $C_0(G_{H^0}) \rtimes_{\mathrm{rt}} H$, tensorized with the original action α on A.

Proof. Let $\alpha : s^*A \longrightarrow r^*A$ denote the $C_0(G)$-linear isomorphism implementing the action of G on A. Now, we can consider the pushforward along the source maps to obtain a $C_0(G^0)$-linear isomorphism $\alpha = s_*\alpha : s_*s^*A \longrightarrow s_*r^*A$. Now, s_*s^*A is just the balanced tensor product $C_0(G) \otimes_{G^0} A$ with the canonical $C_0(G^0)$-algebra structure, while $s_*r^*A = \Gamma_0(G, r^*\mathcal{A})$ is equipped with the $C_0(G^0)$-algebra structure obtained by the formula $(\varphi \cdot f)(g) = \varphi(s(g))f(g)$ for $\varphi \in C_0(G^0)$ and $f \in \Gamma_0(G, r^*\mathcal{A})$. Note that this differs from the canonical structure it obtains as a balanced tensor product! With the structure defined above we can identify the fibre over a point $x \in G^0$ as $\Gamma_0(G, r^*\mathcal{A})_x = \Gamma_0(G_x, r^*\mathcal{A})$ and it makes sense to consider the action $\mathrm{rt} \otimes \mathrm{id}_A$ defined by

$$(\mathrm{rt} \otimes \mathrm{id}_A)_g(f)(h) = f(hg).$$

Summing up the discussion, we see that α implements an isomorphism of groupoid dynamical systems

$$(C_0(G)_s \otimes_{\rho\, G^0} A, G, \mathrm{rt} \otimes \alpha) \to (C_0(G)_r \otimes_{\rho\, G^0} A, G, \mathrm{rt} \otimes \mathrm{id}_A).$$

Now, if we restrict these systems to the subgroupoid H we obtain an isomorphism

$$(C_0(G_{H^0})_s \otimes_{\rho\, H^0} \mathrm{Res}_G^H A, H, \mathrm{rt} \otimes \alpha) \to (C_0(G_{H^0})_r \otimes_{\rho\, G^0} A, H, \mathrm{rt} \otimes \mathrm{id}_A).$$

In particular, we obtain an isomorphism between the crossed products and hence conclude

$$\begin{aligned}\mathrm{Ind}_H^G \mathrm{Res}_G^H A &= (C_0(G_{H^0})_s \otimes_{\rho\, H^0} \mathrm{Res}_G^H A) \rtimes_{r, \mathrm{rt} \otimes \alpha} H \\ &\cong (C_0(G_{H^0})_r \otimes_{\rho\, G^0} A) \rtimes_{r, \mathrm{rt} \otimes \mathrm{id}_A} H \\ &\cong (C_0(G_{H^0}) \rtimes_{\mathrm{rt}} H) \otimes_{G^0} A.\end{aligned}$$

□

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Choosing $H = G$ in the result above yields an isomorphism

$$\mathrm{Ind}_G^G \mathrm{Res}_G^G(B) \cong (C_0(G) \rtimes_{\mathrm{rt}} G) \otimes_{G^0} B \cong \mathcal{K}(L^2(G)) \otimes_{G^0} B.$$

We now prepare to prove the adjunction by defining some auxiliary maps. From now on, we assume $H \subseteq G$ to be an open subgroupoid. We get an induced embedding

$$C_0(G_{H^0}) \rtimes_{\mathrm{rt}} H \hookrightarrow C_0(G) \rtimes_{\mathrm{rt}} G$$

and hence, using the previous lemma, an embedding

$$\kappa\colon \mathrm{Ind}_H^G \mathrm{Res}_G^H(B) \longrightarrow \mathcal{K}(L^2(G)) \otimes_{G^0} B.$$

We can promote X^G to a KK^G-equivalence

$$X_A^G \in \mathrm{KK}^G(\mathrm{Ind}_G^G \mathrm{Res}_G^G(A), A)$$

given by the right A-module $L^2(G)_r \otimes_\rho A$, where A acts pointwise as 'constant functions'. The representation of the crossed product $r^*A \rtimes G \cong \mathrm{Ind}_G^G \mathrm{Res}_G^G(A)$ is the integrated form of the covariant pair given by the right regular representation of G, and pointwise multiplication of functions in r^*A. We will denote this by $M_A \rtimes R_G$.

Now, let $B \in \mathrm{KK}^H$ and recall that

$$\mathrm{Res}_G^H \mathrm{Ind}_H^G(B) = (C_0(G|_{H^0}) \otimes_{H^0} B) \rtimes H.$$

Then the inclusion $C_0(H) \subseteq C_0(G|_{H^0})$ induces a map

$$\iota\colon \mathrm{Ind}_H^H B \cong (C_0(H) \otimes_{H^0} B) \rtimes H \to (C_0(G|_{H^0}) \otimes_{H^0} B) \rtimes H = \mathrm{Res}_G^H \mathrm{Ind}_H^G(B).$$

Theorem 2.3. *Let G be a locally compact Hausdorff groupoid with Haar system. For every open subgroupoid $H \subseteq G$, there is an adjunction*

$$(\epsilon, \eta)\colon \mathrm{Ind}_H^G \dashv \mathrm{Res}_G^H$$

with counit and unit

$$\epsilon\colon \mathrm{Ind}_H^G \mathrm{Res}_G^H \to 1_{\mathrm{KK}^G}$$
$$\eta\colon 1_{\mathrm{KK}^H} \to \mathrm{Res}_G^H \mathrm{Ind}_H^G$$

described as follows:

$$\epsilon_A = X_A^G \circ \kappa$$
$$\eta_B = \iota \circ \left(X_B^H\right)^{op}.$$

Here below we isolate a couple of technical lemmas which will be useful in the proof of the adjunction. The first lemma is just an observation on the compatibility of the canonical element X_A^G with restriction and induction.

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Lemma 2.4. *Let $H \subseteq G$ be an open subgroupoid and $A \in \mathrm{KK}^G$. Then we have $X^H_{\mathrm{Res}^H_G A} = \sigma_{\mathrm{Res}^H_G A}(X^H_{C_0(H^0)})$ and $\mathrm{Res}^H_G(X^G_A) = \sigma_{\mathrm{Res}^H_G A}(\mathrm{Res}^H_G(X^G_{C_0(G^0)}))$.*

Proof. The first equality is immediate from the definition of X^G_A and the isomorphism $\mathrm{Ind}^G_G \mathrm{Res}^G_G(A) \cong \mathcal{K}(L^2(G)) \otimes_{G^0} A$ explained above. The second equality follows from the first and the fact that restriction and tensorization commute. □

Let $L^2(G,B)$ denote the completion of $\Gamma_c(G, r^*\mathcal{B})$ with respect to the B-valued inner product $\langle \xi_1, \xi_2 \rangle(x) = \int_{G^x} \xi_1(g)^* \xi_2(g)\, d\lambda^x(g)$. Note that $L^2(G,B)$ is canonically isomorphic to the B-module $L^2(G) \otimes_{G^0} B$ introduced above.

Let us make a point on notation before continuing the proof. So far, we have used A and $\mathcal{A}$ to denote a $C_0(X)$-C^*-algebra and its corresponding C^*-bundle. However, this difference in font is not very convenient when A is replaced by a more complicated algebra, for example, $A = C_0(G) \rtimes H$. In the sequel, we suppress this notational distinction, as the context suffices to disambiguate the usage.

Lemma 2.5. *Let $H \subseteq G$ be an open subgroupoid and $B \in \mathrm{KK}^H$. Then there is an isometric G-equivariant homomorphism*

$$\Phi\colon \mathrm{Ind}^G_H L^2(H,B) \longrightarrow L^2(G, \mathrm{Ind}^G_H B)$$

of Hilbert $\mathrm{Ind}^G_H B$-modules.

Proof. Let us first describe the module $\mathrm{Ind}^G_H L^2(H,B)$ more concretely. We have a canonical isomorphism $L^2(H,B) \otimes_B (C_0(G_{H^0}) \otimes_{H^0} B) \cong L^2(H, C_0(G_{H^0}) \otimes_{H^0} B)$ given by $\xi \otimes f \mapsto [h \mapsto \xi(h) f]$. Hence, we can write $\mathrm{Ind}^G_H L^2(H,B)$ as $L^2(H, C_0(G_{H^0}) \otimes_{H^0} B) \rtimes H$. So for a function $\xi \in \Gamma_c(H, r^* L^2(H, C_0(G_{H^0}) \otimes_{H^0} B))$, we define $\Phi(\xi) \in L^2(G, \mathrm{Ind}^G_H B)$ as

$$\Phi(\xi)(g,h,x) = \left\{ \begin{array}{ll} \beta_{x^{-1}g}(\xi(g^{-1}xh, g^{-1}x, g)), & g^{-1}x \in H \\ 0, & \text{otherwise} \end{array} \right\},$$

where $g \in G$, $h \in H$ and $x \in G^{r(g)}_{r(h)}$.

Given $\xi_1, \xi_2 \in \Gamma_c(H, r^* L^2(H, C_0(G_{H^0}) \otimes_{H^0} B))$, we compute (for $h \in H$ and $x \in G_{r(h)}$) that $\langle \Phi(\xi_1), \Phi(\xi_2) \rangle(h,x)$ equals

$$\begin{aligned}
&\int_G [\Phi(\xi_1)(g)^* \Phi(\xi_2)(g)]\,(h,x)\, d\lambda^{r(x)}(g) \\
&= \int_G \int_H (\mathrm{rt} \otimes \beta)_{\tilde{h}} (\Phi(\xi_1)(g, \tilde{h}^{-1})^* \Phi(\xi_2)(g, \tilde{h}^{-1}h))(x)\, d\lambda^{r(h)}(\tilde{h})\, d\lambda^{r(x)}(g) \\
&= \int_G \int_H \beta_{\tilde{h}} (\Phi(\xi_1)(g, \tilde{h}^{-1}, x\tilde{h})^* \Phi(\xi_2)(g, \tilde{h}^{-1}h, x\tilde{h}))\, d\lambda^{r(h)}(\tilde{h})\, d\lambda^{r(x)}(g) \\
&= \int_{xH} \int_H \beta_{x^{-1}g} (\xi_1(g^{-1}x, g^{-1}x\tilde{h}, g)^* \xi_2(g^{-1}xh, g^{-1}x\tilde{h}, g))\, d\lambda^{r(h)}(\tilde{h})\, d\lambda^{r(x)}(g).
\end{aligned}$$

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At this point, we perform two change of variables and keep computing:

$$\overset{g\mapsto xg}{=} \int_H \int_H \beta_g(\xi_1(g^{-1},g^{-1}\tilde{h},xg)^*\xi_2(g^{-1}h,g^{-1}\tilde{h},xg))\,d\lambda^{r(h)}(\tilde{h})\,d\lambda^{s(x)}(g)$$
$$\overset{\tilde{h}\mapsto g\tilde{h}}{=} \int_H \int_H \beta_g(\xi_1(g^{-1},\tilde{h},xg)^*\xi_2(g^{-1}h,\tilde{h},xg))\,d\lambda^{s(g)}(\tilde{h})\,d\lambda^{s(x)}(g)$$
$$= \int_H \int_H \beta_{g^{-1}}(\xi_1(g,\tilde{h},xg^{-1})^*\xi_2(gh,\tilde{h},xg^{-1}))\,d\lambda^{r(g)}(\tilde{h})\,d\lambda_{s(x)}(g)$$
$$= \int_H (\mathrm{rt}\otimes\beta)_{g^{-1}}(\langle\xi_1(g),\xi_2(gh)\rangle(x)\,d\lambda_{r(h)}(g)$$
$$= \langle\xi_1,\xi_2\rangle(h,x).$$

This verifies that Φ extends to an isometry. Now, we proceed to checking that Φ is a right module map. Below, we have $\xi\in\Gamma_c(H,r^*L^2(H,C_0(G_{H^0})\otimes_{H^0}B))$ as before, and the element f belongs to $\Gamma_c(H,r^*(C_0(G_{H^0})_s\otimes_\rho B))$.

$$\begin{aligned}(\Phi(\xi)f)(g,h,x) &= \Phi(\xi)(g,h,x)f(h,x)\\ &= \int_{H^{r(g)}} \Phi(\xi)(g,\tilde{h},x)\beta_{\tilde{h}}(f(\tilde{h}^{-1}h,x\tilde{h}))\,d\lambda^{r(h)}(\tilde{h})\\ &= \int_{H^{r(g)}} \beta_{x^{-1}g}(\xi(g^{-1}x\tilde{h},g^{-1}x,g))\beta_{\tilde{h}}(f(\tilde{h}^{-1}h,x\tilde{h}))\,d\lambda^{r(h)}(\tilde{h})\\ &\overset{\tilde{h}\mapsto x^{-1}g\tilde{h}}{=} \int_{H^{s(g)}} \beta_{x^{-1}g}(\xi(\tilde{h},g^{-1}x,g)\beta_{\tilde{h}}(f(\tilde{h}^{-1}g^{-1}xh,g\tilde{h})))\,d\lambda^{s(g)}(\tilde{h})\\ &= \beta_{x^{-1}g}((\xi f)(g^{-1}xh,g^{-1}x,g))\\ &= \Phi(\xi f)(g,h,x).\end{aligned}$$

