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Asymptotic Continuous Orbit Equivalence of Smale Spaces and Ruelle Algebras

Kengo Matsumoto

Abstract. In the first part of the paper, we introduce notions of asymptotic continuous orbit equivalence and asymptotic conjugacy in Smale spaces and characterize them in terms of their asymptotic Ruelle algebras with their dual actions. In the second part, we introduce a groupoid C^*-algebra that is an extended version of the asymptotic Ruelle algebra from a Smale space and study the extended Ruelle algebras from the view points of Cuntz–Krieger algebras. As a result, the asymptotic Ruelle algebra is realized as a fixed point algebra of the extended Ruelle algebra under certain circle action.

1 Introduction

D. Ruelle initiated a study of a basic class of hyperbolic dynamical systems, called Smale spaces, from a view point of noncommutative operator algebras in [33, 34]. Smale spaces are, roughly speaking, hyperbolic dynamical systems with local product structure. His definition of Smale space was motivated by the work of S. Smale [36], R. Bowen [2, 3], and others. Two-sided subshifts of finite type are typical examples of Smale spaces. Ruelle introduced non-commutative algebras from Smale spaces and studied equilibrium states on them. After the Ruelle's papers, Ian F. Putnam [25–28], Putnam–Spielberg [29] and Kaminker–Putnam–Spielberg [10] (*cf.* K. Thomsen [37], etc.) investigated more detail on various kinds of C^*-algebras associated with Smale spaces from the view points of groupoids and structure theory of C^*-algebras. For a Smale space (X,ϕ), Putnam considered the following six kinds of C^*-algebras written in [25, 26]:

$$S(X,\phi),\quad U(X,\phi),\quad A(X,\phi),\quad S(X,\phi)\rtimes\mathbb{Z},\quad U(X,\phi)\rtimes\mathbb{Z},\quad A(X,\phi)\rtimes\mathbb{Z}. \tag{1.1}$$

The symbols S, U, and A correspond to stable, unstable, and asymptotic equivalence relations, respectively. The last three algebras in the above list are crossed products of the first three algebras by $\mathbb{Z}$-actions defined from automorphisms induced by ϕ, respectively. Putnam has written the second three algebras as R_s, R_u, R_a and calls them the stable Ruelle algebra, the unstable Ruelle algebra, and the asymptotic Ruelle algebra ([26]). In this paper, we write them as $\mathcal{R}^s_\phi, \mathcal{R}^u_\phi$, $\mathcal{R}^a_\phi$ to emphasize the original homeomorphism ϕ. He pointed out that if (X,ϕ) is a shift of finite type defined by an irreducible square matrix A with entries in $\{0,1\}$, the algebras

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$S(X,\phi)\rtimes\mathbb{Z}$ and $U(X,\phi)\rtimes\mathbb{Z}$ are isomorphic to the stabilized Cuntz–Krieger algebras $\mathcal{O}_A\otimes\mathcal{K}$ and $\mathcal{O}_{A^t}\otimes\mathcal{K}$, respectively, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space. Putnam and Spielberg [29] (*cf.* Killough and Putnam [11]) also constructed other kinds of C^*-algebras $S(X,\phi,P)$, $U(X,\phi,P)$ and their crossed products $S(X,\phi,P)\rtimes\mathbb{Z}$, $U(X,\phi,P)\rtimes\mathbb{Z}$ from a ϕ-invariant subset $P\subset X$ of periodic points by using étale groupoids defined by restricting stable and unstable equivalence relations to P, respectively. Although there are many different choices for P, they are all Morita equivalent to $S(X,\phi)$, $U(X,\phi)$ and $S(X,\phi)\rtimes\mathbb{Z}$, $U(X,\phi)\rtimes\mathbb{Z}$, respectively. In this paper, we will not deal with these C^*-algebras $S(X,\phi,P)$, $U(X,\phi,P)$, $S(X,\phi,P)\rtimes\mathbb{Z}$, $U(X,\phi,P)\rtimes\mathbb{Z}$.

In this paper we will mainly focus on the algebra $\mathcal{R}^a_\phi$, the last one in (1.1). By Putnam [25], the algebra $\mathcal{R}^a_\phi$ is realized as the groupoid C^*-algebra $C^*(G^a_\phi\rtimes\mathbb{Z})$ of an étale groupoid $G^a_\phi\rtimes\mathbb{Z}$. Its unit space $(G^a_\phi\rtimes\mathbb{Z})^\circ$ is identified with the original space X. We naturally identify $C(X)$ with a subalgebra of $\mathcal{R}^a_\phi$. A Smale space (X,ϕ) is said to be *asymptotically essentially free* if the interior of the set of n-asymptotic periodic points $\{x\in X \mid (\phi^n(x),x)\in G^a_\phi\}$ is empty for every $n\in\mathbb{Z}$ with $n\neq 0$. If (X,ϕ) is irreducible and X is not any finite set, (X,ϕ) is asymptotically essentially free (Lemma 5.2). We know that (X,ϕ) is asymptotically essentially free if and only if the étale groupoid $G^a_\phi\rtimes\mathbb{Z}$ is essentially principal (Lemma 5.3). Hence, if (X,ϕ) is irreducible and the space X is infinite, then the C^*-algebra $\mathcal{R}^a_\phi$ is simple (Proposition 5.4) and the C^*-subalgebra $C(X)$ is maximal abelian in $\mathcal{R}^a_\phi$. Since $C^*(G^a_\phi\rtimes\mathbb{Z})$ is canonically isomorphic to the crossed product $C^*(G^a_\phi)\rtimes\mathbb{Z}$ of the groupoid C^*-algebra $C^*(G^a_\phi)$, which is the C^*-algebra $A(X,\phi)$, the third one in (1.1) with an integer group action coming from the original transformation ϕ on X, the algebra $\mathcal{R}^a_\phi$ has the dual action written ρ^ϕ_t of the circle group $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. Throughout the paper we assume that the space X is infinite.

In the first part of this paper, we introduce a notion of asymptotic continuous orbit equivalence in Smale spaces, which will be defined in Section 2. Roughly speaking, two Smale spaces are asymptotically continuous orbit equivalent if they are continuous orbit equivalent up to asymptotic equivalence. We will show that spaces being asymptotic continuous orbit equivalent in Smale spaces is equivalent to their associated étale groupoids being isomorphic. It corresponds to the fact that continuous orbit equivalence of one-sided topological Markov shifts is equivalent to their associated étale groupoids being isomorphic (*cf.* [19,21,22]). If two Smale spaces (X,ϕ) and (Y,ψ) are asymptotically continuous orbit equivalent, written $(X,\phi)\underset{\text{ACOE}}{\sim}(Y,\psi)$, then there exists a homeomorphism $h\colon X\to Y$ having certain continuous homomorphisms $c_\phi\colon G^a_\phi\rtimes\mathbb{Z}\to\mathbb{Z}$ and $c_\psi\colon G^a_\psi\rtimes\mathbb{Z}\to\mathbb{Z}$. The continuous homomorphisms define unitary representations $U_t(c_\phi)$ on $l^2(G^a_\phi\rtimes\mathbb{Z})$ and $U_t(c_\psi)$ on $l^2(G^a_\psi\rtimes\mathbb{Z})$ of $\mathbb{T}$, which give rise to actions $\mathrm{Ad}(U_t(c_\phi))$ on $\mathcal{R}^a_\phi$ of $\mathbb{T}$ and $\mathrm{Ad}(U_t(c_\psi))$ on $\mathcal{R}^a_\psi$ of $\mathbb{T}$, respectively. In Sections 3 and 5, we will prove the following theorem.

Theorem 1.1 (Theorems 3.4 and 5.7) *Let (X,ϕ) and (Y,ψ) be irreducible Smale spaces. Then the following assertions are equivalent.*

(i) *(X,ϕ) and (Y,ψ) are asymptotically continuous orbit equivalent.*

(ii) *The groupoids $G^a_\phi \rtimes \mathbb{Z}$ and $G^a_\psi \rtimes \mathbb{Z}$ are isomorphic as étale groupoids.*

(iii) *There exists an isomorphism $\Phi\colon \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ and*

$$\Phi \circ \rho^\phi_t = \mathrm{Ad}(U_t(c_\psi)) \circ \Phi, \qquad \Phi \circ \mathrm{Ad}(U_t(c_\phi)) = \rho^\psi_t \circ \Phi \quad \text{for } t \in \mathbb{T}$$

for some continuous homomorphisms $c_\phi\colon G^a_\phi \rtimes \mathbb{Z} \to \mathbb{Z}$ and $c_\psi\colon G^a_\psi \rtimes \mathbb{Z} \to \mathbb{Z}$.

In Section 4, we will prove that stably or unstably asymptotic continuous orbit equivalence of Smale spaces preserves their periodic orbits, so that their zeta functions are related to each other by the associated cocycle functions (Theorem 4.9).

In Section 5, we study asymptotic continuous orbit equivalence in Smale spaces in terms of the dual actions of the associated Ruelle algebras.