To complete the argument, we show that the left action of G commutes with Φ. Let us take $g'\in G$ with $r(g')=r(g)$, and compute

$$\begin{aligned}(g'\Phi(\xi))(g,h,x) &= \Phi(\xi)(g'^{-1}g,h,g'^{-1}x)\\ &= \beta_{x^{-1}g}(\xi(g^{-1}xh,g^{-1}x,g'^{-1}g))\\ &= \beta_{x^{-1}g}((g'\xi)(g^{-1}xh,g^{-1}x,g))\\ &= \Phi(g'\xi)(g,h,x).\end{aligned}$$

The proof is complete. □

Proof of Theorem 2.3. We need to verify the counit-unit equations. We start by proving that for every $A\in\mathrm{KK}^G$ the composition

$$\mathrm{Res}^H_G A \xrightarrow{\eta_{\mathrm{Res}^H_G A}} \mathrm{Res}^H_G \mathrm{Ind}^G_H \mathrm{Res}^H_G A \xrightarrow{\mathrm{Res}^H_G(\epsilon_A)} \mathrm{Res}^H_G A$$

equals the identity in $\mathrm{KK}^H(\mathrm{Res}^H_G A,\mathrm{Res}^H_G A)$: Expanding the definitions of counit and unit in this case, we have $\mathrm{Res}^H_G(\epsilon_A)\circ\eta_{\mathrm{Res}^H_G A}=\mathrm{Res}(X^G_A)\circ\mathrm{Res}(\kappa)\circ\iota\circ(X^H_{\mathrm{Res}^H_G A})^{\mathrm{op}}$. Following the definitions, it is then easily seen that after identifying

$$\operatorname{Ind}_H^H(\operatorname{Res}_G^H A) = (C_0(H) \rtimes_{\mathrm{rt}} H) \otimes \operatorname{Res}_G^H A$$
$$\operatorname{Res}_G^H(\operatorname{Ind}_G^G A) \cong \operatorname{Res}_G^H((C_0(G) \rtimes_{\mathrm{rt}} G) \otimes_{G^0} A) \cong (C_0(G^{H^0}) \rtimes_{\mathrm{rt}} G) \otimes_{H^0} \operatorname{Res}_G^H A,$$

the composition $\operatorname{Res}(\kappa) \circ \iota$ is just given by 00E9

$$(C_0(H) \rtimes H) \otimes_{H^0} \operatorname{Res}_G^H A \overset{j \otimes \mathrm{id}}{\longrightarrow} (C_0(G^{H^0}) \rtimes G) \otimes_{H^0} \operatorname{Res}_G^H A, \tag{4}$$

where $j : C_0(H) \rtimes H \longrightarrow C_0(G^{H^0}) \rtimes G$ is induced by the inclusion of H as an open subgroupoid. Using Lemma 2.4, we have

$$\begin{aligned}
\operatorname{Res}_G^H(\epsilon_A) \circ \eta_{\operatorname{Res}_G^H A} &= \operatorname{Res}(X_A^G) \circ \operatorname{Res}(\kappa) \circ \iota \circ (X^H_{\operatorname{Res}_G^H A})^{\mathrm{op}} \\
&= \sigma_{\operatorname{Res}_G^H A}(\operatorname{Res}_G^H(X^G_{C_0(G^0)}) \circ \sigma_{\operatorname{Res}_G^H A}(j) \circ \sigma_{\operatorname{Res}_H^G A}((X^H_{C_0(H^0)})^{\mathrm{op}}) \\
&= \sigma_{\operatorname{Res}_G^H A}\left(\operatorname{Res}_G^H(X^G_{C_0(G^0)}) \circ j \circ (X^H_{C_0(H^0)})^{\mathrm{op}}\right).
\end{aligned}$$

Hence, it is enough to show that the conclusion holds for $A = C_0(G^0)$. In this case, we can further use the isomorphisms $C_0(H) \rtimes_{\mathrm{rt}} H \cong \mathcal{K}(L^2(H))$ and $C_0(G^{H^0}) \rtimes_{\mathrm{rt}} G \cong \mathcal{K}(L^2(G^{H^0}))$ to replace the map in Equation (4) by the canonical map

$$i \colon \mathcal{K}(L^2(H)) \to \mathcal{K}(L^2(G^{H^0}))$$

and the required verification is easily seen to be reduced to showing that the (interior) Kasparov product

$$[(X^H_{C_0(H^0)})^{\mathrm{op}}] \widehat{\otimes}_{\mathcal{K}(L^2(H))} i^*[\operatorname{Res}_G^H(X^G_{C_0(G^0)})]$$

equals the class of identity $id_{C_0(H^0)}$ in $\mathrm{KK}^H(C_0(H^0), C_0(H^0))$.

The element $\operatorname{Res}_G^H(X^G_{C_0(G^0)}) \in \mathrm{KK}^H(\mathcal{K}(L^2(G^{H^0})), C_0(H^0))$ can be represented by the triple $(L^2(G^{H^0}), \Phi, 0)$, where Φ is the canonical action. Consequently, $i^*[\operatorname{Res}_G^H(X^G_{C_0(G^0)})$ is represented by $(L^2(G^{H^0}), \Phi \circ i, 0)$. The representation $\Phi \circ i$ fails to be nondegenerate, but we can replace $L^2(G^{H^0})$ by its 'nondegenerate closure' $\overline{\Phi \circ i(\mathcal{K}(L^2_s(H)))L^2(G^{H^0})}$ without changing its KK^H-class (see [6, Proposition 18.3.6]). This module is easily seen to be (isomorphic to) $L^2(H)$. Therefore, $i^*[\operatorname{Res}_G^H(X^G_{C_0(G^0)})] = [X^H_{C_0(H^0)}]$ and the desired equality follows from

$$[(X^H_{C_0(H^0))})^{\mathrm{op}}] \widehat{\otimes}_{\mathcal{K}(L^2(H))} [X^H_{C_0(H^0))}] = 1 \in \mathrm{KK}^H(C_0(H^0), C_0(H^0)).$$

The next verification in order regards the composition

$$\operatorname{Ind}_H^G(A) \xrightarrow{\operatorname{Ind}_H^G(\eta_A)} \operatorname{Ind}_H^G \operatorname{Res}_G^H \operatorname{Ind}_H^G(A) \xrightarrow{\epsilon_{\operatorname{Ind}_H^G(A)}} \operatorname{Ind}_H^G(A). \tag{5}$$

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The map $\kappa \circ \operatorname{Ind}_H^G(\iota)$ gives an inclusion

$$\begin{array}{c}\Big(C_0(G_{H^0})_s\otimes_r\Big[\Big(C_0(H)_s\otimes_\rho A\Big)\rtimes_{\mathrm{rt}\otimes\alpha} H\Big]\Big)\rtimes H\\ \Big\downarrow\\ \Big(C_0(G)_s\otimes_r\Big[\Big(C_0(G_{H^0})_s\otimes_\rho A\Big)\rtimes H_{\mathrm{rt}\otimes\alpha}\Big]\Big)\rtimes G.\end{array}$$

By using the isomorphisms introduced in Lemma 2.2 above, we can replace the previous inclusion into the more convenient map

$$\begin{array}{c}\Big(C_0(G_{H^0})_s\otimes_{r\otimes\rho}\Big[\Big(C_0(H)_r\otimes_\rho A\Big)\rtimes_{\mathrm{rt}\otimes\mathrm{id}} H\Big]\Big)\rtimes H\\ \Big\downarrow i\\ \Big(C_0(\overset{\gamma}{G})_r\otimes_r\Big[\Big(C_0(\overset{\nu}{G}_{H^0})_s\otimes_\rho A\Big)\rtimes \overset{\mu}{H}\Big]\Big)\rtimes_{\mathrm{rt}\otimes\mathrm{id}} \overset{\eta}{G}.\end{array}$$

Above, the Greek letters indicate our choice of notation for the variable on the given groupoid. These will be useful in a moment.

Recall the action on A is denoted by α. Suppressing notation for the inclusions $H\subseteq G$ and $C_0(H)\subseteq C_0(G)$, the map i can be understood by

$$i(f)(\eta,\gamma,\mu,\nu)=\alpha_{\nu^{-1}\gamma}(f(\eta,\gamma,\mu,\gamma^{-1}\nu)), \tag{6}$$

where f is in $\Gamma_c(H,r^*(C_0(G_{H^0})_s\otimes_{r\otimes\rho}(C_0(H)_r\otimes_\rho A)\rtimes_{\mathrm{rt}\otimes\mathrm{id}} H))$. Note that the right-hand side is zero unless $\gamma^{-1}\nu\in H$ and $\eta\in H$ (note $\gamma\in G_{H^0}$ follows). The composition in Equation (5) can be computed via the Kasparov product (over the domain of i)

$$[\operatorname{Ind}_H^G((X_A^H)^{\mathrm{op}})]\widehat{\otimes}\, i^*[X^G_{\operatorname{Ind}_H^G A}].$$

We claim that

$$i^*[X^G_{\operatorname{Ind}_H^G A}]=\operatorname{Ind}_H^G(X_A^H)$$

The class $i^*[X^G_{\operatorname{Ind}_H^G A}]$ is represented by the Kasparov triple

$$\Big(L^2(G,\operatorname{Ind}_H^G A),(M_{\operatorname{Ind}_H^G A}\rtimes R_G)\circ i,0\Big)$$

while the class $\operatorname{Ind}_H^G(X_A^H)$ is represented by

$$\Big(\operatorname{Ind}_H^G L^2(H,A),\operatorname{Ind}_H^G(M_A\rtimes R_H),0\Big).$$

Consider the isometric embedding

$$\Phi\colon\ \operatorname{Ind}_H^G L^2(H,A)\longrightarrow L^2(G,\operatorname{Ind}_H^G A)$$

from Lemma 2.5. We first verify that Φ intertwines the left actions of $\operatorname{Ind}_H^G\operatorname{Ind}_H^H A$. To this end, recall that for $f\in\Gamma_c(H,r^*(C_0(G_{H^0})\otimes_{G^0}\operatorname{Ind}_H^H A))$, we have that $(\operatorname{Ind}_H^G(M_A\rtimes R_H)(f)\xi)(g,h,x)$ equals

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$$\int_H \int_H f(h_1,x,h_2,h)\alpha_{h_1}(\xi(h_1^{-1}g,h_1^{-1}hh_2,xh_1))\,d\lambda^{s(h)}(h_2)\,d\lambda^{s(x)}(h_1).$$

Hence, considering elements $\xi \in \Gamma_c(H,r^*L^2(H,C_0(G_{H^0})\otimes_{H^0} B))$ and $f \in \Gamma_c(H,r^*(C_0(G)_s\otimes_r(C_0(H)_s\otimes_\rho A)\rtimes H))$, we compute

$$\begin{aligned}\Phi(\mathrm{Ind}_H^G(M_A\rtimes R_H)(f)\xi)(g,h,x) &= \alpha_{x^{-1}g}((\mathrm{Ind}_H^G(M_A\rtimes R_H)(f)\xi)(g^{-1}xh,g^{-1}x,g))\\ &= \int_H\int_H \alpha_{x^{-1}g}(f(h_1,g,h_2,g^{-1}x)\alpha_{h_1}(\xi(h_1^{-1}g^{-1}xh,h_1^{-1}g^{-1}xh_2,gh_1)))\,d\lambda^{s(x)}(h_2)\,d\lambda^{s(g)}(h_1)\\ &= \int_H\int_H i(f)(h_1,g,h_2,x)\alpha_{h_2}(\Phi(\xi)(gh_1,h_2^{-1}h,xh_2))\,d\lambda^{s(x)}(h_2)\,d\lambda^{s(g)}(h_1)\\ &= \Big((M_{\mathrm{Ind}_H^G A}\rtimes R_G)(i(f))\Phi(\xi)\Big)(g,h,x).\end{aligned}$$

Since the representation $\mathrm{Ind}_H^G(M_A\rtimes R_H)$ is nondegenerate, it follows immediately that $\mathrm{Img}(\Phi)\subseteq \overline{((M_{\mathrm{Ind}_H^G A}\rtimes R_G)\circ i)L^2(G,\mathrm{Ind}_H^G A)}$. In fact, since $\mathrm{Img}(\Phi)$ is closed, in order to have equality it suffices to show the image is dense. From the definition of i in Equation (6), we see that

$$\overline{((M_{\mathrm{Ind}_H^G A}\rtimes R_G)\circ i)L^2(G,\mathrm{Ind}_H^G A)} \subseteq \overline{L^2(G_{H^0},\mathrm{Ind}_H^G A)\cap F},$$

where F is spanned by those L^2-functions such that $f(g,h,x)=0$ unless $g^{-1}x\in H$ (notation from Lemma 2.5). With this, the surjectivity is clear from the formula for Φ in Lemma 2.5. Since the element $i^*[X^G_{\mathrm{Ind}_H^G A}]$ can equally well be represented by the submodule $\overline{((M_{\mathrm{Ind}_H^G A}\rtimes R_G)\circ i)L^2(G,\mathrm{Ind}_H^G A)}$ (see [6, Proposition 18.3.6]), we conclude that $i^*[X^G_{\mathrm{Ind}_H^G B}]=\mathrm{Ind}_H^G X_A^H$ and hence

$$[\mathrm{Ind}_H^G((X_A^H)^{\mathrm{op}})]\,\widehat{\otimes}\, i^*[X^G_{\mathrm{Ind}_H^G A}] = 1_{\mathrm{Ind}_H^G A},$$

as desired. □

2.3. Compatibility with other functors

Let $f\colon Y\to X$ be a continuous map, and A and B be $C_0(X)$-algebras. There is a natural isomorphism $f^*(A\otimes_X B)=f^*(A)\otimes_Y f^*(B)$ because both algebras are naturally isomorphic to restrictions of $C_0(Y\times Y)\otimes A\otimes B$ to the same copy of $Y\times X$ in the topological space $Y\times Y\times X\times X$ (cf. [12, Lemma 6.4])

Lemma 2.6. *There is a natural isomorphism*

$$\mathrm{Ind}_H^G(A)\otimes_{G^0} B\cong \mathrm{Ind}_H^G(A\otimes_{H^0}\mathrm{Res}_G^H(B)).$$

In particular, $\mathrm{Ind}_H^G\circ f^*\cong f^*\circ \mathrm{Ind}_H^G$.

Proof. Let ϕ be the restriction of the source map to G_{H^0}. We have

$$\phi^*(A\otimes_{H^0}\mathrm{Res}_G^H(B))\cong \phi^*A\otimes_{G_{H^0}}\phi^*\mathrm{Res}_G^H(B)$$

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by the observation above. Now, pushing forward along ϕ again we obtain an isomorphism of H-C^*-algebras

$$(\phi_*(\phi^*(A \otimes_{H^0} \operatorname{Res}_G^H(B))) \cong \phi_*(\phi^* A \otimes_{G_{H^0}} \phi^* \operatorname{Res}_G^H(B)) \cong \phi_*\phi^* A \otimes_{H^0} \operatorname{Res}_G^H B.$$

Now, when we take crossed products by H for the leftmost system, we get $\operatorname{Ind}_H^G(A \otimes_{H^0} \operatorname{Res}_G^H(B))$ by definition. The rightmost system is just $C_0(G_{H^0}) \otimes_{H^0} A \otimes_{H^0} \operatorname{Res}_G^H B$ with the diagonal H-action $\mathrm{rt} \otimes \alpha \otimes \operatorname{Res}_G^H(\beta)$. So upon using commutativity of the tensor product and applying Lemma 2.2, we may replace it by the action $\mathrm{rt} \otimes \alpha \otimes \mathrm{id}_B$.

Summing up, after taking crossed products by H we arrive at the desired conclusion:

$$\phi_*(\phi^*(A \otimes_X \operatorname{Res}_G^H(B))) \rtimes H \cong \phi_*\phi^* A \rtimes H \otimes_{G^0} B,$$

where $\phi_*\phi^* A \rtimes H = \operatorname{Ind}_H^G(A)$ by definition. □

We conclude this section by listing other compatibility relations, which are straightforward as each of them involves a forgetful functor.

$$\operatorname{Res}_G^H(A \otimes_X B) \cong \operatorname{Res}_G^H(A) \otimes_X \operatorname{Res}_G^H(B)$$
$$\operatorname{Ind}_H^G \circ f_* \cong f_* \circ \operatorname{Ind}_H^G \qquad \operatorname{Res}_G^H \circ f_* \cong f_* \circ \operatorname{Res}_G^H \qquad \operatorname{Res}_G^H \circ f^* \cong f^* \circ \operatorname{Res}_G^H.$$

3. The strong Baum–Connes conjecture

In this section, we formulate the strong Baum–Connes conjecture for *étale* groupoids by using the framework developed in the previous section.

As a start, a natural idea is identifying a 'probing' class of objects $\mathcal{P}r \subseteq \mathrm{KK}^G$, that we understand somewhat better than a generic object of KK^G, and for which we can prove the equality of categories $\langle \mathcal{P}r \rangle = \mathrm{KK}^G$.

Definition 3.1. We say that G is *proper* if the *anchor map* $(r,s)\colon G \to X \times X$ is proper. Furthermore, if Z is a a second countable, locally compact, Hausdorff G-space, we say that G *acts properly* on Z if $Z \rtimes G$ is proper. A G-algebra A is called proper if there is a proper G-space Z such that A is a $Z \rtimes G$-algebra.

We let $\mathcal{P}r$ denote the class of proper objects in KK^G.

Evidently, a commutative G-C^*-algebra is proper if and only if its spectrum is a proper G-space.

Recall that G is called étale if its source and range maps are local homeomorphisms. A *bisection* is an open $W \subseteq G$ such that $s|_W, r|_W$ are homeomorphisms onto an open in X. Hereafter, it is assumed that G is étale.

Recall that a map $f : X \to Y$ is proper at $y \in Y$ if

- the fiber at y is compact,
- any open containing the fiber also contains a tube (a tube is the preimage of an open neighborhood of y).

A map is proper if and only if it is proper at each point. The proposition below clarifies the local picture of proper actions (cf. [42, Theorem 4.1.1] and [60, Proposition 2.42]).

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Proposition 3.2. *Suppose G acts properly on Z and denote by $\rho\colon Z \to X$ the moment map. Then for each $z \in Z$ there are open neighborhoods U^ρ,U, respectively, of $z \in Z$ and $\rho(z) \in X$, satisfying:*

- *The fixgroup $\Gamma_z := \{g \in G \mid gz = z\}$ acts on U;*
- *There exists an isomorphism from $\Gamma_z \ltimes U$ onto an open subgroupoid H_z of $G|_U$;*
- *The G-action restricted to U^ρ is induced from $\Gamma_z \ltimes U$; in other words, the groupoid $(G \ltimes Z)|_{U^\rho}$ equals $(\Gamma_z \ltimes U) \ltimes U^\rho$.*

Proof. Since the G-action on Z is proper, Γ_z is a finite subgroup of the isotropy group $G^{\rho(z)}_{\rho(z)}$. For each $g \in \Gamma_z$, choose an open bisection W_g around g. Since G is Hausdorff and Γ_z is finite, we may assume that the W_g are pairwise disjoint. For any two $g,h \in \Gamma_z$, there is an open neighborhood V of $\rho(z)$ such that $W_{gh} \cap G|_V$ and $(W_g W_h) \cap G|_V$ are nonempty and equal because both are bisections containing gh. Likewise, for each g in Γ_z there is an open neighborhood V of $\rho(z)$, where $W_{g^{-1}} \cap G|_V$ and $(W_g)^{-1} \cap G|_V$ are nonempty and equal. Ranging over the group Γ_z, we collect a finite number of V's whose intersection we denote by U. Notice U is an open neighborhood of $\rho(z)$. We now replace all the W_g's by $W_g \cap r^{-1}(U) \cap s^{-1}(U)$. Then we can define an action of Γ_z on U by setting $g \cdot x := r(s^{-1}_{|W_g}(x))$, that is, g acts by the partial homeomorphism $U \to U$ associated with the bisection W_g. This is then indeed a well-defined action by the construction of the W_g above. We have a canonical continuous groupoid homomorphism

$$\Phi : \Gamma_z \ltimes U \to G,\ \Phi(g,x) = s^{-1}_{|W_g}(x).$$

Since the W_g were chosen pairwise disjoint this is in fact an isomorphism of topological groupoids onto the union $H := \bigsqcup_{g \in \Gamma_z} W_g$.

Define $U' := \rho^{-1}(U)$. Because G acts on Z, and H is a subgroupoid of G, the notation $U' \rtimes H$ makes sense, and it indicates an open subgroupoid of the restriction $(Z \rtimes G)|_{U'}$. The action of G on Z is proper; in particular, the anchor map of the groupoid $Z \rtimes G$ is proper at z. Now, $U' \rtimes H$ is an open containing the fiber of the anchor map at z; therefore, it contains a tube. In other words, there is an open neighborhood of z, say U^ρ (we may assume it is also contained in U'), such that the restriction $(Z \rtimes G)|_{U^\rho}$ (i.e., the tube at U^ρ) is contained in $U' \rtimes H$. This means that the groupoid that G induces on U^ρ only involves arrows belonging to H (recall that H is isomorphic to $U \rtimes \Gamma$). ☐

Remark 3.3. As a simple corollary of Proposition 3.2, the range map $r\colon s^{-1}(U^\rho) \to Z$ descends to a G-equivariant homeomorphism

$$G \times_H U^\rho \to G \cdot U^\rho = V. \tag{7}$$

Moreover, the space $s^{-1}(U^\rho)$ provides a principal bibundle implementing an equivalence between $(G \rtimes Z)|_{U^\rho}$ and $(G \rtimes Z)|_V$ in the sense of [43] (cf. [19]). Hence, the induction functor $\mathrm{KK}^{(G \rtimes Z)|_{U^\rho}} \to \mathrm{KK}^{(G \rtimes Z)|_V}$ is essentially surjective [30], that is, if A is a G-algebra over Z, then $A|_V$ is isomorphic to $\mathrm{Ind}^{(G \rtimes Z)|_V}_{(G \rtimes Z)|_{U^\rho}}(A|_{U^\rho})$. We can forget the $C_0(Z)$-structure and obtain $A|_V \cong \mathrm{Ind}^G_H(A|_{U^\rho})$ in KK^G.

In Definition 3.1 for a proper G-algebra, we can always assume Z to be a realization of $\underline{E}G$, the classifying space for proper actions of G. Indeed, if $\phi\colon Z \to \underline{E}G$ is a G-equivariant

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continuous map, then $\phi^*\colon C_0(\underline{E}G) \to M(C_0(Z))$ can be precomposed with the structure map $C_0(Z) \to ZM(A)$, making A into an $\underline{E}G \rtimes G$-algebra.

Note that if G is locally compact, σ-compact, Hausdorff $\underline{E}G$ always exists and is locally compact, σ-compact, and Hausdorff; in our case, G is second countable, hence $\underline{E}G$ is too [59, Proposition 6.15].

A subgroupoid of the form $\Phi(\Gamma_z \ltimes U) \subseteq G$, as in Proposition 3.2, will be called a *compact action* around $\rho(z)$. Given a proper G-algebra over $Z = \underline{E}G$, for any $z \in Z$ we can find an open neighborhood as in Equation (7). These open cover Z, and we can extract a countable subcover $\mathcal{V}$ (being second countable, Z is a Lindelöf space). Corresponding to this subcover, we get a countable collection of compact actions which we denote by $\mathcal{F}$. Define the full subcategory of *compactly induced objects*,

$$\mathcal{CI} = \{\mathrm{Ind}_Q^G(B) \mid B \in \mathrm{KK}^Q, Q \in \mathcal{F}\}.$$

We define a homological ideal $\mathcal{I}$ as the kernel of a single functor

$$\begin{aligned} F\colon \mathrm{KK}^G &\to \prod_{Q\in\mathcal{F}} \mathrm{KK}^Q \\ A &\mapsto (\mathrm{Res}_G^Q(A))_{Q\in\mathcal{F}}. \end{aligned} \tag{8}$$

The functor F commutes with direct sums because each restriction functor clearly does. Hence, $\mathcal{I}$ is compatible with countable direct sums. The proof below follows the blueprint in [39, Theorem 7.3], we reproduce it here for completeness.