In Section 6, we will introduce a notion of asymptotic conjugacy between Smale spaces (X,ϕ) and (Y,ψ), written $(X,\phi) \underset{a}{\cong} (Y,\psi)$. Roughly speaking, two Smale spaces are asymptotically conjugate if they are topologically conjugate up to asymptotic equivalences. This is stronger than asymptotic continuous orbit equivalence but weaker than topological conjugacy. The notion of asymptotic conjugacy in this paper is not the same as the notion of eventual conjugacy, which is used in dynamical systems (*cf.* [13, Definition 7.7.14]). Let $d_\phi\colon G^a_\phi \rtimes \mathbb{Z} \to \mathbb{Z}$ and $d_\psi\colon G^a_\psi \rtimes \mathbb{Z} \to \mathbb{Z}$ be the continuous homomorphisms of étale groupoids defined by

$$d_\phi(x,n,z) = n, \qquad d_\psi(y,m,w) = m \quad \text{for } (x,n,z) \in G^a_\phi \rtimes \mathbb{Z},\ (y,m,w) \in G^a_\psi \rtimes \mathbb{Z}.$$

We will characterize asymptotic conjugacy in terms of the Ruelle algebras with their dual actions in the following way.

Theorem 1.2 (Theorem 6.4) *Let (X,ϕ) and (Y,ψ) be irreducible Smale spaces. Then the following assertions are equivalent.*

(i) *(X,ϕ) and (Y,ψ) are asymptotically conjugate.*

(ii) *There exists an isomorphism $\varphi\colon G^a_\phi \rtimes \mathbb{Z} \to G^a_\psi \rtimes \mathbb{Z}$ of étale groupoids such that $d_\psi \circ \varphi = d_\phi$.*

(iii) *There exists an isomorphism $\Phi\colon \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ and $\Phi \circ \rho^\phi_t = \rho^\psi_t \circ \Phi$ for $t \in \mathbb{T}$.*

The asymptotic Ruelle algebra $\mathcal{R}^a_\phi$ has a translation invariant faithful tracial state coming from a maximal measure called the *Bowen measure* on X. Hence, the algebra $\mathcal{R}^a_\phi$ is never purely infinite. In Section 7, we will introduce a unital, purely infinite version of $\mathcal{R}^a_\phi$. The introduced C^*-algebra is denoted by $\mathcal{R}^{s,u}_\phi$ and called the extended asymptotic Ruelle algebra. It has a natural $\mathbb{T}^2$-action denoted by $\rho^{s,u}_\phi$. The fixed point algebra $(\mathcal{R}^{s,u}_\phi)^{\delta^\phi}$ of $\mathcal{R}^{s,u}_\phi$ under the diagonal $\mathbb{T}$-action defined by $\delta^\phi_z = \rho^{s,u}_{\phi,(z,z)}$, $z \in \mathbb{T}$ is isomorphic to the original asymptotic Ruelle algebra $\mathcal{R}^a_\phi$ (Theorem 7.9).

In Sections 8 and 9, we will apply the above discussions to shifts of finite type, which we call topological Markov shifts, from the view point of Cuntz–Krieger algebras. For an irreducible and not permutation square matrix A with entries in $\{0,1\}$, let us denote by $(\overline{X}_A, \overline{\sigma}_A)$ the associated two-sided topological Markov shift. The dynamical system is a typical example of a Smale space as in [25, 26, 33]. Consider the asymptotic Ruelle algebra $\mathcal{R}^a_{\overline{\sigma}_A}$ and the extended asymptotic Ruelle algebra $\mathcal{R}^{s,u}_{\overline{\sigma}_A}$ for the

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topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$, respectively. Let ρ^{A^t} and ρ^A be the gauge actions on the Cuntz–Krieger algebras $\mathcal{O}_{A^t}$ and $\mathcal{O}_A$, respectively, where A^t is the transpose of the matrix A. We put the diagonal gauge action $\delta^A_r = \rho^{A^t}_r \otimes \rho^A_r, r \in \mathbb{T}$ on $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$.

Theorem 1.3 (Theorem 9.6 and Corollary 9.7) *Let $(\overline{X}_A, \overline{\sigma}_A)$ be the Smale space of the two-sided topological Markov shift defined by an irreducible non-permutation matrix A with entries in $\{0,1\}$. Then there exists a projection E_A in the tensor product C^*-algebra $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$ such that $\delta^A_r(E_A) = E_A$ for all $r \in \mathbb{T}$ and the extended asymptotic Ruelle algebra $\mathcal{R}^{s,u}_{\overline{\sigma}_A}$ is isomorphic to the C^*-algebra $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$, denoted by $\mathcal{R}^{s,u}_A$. The asymptotic Ruelle algebra $\mathcal{R}^a_{\overline{\sigma}_A}$ is isomorphic to the fixed point algebra $(\mathcal{R}^{s,u}_A)^{\delta^A}$ of $\mathcal{R}^{s,u}_A$ under the diagonal gauge action δ^A.*

For the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$, we denote by $\mathcal{R}^{s,u}_A$ the extended asymptotic Ruelle algebra $\mathcal{R}^{s,u}_{\overline{\sigma}_A}$, which is identified with the C^*-algebra $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$, and by $\mathcal{R}^a_A$ the asymptotic Ruelle algebra $\mathcal{R}^a_{\overline{\sigma}_A}$, which is identified with the fixed point algebra of $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A$ under the diagonal gauge action δ^A by the above theorem.

In Section 10, we will present the K-groups of the asymptotic Ruelle algebras $\mathcal{R}^a_\phi$ for some topological Markov shifts. In Putnam [25] and Killough and Putnam [11], the K-theory formula for the asymptotic Ruelle algebras $\mathcal{R}^a_A$ for the topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ has been provided. We will use Putnam's formula in [25] to compute the K-groups of the C^*-algebra $\mathcal{R}^a_A$ for the $N \times N$ matrix

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

with all entries being 1's, so that the topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ is the full N-shift. Let us denote by $\mathcal{R}^a_N$ the asymptotic Ruelle algebra $\mathcal{R}^a_A$ for the matrix A. The C^*-algebra $\mathcal{R}^a_N$ is a simple A$\mathbb{T}$-algebra of real rank zero with a unique tracial state, written τ_N. We will show that $K_0(\mathcal{R}^a_N) \cong K_1(\mathcal{R}^a_N) \cong \mathbb{Z}[\frac{1}{N}]$ (Proposition 10.3) and $\tau_{N*}(K_0(\mathcal{R}^a_N)) = \mathbb{Z}[\frac{1}{N}]$ (Lemma 10.4). We then see (Proposition 10.5) that two algebras $\mathcal{R}^a_N$ and $\mathcal{R}^a_M$ are isomorphic if and only if $\mathbb{Z}[\frac{1}{N}] = \mathbb{Z}[\frac{1}{M}]$. As a result, we know that the two-sided full 2-shift and the two-sided full 3-shift are not asymptotically continuous orbit equivalent (Corollary 10.6).

In Section 11, we refer to differences among asymptotic continuous orbit equivalence, asymptotic conjugacy and topological conjugacy of Smale spaces, and present an open question related to their differences.

Throughout the paper, we denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of nonnegative integers and the set of positive integers, respectively.

2 Smale Spaces and their Groupoids

Let ϕ be a homeomorphism on a compact metric space (X, d) with metric d. Let us recall the definition of Smale space following D. Ruelle [33, 7.1] and I. F Putnam [25, Section 1]. Our notations differ slightly from those of Ruelle and Putnam. For $\epsilon > 0$, we set

$$\Delta_\epsilon := \{(x,y) \in X \times X \mid d(x,y) < \epsilon\}.$$

Suppose that there exist $\epsilon_0 > 0$ and a continuous map

$$[\,\cdot\,,\,\cdot\,] : (x,y) \in \Delta_{\epsilon_0} \longrightarrow [x,y] \in X$$

having the following three properties called (**SS1**):

(i) $[x,x] = x$ for $x \in X$,

(ii) $[[x,y],z] = [x,[y,z]] = [x,z]$ for $(x,y),(y,z),(x,z),([x,y],z),(x,[y,z]) \in \Delta_{\epsilon_0}$,

(iii) $[\phi(x),\phi(y)] = \phi([x,y])$ for $(x,y),(\phi(x),\phi(y)) \in \Delta_{\epsilon_0}$.

For $0 < \epsilon \le \epsilon_0$, put

$$X^s(x,\epsilon) = \{y \in X \mid [y,x] = y,\ d(x,y) < \epsilon\},$$
$$X^u(x,\epsilon) = \{y \in X \mid [x,y] = y,\ d(x,y) < \epsilon\}.$$

We further require that there exists $0 < \lambda_0 < 1$ such that the following two properties called (**SS2**) hold:

(2.1)
$$d(\phi(y),\phi(z)) \le \lambda_0 d(y,z) \qquad \text{for } y,z \in X^s(x,\epsilon),$$
$$d(\phi^{-1}(y),\phi^{-1}(z)) \le \lambda_0 d(y,z) \qquad \text{for } y,z \in X^u(x,\epsilon).$$

Definition 2.1 (Ruelle [33, 7.1]) A Smale space is a topological dynamical system (X,ϕ) of a homeomorphism ϕ on a compact metric space X with a bracket $[\,\cdot\,,\,\cdot\,]$ satisfying (**SS1**) and (**SS2**) for suitable ϵ_0, λ_0.

By Ruelle [33, 7.1] and Putnam [25, Section 1], there exists ϵ_1 with $0 < \epsilon_1 < \epsilon_0$ such that for any ϵ satisfying $0 < \epsilon < \epsilon_1$, the equalities

$$X^s(x,\epsilon) = \{y \in X \mid d(\phi^n(x),\phi^n(y)) < \epsilon \text{ for all } n = 0,1,2,\dots\},$$
$$X^u(x,\epsilon) = \{y \in X \mid d(\phi^n(x),\phi^n(y)) < \epsilon \text{ for all } n = 0,-1,-2,\dots\}$$

hold.

Lemma 2.2 (Putnam [25, Section 1], Ruelle [33, 7.1]) *For $x,y \in X$ with $(x,y) \in \Delta_{\epsilon_0}$ and $d(x,[y,x]), d(y,[y,x]) < \epsilon_1$,*

$$\{[y,x]\} = X^u(y,\epsilon_1) \cap X^s(x,\epsilon_1).$$

Hence, for $0 < \epsilon < \epsilon_1$ and $x,y,z \in X$, the equality $[y,x] = z$ holds if and only if

$$d(\phi^{-n}(y),\phi^{-n}(z)) < \epsilon, \qquad d(\phi^n(x),\phi^n(z)) < \epsilon \quad \text{for all } n = 0,1,2,\dots.$$

This means that the bracket $[\,\cdot\,,\,\cdot\,]$ is determined by the original dynamics (X,d,ϕ) if it exists. The following lemma is also useful.

Lemma 2.3 (Putnam [25, Section 1], Ruelle [33, 7.1]) *For any ϵ with $0 < \epsilon \le \epsilon_0$ and $x \in X$, we have*

(i) $\phi(X^s(x,\epsilon)) \subset X^s(\phi(x),\epsilon)$,

(ii) $\phi^{-1}(X^u(x,\epsilon)) \subset X^u(\phi^{-1}(x),\epsilon)$.

Following Putnam [25, Section 1], for $x \in X$, we put

$$X^s(x) = \{y \in X \mid \lim_{n\to\infty} d(\phi^n(x), \phi^n(y)) = 0\},$$
$$X^u(x) = \{y \in X \mid \lim_{n\to\infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0\},$$
$$X^a(x) = X^s(x) \cap X^u(x).$$

We note that the inclusion relations $X^s(x, \epsilon_0) \subset X^s(x)$ and $X^u(x, \epsilon_0) \subset X^u(x)$ were shown in [25]. The following lemma is from [25, 33].

Lemma 2.4 (Putnam [25, Section 1], Ruelle [33, 7.1]) *For any ϵ with $0 < \epsilon \le \epsilon_0$ and $x \in X$, we have*

(i) $X^s(x) = \bigcup_{n=0}^{\infty} \phi^{-n}(X^s(\phi^n(x), \epsilon))$.
(ii) $X^u(x) = \bigcup_{n=0}^{\infty} \phi^{n}(X^u(\phi^{-n}(x), \epsilon))$.

Following Putnam [25, Section 1], we put

$$G_\phi^{s,0} = \{(x, y) \in X \times X \mid y \in X^s(x, \epsilon_0)\},$$
$$G_\phi^{u,0} = \{(x, y) \in X \times X \mid y \in X^u(x, \epsilon_0)\},$$
$$G_\phi^{a,0} = G_\phi^{s,0} \cap G_\phi^{u,0},$$

and for $n \in \mathbb{N}$,

$$G_\phi^{s,n} = (\phi \times \phi)^{-n}(G_\phi^{s,0}),$$
$$G_\phi^{u,n} = (\phi \times \phi)^{n}(G_\phi^{u,0}),$$
$$G_\phi^{a,n} = G_\phi^{s,n} \cap G_\phi^{u,n}.$$

All are given the relative topology of $X \times X$.

Since $[y, x] = y$ if and only if $[x, y] = x$, one sees that $y \in X^s(x, \epsilon_0)$ if and only if $x \in X^s(y, \epsilon_0)$. Hence, $(x, y) \in G_\phi^{*,n}$ if and only if $(y, x) \in G_\phi^{*,n}$ for $* = s, u, a$.

We note the following lemma, which is well known and useful.

Lemma 2.5 *For $x, y \in X$ we have $(x, y) \in G_\phi^{a,0}$ if and only if $x = y$. Hence we may identify $G_\phi^{a,0}$ with X as a topological space.*

Proof Take an arbitrary $(x, y) \in G_\phi^{a,0}$. As $(x, y) \in G_\phi^{s,0}$, we see that $y \in X^s(x, \epsilon_0)$, so that $[y, x] = y$, and also as $(x, y) \in G_\phi^{u,0}$, we see that $y \in X^u(x, \epsilon_0)$, so that $[x, y] = y$. Hence, we have

$$x = [x, x] = [x, [y, x]] = [x, y] = y.$$

■

By Lemma 2.3, we know that

$$G_\phi^{*,n} \subset G_\phi^{*,n+1}, \quad * = s, u, a, \quad n = 0, 1, \dots \tag{2.2}$$

Following [25, Section 1], [26, Section 3], and [29, Section 2], we define several equivalence relations on X:

$$G_\phi^s = \bigcup_{n=0}^{\infty} G_\phi^{s,n}, \qquad G_\phi^u = \bigcup_{n=0}^{\infty} G_\phi^{u,n}, \qquad G_\phi^a = \bigcup_{n=0}^{\infty} G_\phi^{a,n}.$$

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By (2.2), the set $G_\phi^* = \bigcup_{n=0}^\infty G_\phi^{*,n}$ is an inductive system of topological spaces. Each $G_\phi^*, * = s, u, a$ is endowed with the inductive limit topology. The following lemma has also been shown by Putnam.

Lemma 2.6 (Putnam [25, Section 1])

(i) $$G_\phi^s = \left\{ (x, y) \in X \times X \mid \lim_{n\to\infty} d(\phi^n(x), \phi^n(y)) = 0 \right\}.$$

(ii) $$G_\phi^u = \left\{ (x, y) \in X \times X \mid \lim_{n\to\infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0 \right\}.$$

(iii) $$G_\phi^a = \left\{ (x, y) \in X \times X \mid \lim_{n\to\infty} d(\phi^n(x), \phi^n(y)) = \lim_{n\to\infty} d(\phi^{-n}(x), \phi^{-n}(y)) = 0 \right\}.$$

Putnam studied three equivalence relations, G_ϕ^s, G_ϕ^u, and G_ϕ^a on X, by regarding them as principal groupoids. He pointed out that the third equivalence relation G_ϕ^a is an étale equivalence relation whereas the first two are not étale in general. He also studied the associated groupoid C^*-algebras $C^*(G_\phi^s)$, $C^*(G_\phi^u)$, and $C^*(G_\phi^a)$, which have been denoted by $S(X,\phi)$, $U(X,\phi)$, and $A(X,\phi)$, respectively. He has pointed out that they are all stably AF-algebras if (X,ϕ) is a shift of finite type. He also studied their semi-direct products by the integer group $\mathbb{Z}$ as groupoids

$$\begin{aligned} G_\phi^s \rtimes \mathbb{Z} &= \{(x,n,y) \in X \times \mathbb{Z} \times X \mid (\phi^n(x), y) \in G_\phi^s\}, \\ G_\phi^u \rtimes \mathbb{Z} &= \{(x,n,y) \in X \times \mathbb{Z} \times X \mid (\phi^n(x), y) \in G_\phi^u\}, \\ G_\phi^a \rtimes \mathbb{Z} &= \{(x,n,y) \in X \times \mathbb{Z} \times X \mid (\phi^n(x), y) \in G_\phi^a\}. \end{aligned}$$

Since the map

$$\gamma : (x,n,y) \in G_\phi^* \rtimes \mathbb{Z} \longrightarrow \big((x, \phi^{-n}(y)), n \big) \in G_\phi^* \times \mathbb{Z}$$

is bijective, the topology of the groupoid $G_\phi^* \rtimes \mathbb{Z}$ is defined by the product topology of $G_\phi^* \times \mathbb{Z}$ through the map γ. Let us denote by $(G_\phi^* \rtimes \mathbb{Z})^\circ$ the unit space of the groupoid $G_\phi^* \rtimes \mathbb{Z}$, which is identified with that of G_ϕ^* and naturally homeomorphic to the original space X through the correspondence $(x,0,x) \in (G_\phi^* \rtimes \mathbb{Z})^\circ \to x \in X$ for $* = s,u,a$. The range map $r\colon G_\phi^* \rtimes \mathbb{Z} \to (G_\phi^* \rtimes \mathbb{Z})^\circ$ and the source map $s\colon G_\phi^* \rtimes \mathbb{Z} \to (G_\phi^* \rtimes \mathbb{Z})^\circ$ are defined by

$$r(x,n,y) = (x,0,x) \quad \text{and} \quad s(x,n,y) = (y,0,y).$$

The groupoid operations are defined by

$$\begin{aligned} (x,n,y) \cdot (x',m,w) &= (x, n+m, w) \quad \text{if } y = x', \\ (x,n,y)^{-1} &= (y,-n,x). \end{aligned}$$

Putnam [25, 26] and Putnam and Spielberg [29] also studied their associated groupoid C^*-algebras $C^*(G_\phi^s \rtimes \mathbb{Z})$, $C^*(G_\phi^u \rtimes \mathbb{Z})$, and $C^*(G_\phi^a \rtimes \mathbb{Z})$, which have been written R_s, R_u, and R_a, respectively in their papers. In this paper we denote them by $\mathcal{R}_\phi^s$, $\mathcal{R}_\phi^u$, and $\mathcal{R}_\phi^a$, respectively, to emphasize the homeomorphism ϕ. We remark that Putnam–Spielberg [29] (*cf.* Killough–Putnam [11]) also constructed other kinds of C^*-algebras, $S(X,\phi,P)$, $U(X,\phi,P)$, and their crossed products, $S(X,\phi,P) \rtimes \mathbb{Z}$,

$U(X,\phi,P)\rtimes\mathbb{Z}$, from a ϕ-invariant subset $P\subset X$ of periodic points by using étale groupoids defined by restricting the stable equivalence relation G_ϕ^s, unstable equivalence relations G_ϕ^u to P, respectively. In this paper, we will not deal with these C^*-algebras $S(X,\phi,P)$, $U(X,\phi,P)$, $S(X,\phi,P)\rtimes\mathbb{Z}$, $U(X,\phi,P)\rtimes\mathbb{Z}$.