Theorem 3.4. *The projective objects for $\mathcal{I}$ are the retracts of direct sums of objects in $\mathcal{CI}$ and the ideal $\mathcal{I}$ has enough projective objects. Therefore, the subcategories in $(\langle\mathcal{CI}\rangle, N_{\mathcal{I}})$ form a pair of complementary subcategories.*

Proof. According to [39, Theorem 3.22], we need to study the (possibly) partially defined left adjoint of the functor F defined in Equation (8). Since each compact action $Q \in \mathcal{F}$ is open in G, the functor Ind_Q^G is left adjoint to Res_G^Q. Thus, we may take the globally defined adjoint

$$F^\dagger((A_Q)_{Q\in\mathcal{F}}) = \bigoplus_{Q\in\mathcal{F}} \mathrm{Ind}_Q^G(A_Q).$$

Since $\mathcal{F}$ is countable and F is compatible with countable direct sums, this definition is legitimate. It follows that $\mathcal{I}$ has enough projective objects which are retracts as described. Indeed, $F^\dagger F(A)$ is projective because the isomorphism

$$\mathrm{KK}^G(\mathrm{Ind}_Q^G \mathrm{Res}_G^Q(A), B) \cong \mathrm{KK}^Q(\mathrm{Res}_G^Q(A), \mathrm{Res}_G^Q(B))$$

is given by $f \mapsto \mathrm{Res}_G^Q(f) \circ \eta_{\mathrm{Res}_G^Q(A)}$, where η is the unit of the adjunction. We then see that if $f \in \mathcal{I}$, then we must have $f = 0$. Similarly, the counits of the adjunctions yield an $\mathcal{I}$-epic morphism $\delta : F^\dagger F(A) \to A$ [41, Definition 21]. In particular, if A is already projective, then δ can be embedded in a split triangle. Split triangles are isomorphic to direct sum triangles [45, Corollary 1.2.7]. □

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Using notation from Section 1.2 and applying the result above, we have $\mathcal{P} = \langle \mathcal{CI} \rangle = \langle P_{\mathcal{I}} \rangle$ and $\mathcal{N} = N_{\mathcal{I}}$. Since we will only be dealing with the homological ideal $\ker(F)$ just described, we will drop the $\mathcal{I}$ from our notation and just write $\mathcal{N}$ instead of $N_{\mathcal{I}}$. The objects in $\mathcal{N} \subseteq \mathrm{KK}^G$ are also referred to as *weakly* contractible. We denote by $P(A)$ the $\mathcal{CI}$-cellular approximation of A. Note $P(A)$ belongs to $\mathcal{P}$.

Corollary 3.5. *We have the following equivalences,*

$$P(A) \cong P(C_0(G^0)) \otimes_{G^0} A \qquad N(A) \cong N(C_0(G^0)) \otimes_{G^0} A.$$

Proof. We have already explained that tensorization via the maximal balanced tensor product functor gives a triangulated functor. Hence, it maps the canonical exact triangle $P(C_0(G^0)) \longrightarrow C_0(G^0) \longrightarrow N(C_0(G^0))$ to an exact triangle

$$P(C_0(G^0)) \otimes_{G^0} A \longrightarrow A \longrightarrow N(C_0(G^0)) \otimes_{G^0} A.$$

If we can show that $- \otimes_{G^0} A$ leaves the subcategories $\langle \mathcal{CI} \rangle$ and $\mathcal{N}$ invariant, the result follows from the uniqueness statement in Proposition 1.12. Let us begin with the contractible objects: For $B \in \mathcal{N}$, since the restriction functor behaves well with respect to the maximal balanced tensor product, we compute

$$\mathrm{Res}_G^Q(\mathrm{id}_{B \otimes_{G^0}^{\max} A}) = \mathrm{Res}_G^Q(\mathrm{id}_B) \otimes_{Q^0} \mathrm{Res}_G^Q(\mathrm{id}_A) = 0,$$

and hence $B \otimes_{G^0}^{\max} A \in \mathcal{N}$.

On the other hand, for every $Q \in \mathcal{F}$ and $B \in \mathrm{KK}^Q$, Lemma 2.6 provides KK^G-equivalences

$$\mathrm{Ind}_Q^G(B) \otimes_{G^0} A \cong \mathrm{Ind}_Q^G(B \otimes_{Q^0} \mathrm{Res}_G^Q(A)) \in \langle \mathcal{CI} \rangle. \qquad \square$$

Definition 3.6. We say that G satisfies the *strong* Baum–Connes conjecture (with coefficients in A) if the natural map $P(A) \rtimes_r G \to A \rtimes_r G$ is a KK-equivalence.

A stronger variant of the formulation above is requiring $P(A) \to A$ to be an isomorphism in KK^G. However, it is known that even the ordinary (weaker) form of the conjecture admits counterexamples [24].

We will need the following deep result proved by J.-L. Tu.

Theorem 3.7 [58]. *Suppose G is a second countable, locally compact, Hausdorff groupoid. If G acts properly on a continuous field of affine Euclidean spaces, then there exists a proper G-C^*-algebra P such that $P \cong C_0(G^0)$ in KK^G.*

This result has the following immediate consequence:

Corollary 3.8. *Suppose G is a second countable, locally compact, Hausdorff groupoid. If G admits a proper action on a continuous field of affine Euclidean spaces, then we have the equality of categories $\langle \mathcal{P}r \rangle = \mathrm{KK}^G$.*

Proof. If $A \in \mathrm{KK}^G$ is any G-C^*-algebra, we have that $A \otimes_{G^0} P$ is proper and KK^G-equivalent to A. $\square$

Our next goal is to show that $\langle \mathcal{CI} \rangle = \langle \mathcal{P}r \rangle$. Let us first treat the proper case:

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Lemma 3.9. *Let G be a proper étale groupoid. Then $C_0(G^0) \in \langle \mathcal{CI} \rangle \subseteq \mathrm{KK}^G$.*

Proof. We have to show that $\mathrm{KK}^G(C_0(G^0), N) = 0$ for every $\mathcal{I}$-contractible object $N \in \mathrm{KK}^G$. Since $C_0(G^0)$ is clearly $C_0(G^0)$-nuclear, we have an isomorphism

$$\mathrm{KK}^G(C_0(G^0), N) \cong E_G(C_0(G^0), N)$$

by Corollary 1.8. Consequently, we can work in the setting of G-equivariant E-theory instead. The upshot is that E-theory satisfies excision. In particular, since G is proper, it is locally induced by compact actions as is explained in Proposition 3.2, that is, we have a countable cover $\mathcal{V}$ of G^0 by G-invariant sets with

$$E_G(C_0(V), N) = \mathrm{KK}^G(C_0(V), N) = 0.$$

As a first step, we aim to replace $\mathcal{V}$ by an increasing sequence. In order to arrange this, we need to show that given $V_0, V_1 \in \mathcal{V}$ we have

$$E_G(C_0(V_0 \cup V_1), N) = 0$$

Let us first observe that $\mathrm{KK}^G(C_0(V_0 \cap V_1), N) = 0$. Following Proposition 3.2, we can write $V_i = GU_i$ such that there exist $H_i \in \mathcal{F}$ with $G \times_{H_i} U_i \cong GU_i = V_i$. Observe that we have $V_0 \cap V_1 = G(U_0 \cap GU_1)$ and that if $g \in G$ satisfies $g(U_0 \cap GU_1) \cap (U_0 \cap GU_1) \neq \emptyset$, then also $gU_0 \cap U_0 \neq \emptyset$ and hence by the construction of H_0, $g \in H_0$. Thus, the canonical map

$$G \times_{H_0} H_0(U_0 \cap GU_1) \to G(U_0 \cap GU_1)$$

is a homeomorphism as it is the restriction of the homeomorphism $G \times_{H_0} U_0 \cong GU_0$. It follows that $C_0(V_0 \cap V_1) \cong C_0(G \times_{H_0} H_0(U_0 \cap GU_1)) = \mathrm{Ind}_{H_0}^G(C_0(H_0(U_0 \cap GU_1)) \in \mathcal{CI}$, and hence $E_G(C_0(V_0 \cap V_1), N) \cong \mathrm{KK}^G(C_0(V_0 \cap V_1), N) = 0$.

The corresponding statement for the union $V_0 \cup V_1$ now follows easily from the long exact sequences in E_G-theory associated with the short exact sequences:

$$0 \longrightarrow C_0(V_0 \cap V_1) \longrightarrow C_0(V_1) \longrightarrow C_0(V_1 \setminus V_0) \longrightarrow 0,$$
$$0 \longrightarrow C_0(V_0) \longrightarrow C_0(V_0 \cup V_1) \longrightarrow C_0(V_1 \setminus V_0) \longrightarrow 0.$$

In each sequence, two out of three groups in the induced long exact sequence vanish and hence so does the third. Replacing V_n by $\bigcup_{i=1}^n V_i$, we can assume that $\mathcal{V} = (V_n)_{n \in \mathbb{N}}$ is an increasing sequence. We clearly have $C_0(G^0) = \varinjlim_n C_0(V_n)$, and since E-theory has countable direct sums we have a Milnor $\lim^1$-sequence (see Lemma 1.16)

$$0 \longrightarrow \varprojlim{}^1 E_G(C_0(V_n), \Sigma N) \longrightarrow E_G(C_0(G^0), N) \longrightarrow \varprojlim E_G(C_0(V_n), N) \longrightarrow 0$$

Since the left and right terms are both zero, this concludes the proof. □

In the argument above, we can replace $C_0(G^0)$ by any KK^G-nuclear G-algebra A.

Theorem 3.10. *The localizing subcategory of KK^G generated by compactly induced objects equals the one generated by proper objects, that is, $\langle \mathcal{CI} \rangle = \langle \mathcal{P}r \rangle$.*

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Proof. Consider the canonical triangle

$$P \xrightarrow{D} C_0(G^0) \xrightarrow{\eta} N \longrightarrow \Sigma P, \tag{9}$$

and let $p: G \ltimes \underline{E}G \to G$ denote the projection homomorphism. The associated functor $p^*: \mathrm{KK}^G \to \mathrm{KK}^{G\ltimes \underline{E}G}$ maps contractible objects to contractible objects. Indeed, since $\underline{E}G$ is a proper G-space, a compact action for $G \ltimes \underline{E}G$ is just given by the restriction to one of the sets U^ρ as in Proposition 3.2. Continuing to use the notation from that proposition, let Q be the open copy of $\Gamma_z \ltimes U$ inside G, a compact action for G! Then the compositions of groupoid homomorphisms $(G \ltimes \underline{E}G)|_{U^\rho} \hookrightarrow G \ltimes \underline{E}G \xrightarrow{p} G$ and $(G \ltimes \underline{E}G)|_{U^\rho} \cong (\Gamma_z \ltimes U) \ltimes U^\rho \xrightarrow{p} \Gamma_z \ltimes U \cong Q \hookrightarrow G$ coincide. The resulting commutative diagram of KK groups gives $\mathrm{Res}_{(G\ltimes \underline{E}G)|_{U^\rho}}(\mathrm{id}_{p^*N}) = \mathrm{Res}_{(G\ltimes \underline{E}G)|_{U^\rho}}(p^*(\mathrm{id}_N)) = p^*(\mathrm{Res}_Q(\mathrm{id}_N)) = 0$ for any contractible object $N \in \mathrm{KK}^G$.

Combining this with Lemma 3.9, we can use the fact that $\langle \mathcal{CI} \rangle$ and $\mathcal{N}$ are complementary to conclude that $p^*(\eta) \in \mathrm{KK}^{G\ltimes \underline{E}G}(C_0(\underline{E}G), p^*N) = 0$.

Now, let $A \in \mathrm{KK}^G$ be an arbitrary proper G-algebra. As explained before, we may assume that A is a $C_0(\underline{E}G)$-algebra. From our observation above, it follows that $p^*(\eta) \otimes_{\underline{E}G} 1_A = 0$. Since the functors p^* and σ_A are both triangulated, we can apply them in this order to obtain a triangle

$$p^*P \otimes_{\underline{E}G} A \to C_0(\underline{E}G) \otimes_{\underline{E}G} A \to p^*N \otimes_{\underline{E}G} A \to \Sigma(p^*P \otimes_{\underline{E}G} A).$$

Note that $C_0(\underline{E}G) \otimes_{\underline{E}G} A \cong A$. Rotating this triangle gives the triangle

$$p^*N \otimes_{\underline{E}G} A \longrightarrow \Sigma(p^*P \otimes_{\underline{E}G} A) \longrightarrow \Sigma A \xrightarrow{0} \Sigma(p^*N \otimes_{\underline{E}G} A),$$

in which the last morphism is zero as indicated. Thus, [45, Corollary 1.2.7] implies that the latter triangle splits, namely $\Sigma(p^*P \otimes_{\underline{E}G} A) \cong (p^*N \otimes_{\underline{E}G} A) \oplus \Sigma A$.