3 Asymptotic Continuous Orbit Equivalence

Let (X,ϕ) be a Smale space. In this section, the symbol d will be used as a two-cocycle. It does not mean the metric on X. A sequence $\{f_n\}_{n\in\mathbb{Z}}$ of integer-valued continuous functions on X is called a *one-cocycle for* ϕ if it satisfies the identity

(3.1) $$f_n(x)+f_m(\phi^n(x))=f_{n+m}(x),\qquad x\in X,\ n,m\in\mathbb{Z}.$$

For a continuous function $f\colon X\to\mathbb{Z}$ and $n\in\mathbb{Z}$, we define

$$f^n(x)=\begin{cases}\sum_{i=0}^{n-1}f(\phi^i(x)) & \text{for } n>0,\\ 0 & \text{for } n=0,\\ -\sum_{i=n}^{-1}f(\phi^i(x)) & \text{for } n<0.\end{cases}$$

It is straightforward to prove the following lemma.

Lemma 3.1 *For $n,m\in\mathbb{Z}$, the identity*

(3.2) $$f^n(x)+f^m(\phi^n(x))=f^{n+m}(x),\qquad x\in X$$

holds. Hence, the sequence $\{f^n\}_{n\in\mathbb{Z}}$ is a one-cocycle for ϕ.

We note that a sequence of functions satisfying (3.1) is determined only by f_1.

In what follows we focus on asymptotic equivalence relations G_ϕ^a. A continuous function $d\colon G_\phi^a\to\mathbb{Z}$ is called a *two-cocycle* if it satisfies the following equalities:

(3.3) $$d(x,z)+d(z,w)=d(x,w),\qquad (x,z),(z,w),(x,w)\in G_\phi^a.$$

The identity (3.3) means that $d\colon G_\phi^a\to\mathbb{Z}$ is a groupoid homomorphism.

Definition 3.2 Smale spaces (X,ϕ) and (Y,ψ) are said to be *asymptotically continuously orbit equivalent* if there exist a homeomorphism $h\colon X\to Y$, continuous functions $c_1\colon X\to\mathbb{Z}$, $c_2\colon Y\to\mathbb{Z}$, and two-cocycle functions $d_1\colon G_\phi^a\to\mathbb{Z}$, $d_2\colon G_\psi^a\to\mathbb{Z}$ such that

(1) $c_1^m(x)+d_1(\phi^m(x),\phi^m(z))=c_1^m(z)+d_1(x,z)$, $\quad (x,z)\in G_\phi^a$, $m\in\mathbb{Z}$;

(2) $c_2^m(y)+d_2(\psi^m(y),\psi^m(w))=c_2^m(w)+d_2(y,w)$, $\quad (y,w)\in G_\psi^a$, $m\in\mathbb{Z}$;

and

(i) for each $n\in\mathbb{Z}$, the pair $(\psi^{c_1^n(x)}(h(x)),h(\phi^n(x)))$, denoted by $\xi_1^n(x)$, belongs to G_ψ^a for each $x\in X$, and the map $\xi_1^n\colon x\in X\to\xi_1^n(x)\in G_\psi^a$ is continuous;

(ii) for each $n\in\mathbb{Z}$, the pair $(\phi^{c_2^n(y)}(h^{-1}(y)),h^{-1}(\psi^n(y)))$, denoted by $\xi_2^n(y)$, belongs to G_ϕ^a for each $y\in Y$, and the map $\xi_2^n\colon y\in Y\to\xi_2^n(y)\in G_\phi^a$ is continuous;

(iii) the pair $(\psi^{d_1(x,z)}(h(x)),h(z))$, denoted by $\eta_1(x,z)$, belongs to G_ψ^a for each $(x,z)\in G_\phi^a$, and the map $\eta_1\colon(x,z)\in G_\phi^a\to\eta_1(x,z)\in G_\psi^a$ is continuous;

(iv) the pair $(\phi^{d_2(y,w)}(h^{-1}(y)), h^{-1}(w))$, denoted by $\eta_2(y,w)$, belongs to G_ϕ^a for each $(y,w) \in G_\psi^a$, and the map $\eta_2\colon (y,w) \in G_\psi^a \to \eta_2(y,w) \in G_\phi^a$ is continuous;

(v) $c_2^{c_1^n(x)}(h(x)) + d_2(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))) = n, \quad x \in X,\ n \in \mathbb{Z}$;

(vi) $c_1^{c_2^n(y)}(h^{-1}(y)) + d_1(\phi^{c_2^n(y)}(h^{-1}(y)), h^{-1}(\psi^n(y))) = n, \qquad y \in Y,\ n \in \mathbb{Z}$;

(vii) $c_2^{d_1(x,z)}(h(x)) + d_2(\psi^{d_1(x,z)}(h(x)), h(z)) = 0, \quad (x,z) \in G_\phi^a$;

(viii) $c_1^{d_2(y,w)}(h^{-1}(y)) + d_1(\phi^{d_2(y,w)}(h^{-1}(y)), h^{-1}(w)) = 0, \qquad (y,w) \in G_\psi^a$.

In this situation, we write $(X,\phi) \underset{\text{ACOE}}{\sim} (Y,\psi)$.

Remark 3.3 (i) Condition (1) above is equivalent to

$$c_1(x) + d_1(\phi(x), \phi(z)) = c_1(z) + d_1(x,z), \quad (x,z) \in G_\phi^a,$$

and condition (2) is similar to (1).

(ii) Conditions (i)–(iv) are equivalent to the following conditions, respectively:

(i) For each $n \in \mathbb{Z}$, there exists a continuous function $k_{1,n}\colon X \to \mathbb{Z}_+$ such that

$$(3.4)\qquad \begin{aligned} &\big(\psi^{k_{1,n}(x)+c_1^n(x)}(h(x)), \psi^{k_{1,n}(x)}(h(\phi^n(x)))\big) \in G_\psi^{s,0},\\ &\big(\psi^{-k_{1,n}(x)+c_1^n(x)}(h(x)), \psi^{-k_{1,n}(x)}(h(\phi^n(x)))\big) \in G_\psi^{u,0}. \end{aligned}$$

(ii) For each $n \in \mathbb{Z}$, there exists a continuous function $k_{2,n}\colon Y \to \mathbb{Z}_+$ such that

$$\begin{aligned} &\big(\phi^{k_{2,n}(y)+c_2^n(y)}(h^{-1}(y)), \phi^{k_{2,n}(y)}(h^{-1}(\psi^n(y)))\big) \in G_\phi^{s,0},\\ &\big(\phi^{-k_{2,n}(y)+c_2^n(y)}(h^{-1}(y)), \phi^{-k_{2,n}(y)}(h^{-1}(\psi^n(y)))\big) \in G_\phi^{u,0}. \end{aligned}$$

(iii) There exists a continuous function $m_1\colon G_\phi^a \to \mathbb{Z}_+$ such that

$$\begin{aligned} &\big(\psi^{m_1(x,z)+d_1(x,z)}(h(x)), \psi^{m_1(x,z)}(h(z))\big) \in G_\psi^{s,0} \text{ for } (x,z) \in G_\phi^a,\\ &\big(\psi^{-m_1(x,z)+d_1(x,z)}(h(x)), \psi^{-m_1(x,z)}(h(z))\big) \in G_\psi^{u,0} \text{ for } (x,z) \in G_\phi^a. \end{aligned}$$

(iv) There exists a continuous function $m_2\colon G_\psi^a \to \mathbb{Z}_+$ such that

$$\begin{aligned} &\big(\phi^{m_2(y,w)+d_2(y,w)}(h^{-1}(y)), \phi^{m_2(y,w)}(h^{-1}(w))\big) \in G_\phi^{s,0} \text{ for } (y,w) \in G_\psi^a,\\ &\big(\phi^{-m_2(y,w)+d_2(y,w)}(h^{-1}(y)), \phi^{-m_2(y,w)}(h^{-1}(w))\big) \in G_\phi^{u,0} \text{ for } (y,w) \in G_\psi^a. \end{aligned}$$

In what follows, we will assume that our Smale space is irreducible, which means that for every ordered pair of open sets $U, V \subset X$, there exists $K \in \mathbb{N}$ such that $\phi^K(U) \cap V \neq \varnothing$.

Theorem 3.4 *Suppose that Smale spaces (X,ϕ) and (Y,ψ) are irreducible. Then the following assertions are equivalent:*

(i) *(X,ϕ) and (Y,ψ) are asymptotically continuous orbit equivalent;*

(ii) *the groupoids $G_\phi^a \rtimes \mathbb{Z}$ and $G_\psi^a \rtimes \mathbb{Z}$ are isomorphic as étale groupoids.*

Proof (ii) ⇒ (i): Suppose that the groupoids $G_\phi^a \rtimes \mathbb{Z}$ and $G_\psi^a \rtimes \mathbb{Z}$ are isomorphic as étale groupoids. There exist homeomorphisms $h\colon (G_\phi^a \rtimes \mathbb{Z})^\circ \to (G_\psi^a \rtimes \mathbb{Z})^\circ$ and $\varphi_h\colon G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ that are compatible with their groupoid operations. Since the

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unit spaces $(G_\phi^a \rtimes \mathbb{Z})^\circ$ and $(G_\psi^a \rtimes \mathbb{Z})^\circ$ are identified with the original spaces X and Y as topological spaces through the identifications

$$(x,0,x) \in (G_\phi^a \rtimes \mathbb{Z})^\circ \longrightarrow x \in X \quad \text{and} \quad (y,0,y) \in (G_\psi^a \rtimes \mathbb{Z})^\circ \longrightarrow y \in Y,$$

respectively, we have a homeomorphism from X onto Y, which is still denoted by $h\colon X \to Y$. As $\varphi_h(x,n,z) \in G_\psi^a \rtimes \mathbb{Z}$ for $(x,n,z) \in G_\phi^a \rtimes \mathbb{Z}$, there exists $c_1(x,n,z) \in \mathbb{Z}$ such that $\varphi_h(x,n,z) = (h(x), c_1(x,n,z), h(z))$. In particular, we have $(x,n,\phi^n(x)) \in G_\phi^a \rtimes \mathbb{Z}$ for $z = \phi^n(x)$, and can define $c_{1,n}(x) = c_1(x,n,\phi^n(x))$ so that

$$\big(h(x), c_{1,n}(x), h(\phi^n(x))\big) \in G_\psi^a \rtimes \mathbb{Z}. \tag{3.5}$$

Now for $x \in X$ and $n,m \in \mathbb{Z}$, we have

$$\big(x, n+m, \phi^{n+m}(x)\big) = \big(x,n,\phi^n(x)\big) \cdot \big(\phi^n(x), m, \phi^{n+m}(x)\big) \text{ in } G_\phi^a \rtimes \mathbb{Z},$$

so that

$$\begin{aligned}
&(h(x), c_{1,n+m}(x), h(\phi^{n+m}(x)))\\
&= \varphi_h(x, n+m, \phi^{n+m}(x))\\
&= \varphi_h(x,n,\phi^n(x))\varphi_h(\phi^n(x), m, \phi^{n+m}(x))\\
&= (h(x), c_{1,n}(x), h(\phi^n(x)))(h(\phi^n(x)), c_{1,m}(\phi^n(x)), h(\phi^{n+m}(x)))\\
&= (h(x), c_{1,n}(x) + c_{1,m}(\phi^n(x)), h(\phi^{n+m}(x))).
\end{aligned}$$

Hence we have

$$c_{1,n+m}(x) = c_{1,n}(x) + c_{1,m}(\phi^n(x)), \tag{3.6}$$

so that the sequence $\{c_{1,n}\}_{n\in\mathbb{Z}}$ of continuous functions is a one-cocycle function on X. By putting $c_1(x) := c_{1,1}(x)$, we see that $c_1^n(x) = c_{1,n}(x)$ by (3.6). By (3.5), we see that $(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))) \in G_\psi^a$. Since the maps below

$$\begin{aligned}
&\big((x,x),n\big) \in G_\phi^a \times \mathbb{Z}\\
\xrightarrow{\gamma^{-1}}\ &\big(x,n,\phi^n(x)\big) \in G_\phi^a \rtimes \mathbb{Z}\\
\xrightarrow{\varphi_h}\ &\big(h(x), c_1^n(x), h(\phi^n(x))\big) \in G_\psi^a \rtimes \mathbb{Z}\\
\xrightarrow{\gamma}\ &\big(h(x), \psi^{-c_1^n(x)}(h(\phi^n(x))), c_1^n(x)\big) \in G_\psi^a \times \mathbb{Z}\\
\xrightarrow{(\psi^{c_1^n(x)} \times \psi^{c_1^n(x)}) \times \mathrm{id}}\ &\big(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x)), c_1^n(x)\big) \in G_\psi^a \times \mathbb{Z}
\end{aligned}$$

are all continuous, the map $\xi_1^n\colon x \in X \to \xi_1^n(x) := (\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))) \in G_\psi^a$ is continuous.

On the other hand, for $(x,z) \in G_\phi^a$ we see that $(x,0,z) \in G_\phi^a \rtimes \mathbb{Z}$. Hence there exists $d_1(x,z) \in \mathbb{Z}$ such that $\varphi_h(x,0,z) = (h(x), d_1(x,z), h(z))$. Since $\varphi_h\colon G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ is continuous, the function $d_1\colon G_\phi^a \to \mathbb{Z}$ is continuous. For $(x,z),(z,w) \in G_\phi^a$, we have $(x,0,w)=(x,0,z)(z,0,w)\in G_\phi^a$, and hence

$$\begin{aligned}\big(h(x),d_1(x,w),h(w)\big) &= \varphi_h(x,0,w)\\ &= \varphi_h(x,0,z)\cdot\varphi_h(z,0,w)\\ &= \big(h(x),d_1(x,z),h(z)\big)\cdot\big(h(z),d_1(z,w),h(w)\big)\\ &= \big(h(x),d_1(x,z)+d_1(z,w),h(w)\big),\end{aligned}$$

so that $d_1(x,w)=d_1(x,z)+d_1(z,w)$ holds, and $d_1\colon G_\phi^a\to\mathbb{Z}$ is a two-cocycle function. Since the maps

$$\begin{aligned} &\big((x,z),0\big)\in G_\phi^a\times\mathbb{Z}\\ \xrightarrow{\gamma^{-1}}\ &(x,0,z)\in G_\phi^a\rtimes\mathbb{Z}\\ \xrightarrow{\varphi_h}\ &\big(h(x),d_1(x,z),h(z)\big)\in G_\psi^a\rtimes\mathbb{Z}\\ \xrightarrow{\gamma}\ &\big((h(x),\psi^{-d_1(x,z)}(h(z)),d_1(x,z)\big)\in G_\psi^a\times\mathbb{Z}\\ \xrightarrow{(\psi^{d_1(x,z)}\times\psi^{d_1(x,z)})\times\mathrm{id}}\ &\big(\psi^{d_1(x,z)}(h(x)),h(z)),d_1(x,z)\big)\in G_\psi^a\times\mathbb{Z}\end{aligned}$$

are all continuous, the map $\eta_1\colon(x,z)\in G_\phi^a\to\eta_1(x,z):=(\psi^{d_1(x,z)}(h(x)),h(z))\in G_\psi^a$ is continuous.

For $(x,n,x'),(x',m,z)\in G_\phi^a\rtimes\mathbb{Z}$, the identity

$$\varphi_h\big((x,n,x')\cdot(x',m,z)\big)=\varphi_h(x,n,x')\cdot\varphi_h(x',m,z)$$

is equivalent to the identity

$$c_1^m(\phi^n(x))+d_1\big(\phi^{m+n}(x),z\big)=c_1^m(x')+d_1\big(\phi^n(x),x'\big)+d_1\big(\phi^m(x'),z\big),$$

which implies the identity

$$c_1^m(x)+d_1\big(\phi^m(x),\phi^m(z)\big)=c_1^m(z)+d_1(x,z),\quad (x,z)\in G_\phi^a,\ m\in\mathbb{Z}.$$

Similarly, we have one-cocycle function $c_2\colon Y\to\mathbb{Z}$ and two-cocycle function $d_2\colon G_\psi^a\to\mathbb{Z}$ for the homeomorphism $\varphi_h^{-1}\colon G_\psi^a\rtimes\mathbb{Z}\to G_\phi^a\rtimes\mathbb{Z}$. Since

$$h^{-1}=(\varphi_h)^{-1}|_{(G_\psi^a\rtimes\mathbb{Z})^\circ}\colon(G_\psi^a\rtimes\mathbb{Z})^\circ=Y\longrightarrow(G_\phi^a\rtimes\mathbb{Z})^\circ=X,$$

we see that $\varphi_h^{-1}=\varphi_{h^{-1}}$. By the identity

$$\big(x,n,\phi^n(x)\big)=(\varphi_h^{-1}\circ\varphi_h)\big(x,n,\phi^n(x)\big)\qquad\text{for } x\in X,\ n\in\mathbb{Z},$$

we have

$$\begin{aligned}&(\varphi_h^{-1}\circ\varphi_h)\big(x,n,\phi^n(x)\big)\\ &\quad=\varphi_h^{-1}\big(h(x),c_1^n(x),h(\phi^n(x))\big)\\ &\quad=\varphi_h^{-1}\big(h(x),c_1^n(x),\psi^{c_1^n(x)}(h(x))\big)\varphi_h^{-1}\big(\psi^{c_1^n(x)}(h(x)),0,h(\phi^n(x))\big)\\ &\quad=\big(x,c_2^{c_1^n(x)}(h(x)),h^{-1}(\psi^{c_1^n(x)}(h(x))\big)\\ &\qquad\cdot\big(h^{-1}(\psi^{c_1^n(x)}(h(x))\big),d_2\big(\psi^{c_1^n(x)}(h(x)),h(\phi^n(x))),\phi^n(x)\big)\\ &\quad=\big(x,c_2^{c_1^n(x)}(h(x))+d_2(\psi^{c_1^n(x)}(h(x)),h(\phi^n(x))),\phi^n(x)\big),\end{aligned}$$