In particular, after suspending once more we obtain a retraction $A \longrightarrow p^*P \otimes_{\underline{E}G} A$, that is, a right inverse of $p^*D \otimes_{\underline{E}G} 1_A : p^*P \otimes_{\underline{E}G} A \longrightarrow A$. Now, applying the forgetful functor p_* gives a retraction $A \longrightarrow p_*(p^*P \otimes_{\underline{E}G} A) \cong P \otimes_{G^0} A \cong P(A)$. Since $\langle \mathcal{CI} \rangle$ is a thick subcategory of KK^G it follows that $A \in \langle \mathcal{CI} \rangle$. □

Remark 3.11. In general, we do not know if any object in $\langle \mathcal{CI} \rangle$ is equivalent in KK^G to a proper G-C^*-algebra. However, if the cellular approximation $P = P(C_0(G^0))$ happens to be proper (e.g., in the setting of Theorem 3.7), then the previous statement clearly holds, because for any $A \in \langle \mathcal{CI} \rangle$, we have that $P \otimes_{G^0} A \cong A$ is a proper G-C^*-algebra (cf. [21, Corollary 4.37] and [41, Section 7].)

The corollary below identifies the localization category in terms of the more classical RKK^G-functor. Recall a morphism $f: A \to B$ in KK^G is called a *weak* equivalence if $F(f)$ is an isomorphism, where F is the functor in Equation (8). For instance, the natural map $D_A: P(A) \to A$ is a weak equivalence.

Theorem 3.12. *Let $p: \underline{E}G \to G^0$ be the moment map underlying the G-action. The functor $p^*: \mathrm{KK}^G \to \mathrm{RKK}^G(\underline{E}G)$ is an isomorphism of categories up to localization*

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at $N_{\mathcal{I}}$. More precisely, the indicated maps in the following commutative diagram are isomorphisms.

$$\begin{array}{ccc} \mathrm{KK}^G(P(A),B) & \xleftarrow{\;D_A^*\;} & \mathrm{KK}^G(A,B) \\ {\scriptstyle p^*}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle p^*} \\ \mathrm{RKK}^G(\underline{E}G;P(A),B) & \xleftarrow[\;p^*(D_A)^*\;]{\cong} & \mathrm{RKK}^G(\underline{E}G;A,B). \end{array}$$

Proof. Let us first consider the bottom map. Since $\mathrm{RKK}^G(\underline{E}G;-,B)$ is a cohomological functor, the claim follows from the inclusion $N_{\mathcal{I}} \subseteq \ker(p^*)$. If A is weakly contractible, then $p^*(A)$ is both weakly contractible and proper, hence $\mathrm{KK}^{G\ltimes \underline{E}G}(p^*A,p^*A)=0$ by Proposition 1.12. Thus, $p^*(A)=0$.

Secondly, let us turn to the vertical map. Both the top and the bottom groups are functorial in the first slot and compatible with direct sums, hence the class of objects for which p^* is an isomorphism is localizing. Thus, we can assume $P(A)=\mathrm{Ind}_H^G(D)$ for some compact action $H\subseteq G$. Then, by using the induction-restriction adjunction and exchanging p^* and Ind_H^G, we can reduce ourselves to proving that

$$p^*\colon\ \mathrm{KK}^H(D,\mathrm{Res}_G^H(B)) \to \mathrm{RKK}^H(U^\rho;D,\mathrm{Res}_G^H(B)) \tag{10}$$

is an isomorphism (we are using notation from Equation (7)). The subgroupoid H is a compact action and it satisfies a strong form of the Baum–Connes conjecture; in particular it admits a Dirac-dual-Dirac triple as in [21, Definition 4.38]. Then [21, Theorem 4.34 & 4.39] imply that Equation (10) is an isomorphism. More concretely, if P' a proper C^*-algebra which is also the cellular approximation of $C(H^{(0)})$, then the inverse map is given by $[x]\mapsto p_*(P'\otimes_{U^\rho}[x])$ (cf. [21, Lemma 4.31]). □

Remark 3.13. The second part of the proof above should be viewed as a statement about the H-equivariant 'contractibility' of $\underline{E}G$ (cf. [40, Theorem 7.1] and [58, Theorem 11.3]). Concerning the map in Equation (10), if the G-cellular approximation P was KK^G-equivalent to a proper C^*-algebra, then the map $[x]\mapsto p_*(P\otimes_{\underline{E}G}[x])$ would provide an inverse already in KK^G. This holds for many groupoids, as is shown by Theorem 3.7; however, by passing to H via the adjunction, we do not need to assume that P is proper in the theorem above.

The relation to the ordinary Baum–Connes conjecture is explained by means of the following result (compare with [21, Theorem 6.12]; see also [40] for action groupoids). The left-hand side of the Baum–Connes assembly map (with coefficients in A) is often denoted $K_*^{\mathrm{top}}(G;A)$ and is defined as $\varinjlim_{Y\subseteq \underline{E}G}\mathrm{KK}^G(C_0(Y),A)$, the limit ranging over the directed set of G-invariant G-compact subspaces of $\underline{E}G$.

Theorem 3.14. *Let $A\in\mathrm{KK}^G$ be a G-C^*-algebra, and denote by μ_A^G the associated assembly map. Let $D_A\colon P(A)\to A$ be the natural KK^G-morphism. The indicated maps in the following commuting diagram are isomorphisms.*

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G;A) & \xrightarrow{\mu_A^G} & K_*(A \rtimes G) \\ \cong \downarrow & & \uparrow D_A \rtimes G \\ K_*^{\mathrm{top}}(G;P(A)) & \xrightarrow[\cong]{\mu_{P(A)}^G} & K_*(P(A) \rtimes G). \end{array} \tag{11}$$

Proof. The functor $K_*^{\mathrm{top}}(G;-)$ is homological, it commutes with direct sums and by the vertical isomorphism in Theorem 3.12, it is functorial for maps in $\mathrm{RKK}^G(\underline{E}G;A,B)$. The same theorem also implies $p^*(D_A)$ is invertible, thus the left map in the diagram above is an isomorphism. Now, $\mu_{P(A)}^G$ is an isomorphism if the Baum–Connes conjecture holds for compactly induced coefficient algebras. This is proved in [14] (see also [16] and [21, Theorem 4.48]). □

Combining $\langle \mathcal{CI} \rangle = \langle \mathcal{P}r \rangle$ and Tu's Theorem 3.7, we obtain the following.

Corollary 3.15. *Suppose G is a second countable, locally compact, Hausdorff groupoid. Assume that there exists a proper G-C^*-algebra P such that $P \cong C_0(G^0)$ in KK^G. Then G satisfies the strong Baum–Connes conjecture with coefficients.*

The previous corollary applies in particular to all amenable groupoids and more generally to all *a-T-menable* groupoids (a-T-menability is also known as the *Haagerup property*) by [58, Proposition 3.8].

The following lemma shows that we can use Theorem 3.10 to rephrase the definition of $\mathcal{N}$ as the category of contractible objects with respect to the kernel of the joint restriction functor to all proper open subgroupoids (instead of just the compact actions).

Lemma 3.16. *Let $B \in \mathrm{KK}^G$. Then $B \in \mathcal{N}$ if and only if $\mathrm{Res}_G^H(\mathrm{id}_B) = 0$ for all proper open subgroupoids $H \subseteq G$.*

Proof. Suppose that $B \in \mathcal{N}$. By Theorem 3.10 and the fact that $(\langle \mathcal{CI} \rangle, \mathcal{N})$ is a pair of complementary subcategories, we get that $\mathrm{KK}^G(A,B) = 0$ for all $A \in \mathcal{P}r$. If $H \subseteq G$ is a proper open subgroupoid, then $\mathrm{Ind}_H^G D \in \mathcal{P}r$ for all $D \in \mathrm{KK}^H$. Using the induction-restriction adjunction, we get that

$$\mathrm{KK}^H(D, \mathrm{Res}_G^H B) \cong \mathrm{KK}^G(\mathrm{Ind}_H^G D, B) = 0$$

for all $D \in \mathrm{KK}^H$. If we apply this to $D = \mathrm{Res}_G^H(B)$, we get, in particular, that $\mathrm{Res}_G^H(\mathrm{id}_B) = \mathrm{id}_{\mathrm{Res}_G^H B} = 0$. The converse follows from the definition of $\mathcal{N}$ and the fact that each $Q \in \mathcal{F}$ is a proper open subgroupoid of G. □

4. Applications

4.1. The UCT

The article [12] established a connection between the Baum–Connes conjecture for groupoids and the Künneth formula for groupoid crossed products. Now, the UCT introduced in [55] is formally stronger than the Künneth formula, so philosophically

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speaking it may not come as a surprise that a similar relation exists between the strong Baum–Connes conjecture and the UCT.

Proposition 4.1. *Let (A,G,α) be a groupoid dynamical system with A type I. Then $P(A)\rtimes_r G$ satisfies the UCT. If furthermore G satisfies the strong Baum–Connes conjecture, then $A\rtimes_r G$ satisfies the UCT.*

Proof. If A is a type I C*-algebra and H is a proper groupoid, the crossed product $A\rtimes H$ is type I by [58, Proposition 10.3]. Given A as in the claim, and $H\subseteq G$ a proper open subgroupoid acting on A, then $C_0(G_{H^0})\otimes A$ is type I, $C_0(G_{H^0})\otimes_{H^0} A$ is type I (because it is a quotient) and $L_H(A):=\mathrm{Ind}_H^G\mathrm{Res}_G^H(A)$ is type I as well. Hence, $L_H(A)$ belongs to the bootstrap class. Since $L_H(A)\rtimes_r G$ is Morita equivalent to $A\rtimes_r H$ and $P(A)\rtimes_r G$ belongs to the localising subcategory of KK generated by

$$\{L_{H_1}\cdots L_{H_n}(A)\rtimes G\mid n\in\mathbb{N}, H_i\subseteq G \text{ proper and open}\},$$

it follows that $P(A)\rtimes_r G$ belongs to the bootstrap class as well.

Since the bootstrap class is closed under KK-equivalence, the strong Baum–Connes conjecture yields the result. □

We do in particular obtain the following corollary, generalising [3, 28]. To state it, recall that a twist over G is a central extension

$$G^0\times\mathbb{T}\to\Sigma\xrightarrow{j}G,$$

and that one can associate the twisted groupoid C^*-algebra $C_r^*(G,\Sigma)$ to these data (see [54] for the details of this construction).

Corollary 4.2. *Let Σ be a twist over an étale groupoid G. If G satisfies the strong Baum–Connes conjecture, then $C_r^*(G,\Sigma)$ satisfies the UCT.*

Proof. Apply the stabilisation trick [61, Proposition 5.1] to replace $C_r^*(G,\Sigma)$ up to Morita-equivalence by $K(H)\rtimes_r G$, where $K(H)$ denotes the algebra of compact operators on a suitable Hilbert $C_0(G^0)$-module. As $K(H)$ is type I, the previous proposition applies. □

4.2. The going-down principle

We generalize some results obtained by the first author for ample groupoids [10] to the general étale case.

Theorem 4.3. *Suppose there is an element $f\in\mathrm{KK}^G(A,B)$ such that*

$$\mathrm{KK}^H(D,\mathrm{Res}_G^H(A))\xrightarrow{-\widehat{\otimes}\,\mathrm{Res}_G^H(f)}\mathrm{KK}^H(D,\mathrm{Res}_G^H(B))$$

is an isomorphism for all $H\in\mathcal{F}$ and separable H-C^*-algebras D. Then f is a weak equivalence, and in particular the Kasparov product induces an isomorphism

$$-\widehat{\otimes}_A f: K_*^{\mathrm{top}}(G;A)\to K_*^{\mathrm{top}}(G;B). \tag{12}$$

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Proof. Using the induction-restriction adjunction the hypothesis is equivalent to the following map being an isomorphism for any $\tilde{D} \in \mathcal{CI}$,

$$\mathrm{KK}^G(\tilde{D}, A) \xrightarrow{-\widehat{\otimes} f} \mathrm{KK}^G(\tilde{D}, B).$$

Applying the functor $\mathrm{KK}^G(\tilde{D}, -)$ to a mapping cone triangle for f and using the five lemma we deduce that $\mathrm{KK}^G(\tilde{D}, \mathrm{Cone}(f)) \cong 0$ for all $\tilde{D}$ in $\langle \mathcal{CI} \rangle$. Now, by Theorem 1.12 we get $\mathrm{Cone}(f) \in N_{\mathcal{I}}$. The rest follows from Theorems 3.12 and 3.14. □

If we are only interested in studying the assembly map, then we might want to prove Equation (12) without necessarily proving that A and B have isomorphic cellular approximations. The following result is a version of the previous one 'after $K_*(- \rtimes G)$', and it can be proved with slightly weaker assumptions.