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so that the identity

$$c_2^{c_1^n(x)}(h(x)) + d_2\big(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))\big) = n$$

holds, and similarly

$$c_1^{c_2^n(y)}(h^{-1}(y)) + d_1\big(\phi^{c_2^n(y)}(h^{-1}(y)), h^{-1}(\psi^n(y))\big) = n, \quad y \in Y,\ n \in \mathbb{Z}.$$

For $(x,z) \in G_\phi^a$, the identity $(x,0,z) = (\varphi_h^{-1} \circ \varphi)(x,0,z)$ holds, so that we have

$$\begin{aligned}(\varphi_h^{-1} \circ \varphi_h)(x,0,z) &= \varphi_h^{-1}\big(h(x), d_1(x,z), h(z)\big)\\ &= \varphi_h^{-1}\big(h(x), d_1(x,z), \psi^{d_1(x,z)}(h(x))\big)\,\varphi_h^{-1}\big(\psi^{d_1(x,z)}(h(x)), 0, h(z)\big)\\ &= \big(x, c_2^{d_1(x,z)}(h(x)), h^{-1}(\psi^{d_1(x,z)}(h(x))\big)\\ &\quad\cdot\big(h^{-1}(\psi^{d_1(x,z)}(h(x))\big), d_2\big(\psi^{d_1(x,z)}(h(x)), h(z)\big), z\big)\\ &= \big(x, c_2^{d_1(x,z)}(h(x)) + d_2(\psi^{d_1(x,z)}(h(x)), h(z)), z\big).\end{aligned}$$

Hence we have

$$c_2^{d_1(x,z)}(h(x)) + d_2(\psi^{d_1(x,z)}(h(x)), h(z)) = 0, \qquad (x,z) \in G_\phi^a,$$

and similarly

$$c_1^{d_2(y,w)}(h^{-1}(y)) + d_1(\phi^{d_2(y,w)}(h^{-1}(y)), h^{-1}(w)) = 0, \qquad (y,w) \in G_\psi^a.$$

Therefore, we see that $(X,\phi) \underset{\text{ACOE}}{\sim} (Y,\psi)$.

(i) ⇒ (ii): Assume that $(X,\phi) \underset{\text{ACOE}}{\sim} (Y,\psi)$ and take a homeomorphism $h\colon X \to Y$, continuous functions $c_1\colon X \to \mathbb{Z}$, $c_2\colon Y \to \mathbb{Z}$, and two-cocycle functions $d_1\colon G_\phi^a \to \mathbb{Z}$, $d_2\colon G_\psi^a \to \mathbb{Z}$ as in Definition 3.2. Let $(x,n,z) \in G_\phi^a \rtimes \mathbb{Z}$ be an arbitrary element so that $(\phi^n(x), z) \in G_\phi^a$, and we have

$$(x,n,z) = (x,n,\phi^n(x)) \cdot (\phi^n(x), 0, z).$$

Put

$$\varphi_h(x,n,z) = \big(h(x), c_1^n(x), h(\phi^n(x)) \cdot (h(\phi^n(x)), d_1(\phi^n(x), z), h(z)\big). \tag{3.7}$$

By Definition 3.2(i), $(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x)))$ belongs to G_ψ^a. As a consequence, $(h(x), c_1^n(x), h(\phi^n(x)))$ gives an element of $G_\psi^a \rtimes \mathbb{Z}$. As $(\phi^n(x), z) \in G_\phi^a$, we see that by Definition 3.2(iii), $(\psi^{d_1(\phi^n(x),z)}(h(\phi^n(x))), h(z))$ belongs to G_ψ^a, so that $(h(\phi^n(x)), d_1(\phi^n(x), z), h(z))$ gives an element of $G_\psi^a \rtimes \mathbb{Z}$. Hence, $\varphi_h(x,n,z)$ defines an element of the groupoid $G_\psi^a \rtimes \mathbb{Z}$ such that

$$\varphi_h(x,n,z) = (h(x), c_1^n(x) + d_1\big(\phi^n(x), z), h(z)\big).$$

It is straightforward to see that the equality (1) in Definition 3.2 implies

$$\varphi_h\big((x,n,x') \cdot (x',m,z)\big) = \varphi_h(x,n,x') \cdot \varphi_h(x',m,z)$$

for $(x,n,x'), (x',m,z) \in G_\phi^a \rtimes \mathbb{Z}$.

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Since $x \in X \to \xi_1^n(x) = (\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))) \in G_\psi^a$ is continuous by Definition 3.2(i) and

$$\begin{aligned}\gamma^{-1} \circ \big((\psi^{-c_1^n(x)} \times \psi^{-c_1^n(x)}\big) \times \mathrm{id}\big)\big(\xi_1^n(x), c_1^n(x)\big) \\ = \gamma^{-1}\big(h(x), \psi^{-c_1^n(x)}\big(h(\phi^n(x))\big), c_1^n(x)\big) \\ = \big(h(x), c_1^n(x), h(\phi^n(x))\big),\end{aligned}$$

the map $\varphi_h^1 : G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ defined by

$$\varphi_h^1(x,n,z) := \big(h(x), c_1^n(x), h(\phi^n(x))\big)$$

is continuous.

And also the map $\eta_1 : (x,z) \in G_\phi^a \to \eta_1(x,z) = (\psi^{d_1(x,z)}(h(x)), h(z)) \in G_\psi^a$ is continuous by Definition 3.2(iii) and

$$\begin{aligned}\gamma^{-1}\big((\psi^{-d_1(\phi^n(x),z)} \times \psi^{-d_1(\phi^n(x),z)}) \times \mathrm{id}\,\big)\big(\eta_1(\phi^n(x),z), d_1(\phi^n(x),z)\big) \\ =\gamma^{-1}\big(h(\phi^n(x)), \psi^{-d_1(\phi^n(x),z)}(h(z)), d_1(\phi^n(x),z)\big) \\ =\big(h(\phi^n(x)), d_1(\phi^n(x),z), h(z)\big).\end{aligned}$$

Hence the map $\varphi_h^0 : G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ defined by

$$\varphi_h^0(x,n,z) := (h(\phi^n(x)), d_1(\phi^n(x),z), h(z))$$

is continuous. Since $\varphi_h(x,n,z) = \varphi_h^1(x,n,z)\varphi_h^0(x,n,z)$ by (3.7), the map $\varphi_h : G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ is continuous.

Similarly, we can define a continuous map $\varphi_{h^{-1}} : G_\psi^a \rtimes \mathbb{Z} \to G_\phi^a \rtimes \mathbb{Z}$ from the homeomorphism $h^{-1} : Y \to X$ and one-cocycle function $c_2 : Y \to \mathbb{Z}$, two-cocycle function $d_2 : G_\psi^a \to \mathbb{Z}$ by setting

$$\varphi_{h^{-1}}(y,m,w) = \big(h^{-1}(y), c_2^m(y) + d_2(\psi^m(y),w), h^{-1}(w)\big) \quad \text{for } (y,m,w) \in G_\psi^a \rtimes \mathbb{Z}.$$

For $(y,m,w) \in G_\psi^a \rtimes \mathbb{Z}$, we put

$$\begin{aligned}(\varphi_{h^{-1}})^1(y,m,w) &=\big(h^{-1}(y), c_2^m(y), h^{-1}(\psi^m(y))\big), \\ (\varphi_{h^{-1}})^0(y,m,w) &=\big(h^{-1}(\psi^m(y)), d_2(\psi^m(y),w), h^{-1}(w)\big),\end{aligned}$$

so that

$$\varphi_{h^{-1}}(y,m,w) = (\varphi_{h^{-1}})^1(y,m,w) \cdot (\varphi_{h^{-1}})^0(y,m,w) \qquad \text{for } (y,m,w) \in G_\psi^a \rtimes \mathbb{Z}.$$

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We will next show that φ_h and $\varphi_{h^{-1}}$ are inverses of each other. For $x \in X, n \in \mathbb{Z}$, we have

$$
\begin{aligned}
&(\varphi_{h^{-1}} \circ \varphi_h)(x, n, \phi^n(x)) \\
&\quad = \varphi_{h^{-1}}\big(h(x), c_1^n(x), h(\phi^n(x))\big) \\
&\quad = \varphi_{h^{-1}}\big(h(x), c_1^n(x), \psi^{c_1^n(x)}(h(x))\big) \cdot \varphi_h^{-1}\big(\psi^{c_1^n(x)}(h(x)), 0, h(\phi^n(x))\big) \\
&\quad = \big(x, c_2^{c_1^n(x)}(h(x)), h^{-1}(\psi^{c_1^n(x)}(h(x)))\big) \\
&\qquad \cdot \big(h^{-1}(\psi^{c_1^n(x)}(h(x))), d_2\big(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))\big), \phi^n(x)\big) \\
&\quad = \big(x, c_2^{c_1^n(x)}(h(x)) + d_2(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))), \phi^n(x)\big).
\end{aligned}
$$