Theorem 4.4 (cf. [10, Theorem 7.10]). *Let $f \in \mathrm{KK}^G(A_1, A_2)$ be an element such that the induced map*

$$K_*(\jmath_H(\mathrm{Res}_G^H(f))) : K_*(\mathrm{Res}_G^H(A_1) \rtimes H) \to K_*(\mathrm{Res}_G^H(A_2) \rtimes H)$$

is an isomorphism for all proper open subgroupoids $H \subseteq Q$ for all $Q \in \mathcal{F}$. Then

$$K_*(\jmath_G(P(f))) : K_*(P(A_1) \rtimes_r G) \to K_*(P(A_2) \rtimes_r G)$$

is an isomorphism.

The proof requires some preparation. For a subgroupoid $H \subseteq G$ let $L_H := \mathrm{Ind}_H^G \circ \mathrm{Res}_G^H$. Consider the class $\mathcal{P}_0$ of G-algebras of the form $(L_{H_n} \circ \cdots \circ L_{H_1})(C_0(G^0))$ for $n \in \mathbb{N}$ and $H_i \in \mathcal{F}$.

Lemma 4.5. $P(C_0(G^0)) \in \langle \mathcal{P}_0 \rangle$.

Proof. By [39, Proposition 3.18], the $\mathcal{CI}$-cellular approximation $P(C_0(G^0))$ can be computed as the homotopy limit of a phantom castle over $C_0(G^0)$. Hence, it is enough to show that such a phantom castle can be found inside $\langle \mathcal{P}_0 \rangle$. Using the fact that $\langle \mathcal{P}_0 \rangle$ is localising, an inspection of the construction of such a phantom castle in [39] shows that it suffices to show that $C_0(G^0)$ admits a projective resolution by objects in $\langle \mathcal{P}_0 \rangle$. The standard way to construct such a projective resolution is by considering the algebras $(F^\dagger \circ F)^n(C_0(G^0))$ for $n \geq 1$.

We will prove that this resolution is contained in $\langle \mathcal{P}_0 \rangle$ by induction. First, we have $(F^\dagger \circ F)(C_0(G^0)) = \bigoplus_{H \in \mathcal{F}} \mathrm{Ind}_H^G \mathrm{Res}_G^H C_0(G^0) \in \langle \mathcal{P}_0 \rangle$. Assuming now that the claim holds for $n-1$, we compute

$$(F^\dagger \circ F)^n(C_0(G^0)) = \bigoplus_{H \in \mathcal{F}} \mathrm{Ind}_H \mathrm{Res}_H((F^\dagger \circ F)^{n-1}(C_0(G^0))),$$

and the latter is contained in $\langle \mathcal{P}_0 \rangle$ since $L_H(\langle \mathcal{P}_0 \rangle) \subseteq \langle \mathcal{P}_0 \rangle$ (we have $L_H(\mathcal{P}_0) \subseteq \mathcal{P}_0$ by definition of $\mathcal{P}_0$ and hence the general statement follows from the fact that L_H is triangulated and compatible with direct sums). □

Proof of Theorem 4.4. We will show that

$$K_*((\jmath_G(\mathrm{id}_B \otimes_{G^0} f))) : K_*((B \otimes_{G^0} A_1) \rtimes G) \to K_*((B \otimes_{G^0} A_2) \rtimes G) \qquad (13)$$

is an isomorphism for all $B \in \mathcal{P}_0$. Once this is proven, we can complete the proof as follows: since K-theory is a homological functor (compatible with direct sums), these isomorphisms imply that Equation (13) is also an isomorphism for $B \in \langle \mathcal{P}_0 \rangle$ by a routine argument involving the five lemma.

In particular, we can take $B = P(C_0(G^0))$ by the previous lemma. Noting further that $P(A) \rtimes G \cong P(A) \rtimes_r G$ in KK, the proof will be complete. Thus, in what follows we show that Equation (13) is an isomorphism for all $B \in \mathcal{P}_0$.

Step 1: We will first prove that Equation (13) is an isomorphism for $B = L_H(C_0(G^0)) = C_0(G/H)$ whenever $H \subseteq Q$ for some $Q \in \mathcal{F}$.. In this case, we have natural G-equivariant isomorphisms

$$B \otimes_{G^0} A_i \cong \mathrm{Ind}_H^G(C_0(H^0)) \otimes_{G^0} A_i \cong \mathrm{Ind}_H^G(\mathrm{Res}_G^H(A_i))$$

and hence $(B \otimes_{G^0} A_i) \rtimes G$ is Morita equivalent to $\mathrm{Res}_G^H(A_i) \rtimes H$. Thus, this case follows directly from the assumption.

Step 2: Suppose $B = L_K(C_0(X)) = \mathrm{Ind}_K^G C_0(X|_{K^0})$, where X is any second countable proper étale G-space with anchor map $p: X \to G^0$, and $K \in \mathcal{F}$. We claim that Equation (13) is an isomorphism for this choice of B. Let $\mathcal{B}$ be a countable basis for the topology of $X|_{K^0}$ consisting of open subsets of $X|_{K^0}$ on which p restricts to a homeomorphism. Then we can write

$$X|_{K^0} = \bigcup_{S \in \mathcal{B}} KS.$$

Since $\mathcal{B}$ is countable, we may enumerate its elements writing $\mathcal{B} = \{S_n \mid n \in \mathbb{N}\}$. Let $X_n := \bigcup_{i=1}^n KS_n$. Then X_n is an open K-invariant subset of X. Moreover, $C_0(X|_{K^0}) = \varinjlim_n C_0(X_n)$ where the connecting maps are just given by the canonical inclusions. Since the induction functor, tensor products and the maximal crossed product as well as K-theory are all compatible with inductive limits, it suffices to show that Equation (13) is an isomorphism for $B = \mathrm{Ind}_K^G C_0(X_n)$. We will do this by induction on n.

For $n = 1$, observe that for every $S \in \mathcal{B}$ there are identifications $KS \cong K \times_{\mathrm{Stab}(S)} S$, where $\mathrm{Stab}(S)$ is the proper open subgroupoid of K defined as $\mathrm{Stab}(S) = \{g \in K \mid gS \subseteq S\}$. Note that the restriction of the anchor map induces a homeomorphism $S \cong \mathrm{Stab}(S)^0$. It follows that

$$C_0(KS) \cong C_0(K \times_{\mathrm{Stab}(S)} S) \cong \mathrm{Ind}_{\mathrm{Stab}(S)}^K(C_0(\mathrm{Stab}(S)^0)),$$

and using induction in stages we conclude that

$$\mathrm{Ind}_K^G C_0(KS) = \mathrm{Ind}_{\mathrm{Stab}(S)}^G C_0(\mathrm{Stab}(S)^0) = C_0(G/\mathrm{Stab}(S)).$$

Since $\mathrm{Stab}(S)$ is a proper open subgroupoid of $K \in \mathcal{F}$, it follows that Equation (13) is an isomorphism for $B = \mathrm{Ind}_K^G C_0(KS)$ by Step 1 above.

Next, consider a union $KS \cup KT$ for $S,T \in \mathcal{B}$. Then we have two short exact sequences of K-algebras

$$0 \to C_0(KS \cap KT) \to C_0(KS) \to C_0(KS \setminus KT) \to 0$$

and

$$0 \to C_0(KT) \to C_0(KS \cup KT) \to C_0(KS \setminus KT) \to 0.$$

Using that the functors $\mathrm{Ind}_K^G -$, $(- \otimes_{G^0} A_i)$, and $(- \rtimes G)$ are all exact, we can apply them (in this order) to the above sequences and the result remains exact. Hence, we obtain induced six-term exact sequences in K-theory, which can be compared using the maps induced by f. Thus, using the case $n = 1$ above, to prove the claim for the union $KS \cup KT$ for $S,T \in \mathcal{B}$, it suffices to prove it for $KS \cap KT$. To this end, note that

$$KS \cap KT = K(S \cap KT).$$

Considering the subgroupoid $\mathrm{Stab}(S \cap KT)$ of K defined as above, we can employ the same arguments as in the case $n = 1$ to conclude that $C_0(KS \cap KT)) \cong \mathrm{Ind}^K_{\mathrm{Stab}(S\cap KT)}(C_0(\mathrm{Stab}(S \cap KT)^0))$, and hence using induction in stages again, we conclude that Equation (13) is an isomorphism for

$$B = \mathrm{Ind}_K^G C_0(KS \cap KT) = \mathrm{Ind}^G_{\mathrm{Stab}(S\cap KT)} C_0(\mathrm{Stab}(S \cap KT)^0) \cong C_0(G/\mathrm{Stab}(S \cap KT)).$$

Inductively, we can continue in this way to prove the isomorphism in line (13) for all $B = \mathrm{Ind}_K^G C_0(X_n)$ and hence complete step 2 by passing to the inductive limit.

Step 3: We can now prove that Equation (13) is an isomorphism for all $B \in \mathcal{P}_0$ by induction. The base case is contained in Step 1 above. For the induction step, note that $L_{H_n} \cdots L_{H_1}(C_0(G^0)) \cong C_0(G/H_n \times_{G^0} \ldots \times_{G^0} G/H_1)$ and observe that the space $X := G/H_n \times_{G^0} \ldots \times_{G^0} G/H_1$ is an étale proper G-space. Thus, we can just apply Step 2 to complete the proof. □

This result directly allows to generalize several results obtained by the first author for ample groupoids to the general étale case.

4.2.1. Homotopies of twists. Let G be an étale groupoid. A *homotopy of twists* is a twist over $G \times [0,1]$, that is, a central extension of the form

$$G^0 \times [0,1] \times \mathbb{T} \to \Sigma \xrightarrow{j} G \times [0,1].$$

Theorem 4.6. *Let G be a second countable étale groupoid satisfying the Baum–Connes conjecture with coefficients. If Σ is a homotopy of twists over G, then for each $t \in [0,1]$ the canonical map $q_t : C_r^*(G \times [0,1], \Sigma) \to C_r^*(G, \Sigma_t)$ induces an isomorphism in K-theory.*

Proof. The idea of the proof is the same as for the main result in [11]: Using a groupoid version of the Packer–Raeburn stabilisation trick and the going-down principle

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(Theorem 4.4), one only has to prove the result for all proper open subgroupoids of all elements $H \in \mathcal{F}$ in place of G. Recall that all the groupoids $H \in \mathcal{F}$ are (isomorphic to) transformation groupoids of finite groups. Hence, if the original homotopy of twists over G is topologically trivial in the sense that the map j has a continuous section (this means that the twist is equivalent to a continuous 2-cocycle), one can apply an earlier result of Gillaspy [22] to finish the proof. In the setting of ample groupoids treated in [11], the requirement that the twist is topologically trivial is not actually a restriction by [11, Proposition 4.2].

In the étale setting twists are no longer automatically topologically trivial, so instead we use a refinement of the going-down principle. Observe that the constructions and results from the previous section allow some flexibility in choosing the family $\mathcal{F}$ of subgroupoids of G. Indeed, if $\mathcal{F}'$ is another family of subgroupoids of G with the property that every proper action of G is locally induced by members of $\mathcal{F}'$, we can replace $\mathcal{F}$ by $\mathcal{F}'$ in all the results of Section 3 and hence also in Theorem 4.3.

Now, given a homotopy of twists with quotient map $j : \Sigma \to G \times [0,1]$ we claim that there exists a family $\mathcal{F}'$ of compact actions for G as above with the additional property that the restricted twist $j^{-1}(H \times [0,1]) \to H \times [0,1]$ (this is now a homotopy of twists over H) admits a continuous cross section.

Let us explain how this works: By the proof of [11, Proposition 4.2] every $g \in G$ admits an open neighbourhood V such that there exists a local section $V \times [0,1] \to \Sigma$ of j. Now, given a proper action of G we will proceed as in the proof of Proposition 3.2, but (in the notation of that proof) we additionally choose the bisections W_g to be the domains of local sections of j as above. Since the W_g can be assumed to be pairwise disjoint and the remaining construction in the proof of Proposition 3.2 just shrinks them further, we can patch the resulting finitely many local sections $W_g \times [0,1] \to \Sigma$ together to obtain the desired continuous section $H \times [0,1] \to \Sigma$. Since H is of the form $\Gamma \ltimes U$ for a finite group Γ and an open subset $U \subseteq G^0$ we are again the position to apply Gillaspy's result to conclude that q_t induces an isomorphism for all $H \in \mathcal{F}'$. To lift the result from this to all of G, one can follow the arguments in [11] again. □

4.3. Amenability at infinity

Recall that a locally compact Hausdorff groupoid G is called *amenable at infinity*, if there exists a G-space Y with proper momentum map $p : Y \to G^0$ and such that $G \ltimes Y$ is (topologically) amenable.