By the condition of Definition 3.2(v), we have

$$(\varphi_{h^{-1}} \circ \varphi_h)(x, n, \phi^n(x)) = (x, n, \phi^n(x)).$$

We also have for $(x, z) \in G_\phi^a$,

$$
\begin{aligned}
&\varphi_{h^{-1}} \circ \varphi_h(x, 0, z) \\
&\quad = \varphi_{h^{-1}}(h(x), d_1(x, z), h(z)) \\
&\quad = (x, c_2^{d_1(x,z)}(h(x)), h^{-1}(\psi^{d_1(x,z)}(h(x)))) \\
&\qquad \cdot (h^{-1}(\psi^{d_1(x,z)}(h(x))), d_2(\psi^{d_1(x,z)}(h(x)), h(z)), z) \\
&\quad = (x, c_2^{d_1(x,z)}(h(x)) + d_2(\psi^{d_1(x,z)}(h(x)), h(z)), z)
\end{aligned}
$$

By the condition of Definition 3.2(vii), we have

$$(\varphi_{h^{-1}} \circ \varphi_h)(x, 0, z) = (x, 0, z).$$

Therefore, we have for $(x, n, z) \in G_\phi^a \rtimes \mathbb{Z}$,

$$
\begin{aligned}
&(\varphi_{h^{-1}} \circ \varphi_h)(x, n, z) \\
&\quad = \big((\varphi_{h^{-1}} \circ \varphi_h)(x, n, \phi^n(x))\big) \cdot \big((\varphi_{h^{-1}} \circ \varphi_h)(\phi^n(x), 0, z)\big) \\
&\quad = \big(x, n, \phi^n(x)\big) \cdot \big(\phi^n(x), 0, z\big) = (x, n, z).
\end{aligned}
$$

Similarly, we have $(\varphi_h \circ \varphi_{h^{-1}})(y, m, w) = (y, m, w)$ for $(y, m, w) \in G_\psi^a \rtimes \mathbb{Z}$. Hence we have $\varphi_{h^{-1}} = (\varphi_h)^{-1}$ and φ_h gives rise to an isomorphism $G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ of the étale groupoids. ∎

Smale spaces (X, ϕ) and (Y, ψ) are said to be *stably continuous orbit equivalent* if in Definition 3.2, we can replace G_ϕ^a, G_ψ^a with G_ϕ^s, G_ψ^s, respectively, and written $(X, \phi) \underset{\text{SCOE}}{\sim} (Y, \psi)$. Unstably continuous orbit equivalent is similarly defined by replacing G_ϕ^a, G_ψ^a with G_ϕ^u, G_ψ^u, respectively, and written $(X, \phi) \underset{\text{UCOE}}{\sim} (Y, \psi)$. The precise definition of stably continuous orbit equivalent follows.

Definition 3.5 Smale spaces (X, ϕ) and (Y, ψ) are said to be *stably continuous orbit equivalent* if there exist a homeomorphism $h\colon X \to Y$, continuous functions $c_1\colon X \to \mathbb{Z}$, $c_2\colon Y \to \mathbb{Z}$, and two-cocycle functions $d_1\colon G_\phi^s \to \mathbb{Z}$, $d_2\colon G_\psi^s \to \mathbb{Z}$ such that

(1) $c_1^m(x) + d_1(\phi^m(x), \phi^m(z)) = c_1^m(z) + d_1(x, z), \quad (x, z) \in G_\phi^s,\ m \in \mathbb{Z}.$

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(2) $c_2^m(y) + d_2(\psi^m(y), \psi^m(w)) = c_2^m(w) + d_2(y, w), \quad (y, w) \in G_\psi^s, m \in \mathbb{Z}$;

and

(i) for each $n \in \mathbb{Z}$, there exists a continuous function $k_{1,n}\colon X \to \mathbb{Z}_+$ such that
$$\big(\psi^{k_{1,n}(x)+c_1^n(x)}(h(x)), \psi^{k_{1,n}(x)}(h(\phi^n(x)))\big) \in G_\psi^{s,0};$$

(ii) for each $n \in \mathbb{Z}$, there exists a continuous function $k_{2,n}\colon Y \to \mathbb{Z}_+$ such that
$$\big(\phi^{k_{2,n}(y)+c_2^n(y)}(h^{-1}(y)), \phi^{k_{2,n}(y)}(h^{-1}(\phi^n(y)))\big) \in G_\phi^{s,0};$$

(iii) there exists a continuous function $m_1\colon G_\phi^s \to \mathbb{Z}_+$ such that
$$\big(\psi^{m_1(x,z)+d_1(x,z)}(h(x)), \psi^{m_1(x,z)}(h(z))\big) \in G_\psi^{s,0} \text{ for } (x,z) \in G_\phi^s;$$

(iv) there exists a continuous function $m_2\colon G_\psi^s \to \mathbb{Z}_+$ such that
$$\big(\phi^{m_2(y,w)+d_2(y,w)}(h^{-1}(y)), \phi^{m_2(y,w)}(h^{-1}(w))\big) \in G_\phi^{s,0} \text{ for } (y,w) \in G_\psi^s;$$

(v) $c_2^{c_1^n(x)}(h(x)) + d_2(\psi^{c_1^n(x)}(h(x)), h(\phi^n(x))) = n, \quad x \in X,\ n \in \mathbb{Z}$;

(vi) $c_1^{c_2^n(y)}(h^{-1}(y)) + d_1(\phi^{c_2^n(y)}(h^{-1}(y)), h^{-1}(\psi^n(y))) = n, \quad y \in Y,\ n \in \mathbb{Z}$;

(vii) $c_2^{d_1(x,z)}(h(x)) + d_2(\psi^{d_1(x,z)}(h(x)), h(z)) = 0, \quad (x,z) \in G_\phi^s$;

(viii) $c_1^{d_2(y,w)}(h^{-1}(y)) + d_1(\phi^{d_2(y,w)}(h^{-1}(y)), h^{-1}(w)) = 0, \quad (y,w) \in G_\psi^s$.

If we replace $G_\phi^{s,0}, G_\psi^{s,0}, G_\phi^s, G_\psi^s$ with $G_\phi^{u,0}, G_\psi^{u,0}, G_\phi^u, G_\psi^u$, respectively, then (X,ϕ) and (Y,ψ) are said to be *unstably continuous orbit equivalent*.

We can prove the following theorem in a similar fashion to Theorem 3.4.

Theorem 3.6 *Suppose that the Smale spaces (X,ϕ) and (Y,ψ) are irreducible. Then the following conditions are equivalent:*

(i) $(X,\phi) \underset{\text{SCOE}}{\sim} (Y,\psi)$ *(resp.* $(X,\phi) \underset{\text{UCOE}}{\sim} (Y,\psi)$*);*

(ii) *the groupoids $G_\phi^s \rtimes \mathbb{Z}$ and $G_\psi^s \rtimes \mathbb{Z}$ (resp. $G_\phi^u \rtimes \mathbb{Z}$ and $G_\psi^u \rtimes \mathbb{Z}$) are isomorphic as topological groupoids.*

We note that the groupoids $G_\phi^s, G_\psi^s, G_\phi^u, G_\psi^u$ above are the non-étale groupoids appearing in Lemma 2.6, which were defined in [25]. We do not know whether or not the corresponding theorem holds for étale groupoids defined from ϕ-invariant set of stable or unstable equivalence relations appearing in [29].

4 Asymptotic Periodic Orbits of Smale Spaces

Let (X,ϕ) be an irreducible Smale space.

Definition 4.1 An element $x \in X$ is called *an asymptotic periodic point* if there exists $p \in \mathbb{Z}$ with $p \neq 0$ such that $(x, \phi^p(x)) \in G_\phi^a$. We call such p asymptotic period of x. If $|p|$ is the least positive such number, it is said to be the least asymptotic period of x.

We note that the asymptotic period is possibly negative, and hence if p is the least asymptotic period, then $-p$ is also the least asymptotic period.

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We assume that $(X,\phi) \underset{\text{ACOE}}{\sim} (Y,\psi)$ and keep a homeomorphism $h\colon X \to Y$, continuous functions c_1, c_2 and two-cocycle functions d_1, d_2 which give rise to asymptotically continuous orbit equivalence between (X,ϕ) and (Y,ψ).

Lemma 4.2 *If $x \in X$ is an asymptotic periodic point with asymptotic period p, then $h(x)$ is also an asymptotic periodic point with asymptotic period $c_1^p(x)+d_1(\phi^p(x),x)$.*

Proof Since $(x,\phi^p(x)) \in G_\phi^a$ and hence $(x,p,x) \in G_\phi^a \rtimes \mathbb{Z}$, we have

$$\varphi_h(x,p,x) = \big(h(x), c_1^p(x) + d_1(\phi^p(x),x), h(x)\big).$$

As $(X,\phi) \underset{\text{ACOE}}{\sim} (Y,\psi)$, $h(x)$ is an asymptotic periodic point in Y with asymptotic period $c_1^p(x) + d_1(\phi^p(x),x)$. ■

Lemma 4.3 *Let $x \in X$ be an asymptotic periodic point with least asymptotic period p. Let p' be the least asymptotic period of $h(x)$. Then we have the equality*

$$c_2^{np'}(h(x))+d_2\big(\psi^{np'}(h(x)),h(x)\big) = n\cdot\big(c_2^{p'}(h(x))+d_2\big(\psi^{p'}(h(x)),h(x)\big)\big) \tag{4.1}$$

for all $n \in \mathbb{Z}$.