It is called *strongly amenable at infinity* if, in addition, the momentum map p admits a continuous cross section. Since p is a proper map, it induces an equivariant $*$-homomorphism $C_0(G^0) \to C_0(Y)$ and can hence be viewed as a morphism

$$\mathbf{p} \in \mathrm{KK}^G(C_0(G^0), C_0(Y)).$$

It was shown in [2, Lemma 4.9] that if G is strongly amenable at infinity, then the space Y witnessing this can be chosen second countable. Replacing this space further by the space of probability measures on Y supported in fibres we may also assume that each

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fibre (with respect to p) is a convex space and that G acts by affine transformations. The following result is [10, Proposition 8.2]:

Proposition 4.7. *Let G be a second countable étale groupoid, and let Y be a fibrewise convex space on which G acts by affine transformations. Suppose further that the anchor map $p: Y \to G^0$ admits a continuous cross section. If $H \subseteq G$ is a proper open subgroupoid, then the restriction of p to $p^{-1}(H^0)$ is an H-equivariant homotopy equivalence. In particular,* $\mathrm{Res}_G^H(\mathbf{p}) \in \mathrm{KK}^H(C_0(H^0), C_0(p^{-1}(H^0))$ *is invertible.*

We obtain the following consequence:

Theorem 4.8. *Let G be a second countable étale groupoid which is strongly amenable at infinity. Then there exists an element $\eta \in \mathrm{KK}^G(C_0(G^0), P(C_0(G^0)))$ such that $\eta \circ D = \mathrm{id}_{P(C_0(G^0))}$, where D denotes the Dirac morphism for G. In particular, the Baum–Connes assembly map μ_A for G is split injective for all $A \in \mathrm{KK}^G$.*

Proof. It follows immediately from Theorem 4.3 and Proposition 4.7 that $p \in \mathrm{KK}^G(C_0(G^0), C_0(Y))$ is a weak equivalence. Hence, $P(p)$ is an isomorphism in KK^G. Moreover, since G acts amenably on Y, the natural morphism $D_{C_0(Y)}: P(C_0(Y)) \to C_0(Y)$ is an isomorphism in $\mathrm{KK}^{G\ltimes Y}$. Consider the canonical forgetful functor $p_*: \mathrm{KK}^{G\ltimes Y} \to \mathrm{KK}^G$ induced by the anchor map $p: Y \to G^0$. It is not hard to see that p_* is a triangulated functor. Moreover, it maps proper objects to proper objects (if Z is a proper $G \ltimes Y$ space, then Z is also a proper G-space). Hence, by Theorem 3.4 it maps the localizing subcategory generated by the projective objects in $\mathrm{KK}^{G\ltimes Y}$ to the corresponding localizing subcategory generated by projective objects in KK^G.

Then, since the Dirac morphism is determined uniquely up to isomorphism of the associated exact triangles, we may assume that the natural morphism $D_{C_0(Y)} \in \mathrm{KK}^G(P(C_0(Y)), C_0(Y))$ is an isomorphism as well. Let β denote its inverse. Then the composition $\eta := P(p)^{-1} \circ \beta \circ p \in \mathrm{KK}^G(C_0(G^0), P(C_0(G^0)))$ is the desired morphism. The final assertion then follows from the commutative diagram (11). □

An element η as in the theorem above is often called a dual Dirac morphism for G (see [41, Definition 8.1]) and is unique (if it exists).

4.4. Permanence properties

In this section, we will often need to compare the subcategories $\langle \mathcal{CI} \rangle$ and $\mathcal{N}$ for different groupoids. To highlight this, we will slightly adjust our notation and write $\mathcal{N}_G$ for the weakly contractible objects in KK^G and $\mathcal{CI}_G$ for the compactly induced objects.

Sometimes we write 'BC' as a shorthand for 'Baum–Connes conjecture'.

4.4.1. Subgroupoids. Given a second countable étale groupoid G and a subgroupoid $H \subseteq G$, we may ask how the (strong) Baum–Connes conjectures for G and H are related. We need

Lemma 4.9. *Suppose $H \subseteq G$ is a subgroupoid. Then the following hold:*

1. *If $H \subseteq G$ is open, then $\mathrm{Res}_G^H(\mathcal{N}_G) \subseteq \mathcal{N}_H$.*
2. *If H is closed in $G|_{H^0}$, then $\mathrm{Res}_G^H(\langle \mathcal{CI}_G \rangle) \subseteq \langle \mathcal{CI}_H \rangle$.*
3. *If H is open in G and closed in $G|_{H^0}$, then Res_G^H maps a Dirac triangle for G to a Dirac triangle for H.*

Proof. To show the first item suppose H is an open subgroupoid of G and let $N \in \mathcal{N}_G \subseteq \mathrm{KK}^G$. Suppose that Q is a proper open subgroupoid of H. Then Q is also a proper open subgroupoid of G and hence $\mathrm{Res}_H^Q(\mathrm{Res}_G^H(\mathrm{id}_N)) = \mathrm{Res}_G^Q(\mathrm{id}_N) \overset{3.16}{=} 0$. Another application of Lemma 3.16 yields the result.

Next, suppose H is closed in $G|_{H^0}$. Whenever G acts properly on a space Z with anchor map $p: Z \to G^0$, then the action restricts to a proper action of H on $p^{-1}(H^0)$. In particular, it follows that $\mathrm{Res}_G^H(\mathcal{CI}_G) \subseteq \mathrm{Pr}_H$ and hence $\mathrm{Res}_G^H(\langle \mathcal{CI}_G \rangle) \subseteq \langle \mathcal{CI}_H \rangle$ by Theorem 3.10.

The final assertion is a direct consequence of the first two statements. □

Lemma 4.10. *Suppose $H \subseteq G$ is a subgroupoid such that H is closed in $G|_{H^0}$. Then the following hold:*

$$\mathrm{Ind}_H^G : \mathrm{KK}^H \to \mathrm{KK}^G$$

is triangulated, $\mathrm{Ind}_H^G(\mathcal{N}_H) \subseteq \mathcal{N}_G$, and $\mathrm{Ind}_H^G \langle \mathcal{CI}_H \rangle \subseteq \langle \mathcal{CI}_G \rangle$. In particular, it maps Dirac triangles to Dirac triangles.

Proof. Induction in stages gives that a compactly induced object in KK^H is mapped to a proper object in KK^G. Indeed, if $Q \subseteq H$ is a compact action, then $\mathrm{Ind}_H^G(\mathrm{Ind}_Q^H A) = \mathrm{Ind}_Q^G A$. It follows from our assumption that Q is closed in $G|_{Q^0}$, and hence the action of G on G_{Q^0}/Q is proper. It follows immediately that $\mathrm{Ind}_Q^G A$ is a proper G-algebra (see also the induction picture in [10]). Whence, $\mathrm{Ind}_H^G \langle \mathcal{CI}_H \rangle \subseteq \langle \mathcal{CI}_G \rangle$ by Theorem 3.10.

Finally, let $A \in \mathcal{N}_H \subseteq \mathrm{KK}^H$. Then by Lemma 4.9.(2) we have

$$\mathrm{Res}_G^H(P_G(C_0(G^0))) \otimes_{H^0}^{\max} A \cong P_H(C_0(H^0)) \otimes_{H^0}^{\max} A \cong 0.$$

Using Lemma 2.6, we conclude that

$$P_G(C_0(G^0)) \otimes_{G^0} \mathrm{Ind}_H^G A \cong \mathrm{Ind}_H^G(\mathrm{Res}_G^H(P_G(C_0(G^0))) \otimes_{H^0} A) \cong 0$$

as well. □

The following result was already observed by Tu [57] for the classical Baum–Connes conjecture. Unfortunately, his proof relies on [57, Lemma 3.9], which seems to be erroneous. A counterexample where G is the compact space $[0,1]$ (viewed as a trivial groupoid just consisting of units) is exhibited in [17, Example 5.6] and [4, p.36].

Theorem 4.11. *Let G be a second countable groupoid, $H \subseteq G$ be an étale subgroupoid that is closed in $G|_{H^0}$, and $A \in \mathrm{KK}^H$. Then there is a natural KK-equivalence between $P_G(\mathrm{Ind}_H^G A) \rtimes_r G$ and $P_H(A) \rtimes_r H$. Hence, the (strong) Baum–Connes conjecture with coefficients passes to closed subgroupoids and restrictions to open subsets.*

Proof. From the previous lemma, we conclude that $P_G(\mathrm{Ind}_H^G A) \rtimes_r G \cong \mathrm{Ind}_H^G (P_H(A)) \rtimes_r G$. The latter, however, is canonically Morita-equivalent (and hence in particular KK-equivalent) to $P_H(A) \rtimes_r H$. The result about the (strong) Baum–Connes conjecture follows readily. □

4.4.2. Continuity in the coefficient algebra. Let $(A_n)_n$ be an inductive system of G-C^*algebras, and let $A = \varinjlim A_n$ be the inductive limit. In [12, Section 3], it was shown that A carries a canonical G-action making all the structure maps equivariant, that is, the inductive limit exists in the category of G-C^*-algebras.

Proposition 4.12. *Let $(A_n)_n$ be an admissible inductive system of G-C^*algebras, and let $A = \varinjlim A_n$. Then $P(A) \rtimes_r G$ is naturally* KK*-equivalent to ho-lim$(P(A_n) \rtimes_r G)$, and $N(A) \rtimes_r G$ is naturally* KK*-equivalent to ho-lim$(N(A_n) \rtimes_r G)$.*

If furthermore G satisfies the (strong) Baum–Connes conjecture with coefficients in A_n for all $n \in \mathbb{N}$, then G satisfies the (strong) Baum–Connes conjecture with coefficients in A.

Proof. Let us consider the following diagram:

$$\begin{array}{ccccc}
\bigoplus_n P(A_n) & \xrightarrow{\cong} & P(\bigoplus_n A_n) & \xrightarrow{D_{\oplus A_n}} & \bigoplus_n A_n \\
\big\downarrow{\scriptstyle id-S} & & \big\downarrow{\scriptstyle P(id-S)} & & \big\downarrow{\scriptstyle id-S} \\
\bigoplus_n P(A_n) & \xrightarrow{\cong} & P(\bigoplus_n A_n) & \xrightarrow[D_{\oplus A_n}]{} & \bigoplus_n A_n
\end{array}.$$

The horizontal maps in the left-hand square are the natural isomorphisms obtained from the facts that the categories $\langle \mathcal{CI} \rangle$ and $\mathcal{N}$ are closed under direct sums and the Dirac triangle is unique. The square on the right commutes by naturality of the Dirac morphism. By [5, Proposition 1.1.11], the outer square forms the center of a larger diagram, in which each row and column is an exact triangle, and each square commutes (up to a sign), as shown below.

$$\begin{array}{ccccccc}
\Sigma\text{ho-lim}\, N(A_n) & \dashrightarrow & \Sigma\text{ho-lim}\, P(A_n) & \dashrightarrow & \Sigma A & \dashrightarrow & \Sigma\text{ho-lim}\, N(A_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_n \Sigma N(A_n) & \longrightarrow & \bigoplus_n P(A_n) & \longrightarrow & \bigoplus_n A_n & \longrightarrow & \bigoplus N(A_n) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_n \Sigma N(A_n) & \longrightarrow & \bigoplus_n P(A_n) & \longrightarrow & \bigoplus_n A_n & \longrightarrow & \bigoplus N(A_n) \\
\downarrow & & \downarrow & & \downarrow & & \vdots \\
\text{ho-lim}\, \Sigma N(A_n) & \dashrightarrow & \text{ho-lim}\, P(A_n) & \dashrightarrow & A & \dashrightarrow & \text{ho-lim}\, N(A_n)
\end{array}$$

Since the horizontal maps in the middle square are the morphisms defining the homotopy limit uniquely up to isomorphisms, it is clear which objects appear in the first

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and last row. In the diagram above, we have already made use of the fact that the sequence $(A_n)_n$ is admissible by replacing ho-lim A_n by the inductive limit $A = \lim A_n$. Consider now the bottom row of the diagram. Since $\langle \mathcal{CI} \rangle$ and $\mathcal{N}$ are localizing subcategories, they are closed under homotopy direct limits. Hence, by uniqueness, the bottom row is naturally isomorphic to the exact triangle

$$\Sigma N(A) \to P(A) \to A \to N(A).$$

Taking reduced crossed products is a triangulated functor on KK^G, so we can take crossed products throughout the diagram, completing the proof of the first assertion.

Now, if G satisfies the strong Baum–Connes conjecture with coefficients in A_n for each n, then the horizontal arrows in the central square are KK-equivalences (after taking reduced crossed products). It then follows immediately that $\jmath_r^G(D_A)$ is also a KK-equivalence. For the classical version of the Baum–Connes conjecture, first apply the reduced crossed product functor to the diagram above and then note that the two middle columns in the resulting diagram induce a homomorphism of long exact sequences in K-theory. An application of the five lemma yields the result. □

4.4.3. Products and unions of subgroupoids. Let $G = \bigcup G_n$ be a union of a sequence of clopen subgroupoids. We shall need the G_n to be open so that, if $A \in \mathrm{KK}^G$, we can write the crossed product as an inductive limit $A \rtimes_r G = \lim A \rtimes_r G_n$ as well. Since the G_n are also closed, we obtain canonical restriction maps $\Gamma_c(G,\mathcal{A}) \to \Gamma_c(G_n,\mathcal{A})$, which induce completely positive contractions $A \rtimes_r G \to A \rtimes_r G_n$. It follows that the inductive system $(A \rtimes_r G_n)_n$ is admissible and hence in the category KK we can identify the direct limit $A \rtimes_r G$ with the homotopy direct limit ho-lim $A \rtimes_r G_n$.

Proposition 4.13. *Let $(G_n)_n$ be a sequence of clopen subgroupoids of G such that $G = \bigcup_n G_n$. Suppose $A \in \mathrm{KK}^G$ such that G_n satisfies (strong) BC with coefficients in $\mathrm{Res}_G^{G_n}(A)$ for all $n \in \mathbb{N}$. Then G satisfies (strong) BC with coefficients in A.*

Proof. We know from Lemma 4.9 that $\mathrm{Res}_G^{G_n}$ preserves Dirac triangles. It follows that in KK we have identifications

$$P_{G_n}(\mathrm{Res}_G^{G_n}(A)) \rtimes_r G_n \cong (\mathrm{Res}_G^{G_n} P(A)) \rtimes_r G_n,$$

and similarly

$$N_{G_n}(\mathrm{Res}_G^{G_n}(A)) \rtimes_r G_n \cong (\mathrm{Res}_G^{G_n} N(A)) \rtimes_r G_n.$$

By taking limits, we get

$$P(A) \rtimes_r G \cong \text{ho-lim}\, P(A) \rtimes_r G_n \cong \text{ho-lim}\, P_{G_n}(\mathrm{Res}_G^{G_n}(A)) \rtimes_r G_n$$

and similarly

$$N(A) \rtimes_r G \cong \text{ho-lim}\, N_{G_n}(\mathrm{Res}_G^{G_n}(A)) \rtimes_r G_n$$

Recall that G satisfies the (strong) Baum–Connes conjecture with coefficients in A if and only if $N(A) \rtimes_r G$ is K-contractible (or KK-contractible for the strong version). Since

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the categories of K-contractible (resp. KK-contractible) objects are localising, they are closed under homotopy direct limits. The result follows. $\square$

Let us now turn our attention to direct products. Suppose $G = G_1 \times G_2$ is the product of two étale groupoids G_1, G_2. Suppose further that $A_i \in \mathrm{KK}^{G_i}$ for $i = 1,2$. If either A_1 or A_2 is exact, the minimal tensor product $A := A_1 \otimes A_2$ comes equipped with a diagonal action and hence can be viewed as an object in KK^G.

Proposition 4.14. *If G_i satisfies strong BC with coefficients in A_i for $i = 1,2$, then $G_1 \times G_2$ satisfies strong BC with coefficients in $A_1 \otimes A_2$.*

Proof. We claim that $\mathcal{CI}_{G_1} \otimes \mathcal{CI}_{G_2} \subseteq \mathcal{CI}_{G_1 \times G_2}$ and $\mathcal{N}_{G_1} \otimes \mathcal{N}_{G_2} \subseteq \mathcal{N}_{G_1 \times G_2}$. It follows in particular, that, if $P_i \to C_0(G_i^0) \to N_i$ is a Dirac triangle for G_i, $i = 1,2$, then

$$P_1 \otimes P_2 \to C_0((G_1 \times G_2)^0) \to N_1 \otimes N_2$$

is a Dirac triangle for $G = G_1 \times G_2$. Since the minimal tensor product behaves well with respect to reduced crossed products, we have canonical isomorphisms

$$A \rtimes_r G \cong (A_1 \rtimes_r G_1) \otimes (A_2 \rtimes_r G_2)$$
$$P_G(A) \rtimes_r G \cong (P_{G_1}(A_1) \otimes P_{G_2}(A_2)) \rtimes_r G \cong (P_{G_1}(A_1) \rtimes_r G_1) \otimes (P_{G_2}(A_2) \rtimes_r G_2),$$

where the first KK-equivalence follows from the above observation about Dirac triangles. Under these identifications, the Baum–Connes assembly map $P_G(A) \rtimes_r G \to A \rtimes_r G$ decomposes as the exterior tensor product of the Baum–Connes assembly maps $P_{G_i}(A_i) \rtimes_r G_i \to A_i \rtimes_r G_i$. Since the exterior tensor product of KK-equivalences is a KK-equivalence itself, the result follows. $\square$

As a an immediate consequence, we have the following:

Corollary 4.15. *Let $A_1, A_2 \in \mathrm{KK}^G$ such that at least one of the two is exact. Then $A_1 \otimes_{G^0} A_2 \in \mathrm{KK}^G$, where $\otimes_{G^0}$ denotes the balanced minimal tensor product. If we further assume that G satisfies strong BC with coefficients in A_1 and A_2, then G satisfies strong BC with coefficients in $A_1 \otimes_{G^0} A_2$.*

Proof. Proposition 4.14 implies that $G \times G$ satisfies the strong Baum–Connes conjecture with coefficients in $A_1 \otimes A_2$. View G as a closed subgroupoid of $G \times G$ via the diagonal inclusion. Since $\mathrm{Res}^G_{G\times G}(A_1 \otimes A_2) \cong A_1 \otimes_{G^0} A_2$, the result follows from Theorem 4.11. $\square$

The corresponding results for the classical Baum–Connes conjecture require further assumptions since the Künneth formula for the computation of the K-theory of a tensor product does not always hold. A detailed study in this direction has been carried out by Dell'Aiera and the first named author in [12].

Using the methods developed in the present article the results on the classical Baum–Connes conjecture with coefficients in a minimal balanced tensor product presented in [12] can be extended to all étale groupoids.

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4.5. Group bundles

We can now strengthen the results on group bundles obtained in [10].

Theorem 4.16. *Let G be a second countable étale group bundle which is strongly amenable at infinity. We suppose further that G^0 is locally finite-dimensional. Let A be a separable G-algebra which is continuous as a field of C^*-algebras over G^0. If the discrete group G_u^u satisfies BC with coefficients in A_u for every $u \in G^0$, then G satisfies BC with coefficients in A.*

Proof. We will first prove this in the case that G^0 is compact and finite-dimensional. Since we are working with second countable compact Hausdorff spaces the covering dimension of X coincides with the small inductive dimension of X, which we are going to employ. The proof will proceed by induction on the dimension of X. The zero-dimensional case has already been considered in [10, Theorem 8.11]. Assume that $dim(X) = n$ and the result has been shown for all spaces of dimension strictly smaller than n. It is enough to show $(1-\gamma_A)K_*(A \rtimes_r G) = \{0\}$. So let $x \in (1-\gamma_A)K_i(A \rtimes_r G)$. By our assumption that G_u^u satisfies BC with coefficients in A_u and [10, Lemma 8.10], we have $q_{u,*}(x) = 0$ for all $u \in G^0$. Using [15, Lemma 3.4], we can find an open neighbourhood U_u of u in G^0 such that $q_{\overline{U_u},*}(x) = 0$. Next, apply the fact that G^0 has inductive dimension at most n to replace each of the sets U_u by a smaller neighbourhood of u to assume additionally, that $\dim(\overline{U_u} \setminus U_u) \leq n-1$. Using compactness of G^0, we may find a finite subcover say $U_1, \ldots, U_l$ such that $\dim(\overline{U_i} \setminus U_i) \leq n-1$ and $q_{\overline{U_i},*}(x) = 0$ for all $1 \leq i \leq l$. Consider the open set $O := G^0 \setminus \bigcup_{i=1}^l \partial U_i$ and the associated ideal $A_O := C_0(O)A$. Then $C_0(O)(A \rtimes_r G) = A_O \rtimes_r G_O$. Since G is exact, we have a short exact sequence of C^*-algebras

$$0 \to A_O \rtimes G_O \to A \rtimes_r G \to A_Y \rtimes_r G_Y \to 0.$$

We want to consider the induced six-term exact sequence in K-theory. Since the boundaries ∂U_i are closed and at most $(n-1)$-dimensional so is their union $Y := \bigcup_{i=1}^l \partial U_i$. Applying the induction hypothesis yields that $(1-\gamma_{A_Y})K_*(A_Y \rtimes_r G_Y) = 0$. Hence, the six-term exact sequence in K-theory shows that the canonical inclusion map induces an isomorphism

$$(1-\gamma_{A_O})K_i(A_O \rtimes_r G_O) \cong (1-\gamma_A)K_i(A \rtimes_r G).$$

It follows that there exists a unique element $x' \in (1-\gamma_{A_O})K_i(A_O \rtimes_r G_O)$ whose image under the inclusion map is x. Furthermore, O can be decomposed as a finite disjoint union of open sets $O = \bigsqcup_{j=1}^m W_j$ such that each W_j is contained in at least one of the sets U_i by a standard inclusion/exclusion argument. Corresponding to this decomposition is a decomposition of the crossed product $A_O \rtimes_r G_O$ as

$$A_O \rtimes G_O = \bigoplus_{j=1}^m A_{W_j} \rtimes_r G_{W_j}.$$

It follows that $x' = \sum_{j=1}^l x'_j$ where x'_j is in the image of the inclusion map $(1-\gamma_{A_{W_j}})$ $K_i(A_{W_j} \rtimes G_{W_j}) \to (1-\gamma_{A_O})K_i(A_O \rtimes_r G_O)$. Thus, it is enough to show that $x'_j = 0$ for all

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$j = 1, \ldots, l$. To this end, consider the short exact sequence

$$0 \to A_{W_j} \rtimes G_{W_j} \to A_{\overline{W_j}} \rtimes_r G_{\overline{W_j}} \to A_{\partial W_j} \rtimes_r G_{\partial W_j} \to 0. \tag{14}$$

Since $\partial W_j \subseteq \partial U_i$ is a closed subset for some U_i, the boundary of W_j has dimension at most $n-1$. Hence, we can apply the induction hypothesis again to see that $(1 - \gamma_{A_{\partial W_j}}) K_*(A_{\partial W_j} \rtimes_r G_{\partial W_j}) = 0$. The six-term exact sequence in K-theory induced by (14) shows that the inclusion map induces an isomorphism $(1 - \gamma_{A_{W_j}}) K_i(A_{W_j} \rtimes G_{W_j}) \to (1 - \gamma_{A_{\overline{W_j}}}) K_i(A_{\overline{W_j}} \rtimes_r G_{\overline{W_j}})$. The image of x'_j under this map coincides with the image of x under the restriction map $q_{\overline{W_j},*}$. Since $W_j \subseteq U_i$ for some $1 \leq i \leq n$, we get that $q_{\overline{W_j},*}(x) = q_{\overline{W_j},*}(q_{\overline{U_i},*}(x)) = 0$, and this completes the proof for compact and finite-dimensional unit spaces.

Finally, if G^0 is a locally finite-dimensional and locally compact space, write G^0 as an increasing union $\bigcup K_n$ of compact subsets of G^0 such that $K_n \subseteq \operatorname{int}(K_{n+1})$. Using that G^0 is locally finite-dimensional, we may assume that each K_n has finite dimension. The first part of this proof implies that $G|_{K_n}$ satisfies BC with coefficients in $A|_{K_n}$ and $G|_{\partial K_n}$ satisfies BC with coefficients in $A|_{\partial K_n}$. A six-term exact sequence argument (using exactness of G!) then shows that $G|_{\operatorname{int}(K_n)}$ satisfies BC with coefficients in $A|_{\operatorname{int}(K_n)}$ for all $n \in \mathbb{N}$. Now, we can write $A = \lim A|_{\operatorname{int}(K_n)}$. Picking an approximate unit $(\rho_n)_n$ with $\rho_n \in C_c(\operatorname{int}(K_n))$, we can define completely positive contractions $A \to A|_{\operatorname{int}(K_n)}$ by $a \mapsto \rho_n a$ which converge pointwise to the identity. Hence, the sequence $A|_{\operatorname{int}(K_n)}$ is admissible and the result follows from Proposition 4.12. □

The class of infinite-dimensional spaces to which the previous result applies includes all locally compact CW complexes. An example of a compact space that is not covered by the result is the Hilbert cube.

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