Proof Suppose that $(x,\phi^p(x)) \in G_\phi^a$. Put $y = h(x)$ and $q' = c_1^p(x) + d_1(\phi^p(x),x)$. By the preceding lemma, we know that y has asymptotic period q', so that $(y,p',y) \in G_\psi^a \rtimes \mathbb{Z}$. Now suppose that equality (4.1) holds for $n = k$. Since $(y,p',y)(y,kp',y) = (y,(k+1)p',y)$, we get

$$\varphi_{h^{-1}}\big((y,p',y)(y,kp',y)\big) = \varphi_{h^{-1}}\big(y,(k+1)p',y\big). \tag{4.2}$$

The left-hand side of (4.2) equals

$$\begin{aligned}
&\varphi_{h^{-1}}(y,p',y)\varphi_{h^{-1}}(y,kp',y)\\
&\quad= \big(h^{-1}(y), c_2^{p'}(y) + d_2(\psi^{p'}(y),y), h^{-1}(y)\big)\\
&\qquad\cdot \big(h^{-1}(y), c_2^{kp'}(y) + d_2(\psi^{kp'}(y),y), h^{-1}(y)\big)\\
&\quad= \big(h^{-1}(y), c_2^{p'}(y) + d_2(\psi^{p'}(y),y) + c_2^{kp'}(y) + d_2(\psi^{kp'}(y),y), h^{-1}(y)\big).
\end{aligned}$$

The right-hand side of (4.2) equals

$$\varphi_{h^{-1}}(y,(k+1)p',y) = (h^{-1}(y), c_2^{(k+1)p'}(y) + d_2(\psi^{(k+1)p'}(y),y), h^{-1}(y)),$$

so that we have

$$c_2^{p'}(y)+d_2(\psi^{p'}(y),y)+c_2^{kp'}(y)+d_2(\psi^{kp'}(y),y) = c_2^{(k+1)p'}(y)+d_2(\psi^{(k+1)p'}(y),y).$$

By induction, we obtain the desired equalities for all $n \in \mathbb{N}$, and hence for all $n \in \mathbb{Z}$ in a similar way. ■

Lemma 4.4 *If $x \in X$ is an asymptotic periodic point with asymptotic period p, then $h(x)$ is also an asymptotic periodic point with asymptotic period $c_1^p(x)+d_1(\phi^p(x),x)$. If, in particular, p is the least asymptotic period of x, then $c_1^p(x) + d_1(\phi^p(x),x)$ is the least asymptotic period of $h(x)$.*

Proof It suffices to show the "if in particular" part. Suppose that $(x, \phi^p(x)) \in G_\phi^a$ and p is the least asymptotic period of x. We will show that $c_1^p(x) + d_1(\phi^p(x), x)$ is the least asymptotic period of $h(x)$. Let p' be the the least asymptotic period of $h(x)$. Put $q' = c_1^p(x) + d_1(\phi^p(x), x)$, so that $q' = n \cdot p'$ for some $n \in \mathbb{Z}$. We will prove that $n = \pm 1$. We have

$$\begin{aligned}(x, p, x) &= (\varphi_{h^{-1}} \circ \varphi_h)(x, p, x) \\ &= \varphi_{h^{-1}}\big(h(x), q', h(x)\big) \\ &= \big(x, c_2^{q'}(h(x)) + d_2(\psi^{q'}(h(x)), h(x)), x\big).\end{aligned}$$

Hence, $p = c_2^{q'}(h(x)) + d_2(\psi^{q'}(h(x)), h(x))$. As $q' = np'$, the preceding lemma tells us that

$$p = n \cdot (c_2^{p'}(h(x)) + d_2\big(\psi^{p'}(h(x)), h(x)\big)). \tag{4.3}$$

Since p' is (the least) asymptotic period of $h(x)$, we have $(\psi^{p'}(h(x)), h(x)) \in G_\psi^a$, so that by Definition 3.2(iv), we have

$$\phi^{d_2(\psi^{p'}(h(x)), h(x))}\big(h^{-1}(\psi^{p'}(h(x))), h^{-1}(h(x))\big) \in G_\phi^a. \tag{4.4}$$

By Definition 3.2(ii), we have $(\phi^{c_2^{p'}(h(x))}(x), h^{-1}(\psi^{p'}(h(x)))) \in G_\phi^a$ and hence

$$\big(\phi^{c_2^{p'}(h(x)) + d_2(\psi^{p'}(h(x)), h(x))}(x), \phi^{d_2(\psi^{p'}(h(x)), h(x))}\big(h^{-1}(\psi^{p'}(h(x)))\big)\big) \in G_\phi^a. \tag{4.5}$$

By (4.4) and (4.5), we have

$$\big(\phi^{c_2^{p'}(h(x)) + d_2(\psi^{p'}(h(x)), h(x))}(x), x\big) \in G_\phi^a.$$

As p is the least asymptotic period of x, we have

$$c_2^{p'}(h(x)) + d_2(\psi^{p'}(h(x)), h(x)) = p \cdot m' \quad \text{for some } m' \in \mathbb{Z}. \tag{4.6}$$

By (4.3) and (4.6), we have

$$p = n \cdot m' \cdot p,$$

so that we conclude that $n = \pm 1$, and hence $c_1^p(x) + d_1(\phi^p(x), x)$ is the least asymptotic period of $h(x)$. ∎

For an asymptotic periodic point $x \in X$ with asymptotic period p, we put

$$c_h^p(x) := c_1^p(x) + d_1(\phi^p(x), x) \in \mathbb{Z}.$$

If p is the least asymptotic period, the preceding proposition tells us that

$$c_h^{np}(x) = n \cdot c_h^p(x) \quad \text{for } n \in \mathbb{Z}.$$

In this case, as any asymptotic period q of x is written $q = m \cdot p$ for some $m \in \mathbb{Z}$ with $m \neq 0$, we have $c_h^{nmp}(x) = nm \cdot c_h^p(x) = n \cdot c_h^{mp}(x)$, so that $c_h^{nq}(x) = n \cdot c_h^q(x)$.

For a periodic point $x \in X$, the finite set $\{\phi^n(x) \mid n \in \mathbb{Z}\}$ is called a *periodic orbit*. Let us denote by

$$\mathrm{P}_{\mathrm{orb}}(X) := \text{the set of periodic orbits of } (X, \phi).$$

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A periodic point with period p is called a *p-periodic point.* Let $\mathrm{Per}_p(X)$ be the set of p-periodic points of (X,ϕ). The following theorem due to R. Bowen tells us that the set $\mathrm{Per}_p(X)$ is finite for each p, because so is $\mathrm{Per}_p(\overline{X}_A)$.

Theorem 4.5 ([2, Theorem 3.12]) *Let (X,ϕ) be an irreducible Smale space. Then there exists an irreducible subshift of finite type $(\overline{X}_A,\overline{\sigma}_A)$ such that there exists a finite-to-one factor map $\varphi\colon(\overline{X}_A,\overline{\sigma}_A)\to(X,\phi)$.*

For a periodic orbit $\gamma\in\mathrm{P}_{\mathrm{orb}}(X)$, take a periodic point $x\in X$ such that $\gamma=\{\phi^n(x)\mid n\in\mathbb{Z}\}$. The cardinality of the set $\{\phi^n(x)\mid n\in\mathbb{Z}\}$ is called the *length of γ* and written $|\gamma|$. We will show that the periodic orbits $\mathrm{P}_{\mathrm{orb}}(X)$ and $\mathrm{P}_{\mathrm{orb}}(Y)$ are related by their cocycle functions under the condition $(X,\phi)\underset{\mathrm{SCOE}}{\sim}(Y,\psi)$. A point $x\in X$ is called a *stably periodic point* if there exists $p\in\mathbb{Z}$ with $p\neq 0$ such that $(x,\phi^p(x))\in G^s_\phi$. We call such p a *stable period of x*. We note that Lemmas 4.2, 4.3, and 4.4 hold for stably periodic points and stable periods under the condition $(X,\phi)\underset{\mathrm{SCOE}}{\sim}(Y,\psi)$. We provide the following lemma.

Lemma 4.6 *Suppose that $(X,\phi)\underset{\mathrm{SCOE}}{\sim}(Y,\psi)$. Let $x\in X$ be a periodic point in X such that $\phi^p(x)=x$. Put $q=c_1^p(x)$ and assume $q>0$, otherwise take $-p$. Then we have the following:*

(i) *$c_1^{kp}(x)=kq$ for $k\in\mathbb{Z}$.*
(ii) *$\psi^q(h(x))\in Y^s(h(x))$ so that the limit $\lim_{k\to\infty}\psi^{qk}(h(x))$ exists in Y.*
(iii) *Put $\eta_h(x)=\lim_{k\to\infty}\psi^{qk}(h(x))$. Then*

$$\eta_h(\phi^n(x))=\psi^{c_1^n(x)}(\eta_h(x))\qquad \text{for}\quad n\in\mathbb{Z}. \tag{4.7}$$

In particular, $\eta_h(x)$ is a q-periodic point in Y.

(iv) *$\eta_h(x)\in Y^s(h(x))$.*
(v) *If p is the least period of x, then $c_1^p(x)$ is the least period of $\eta_h(x)$.*
(vi) *$c_2^q(\eta_h(x))=p$.*

Proof (i) Since $\phi^p(x)=x$, we have $d_1(\phi^p(x),x)=0$, so that $c_h^p(x)=c_1^p(x)+d_1(\phi^p(x),x)=c_1^p(x)$. Hence, the equality $c_1^p(x)\cdot k=c_1^{kp}(x)$ for $k\in\mathbb{Z}$ is immediate.

(ii) We have $(\psi^q(h(x)),h(x))=(\psi^{c_1^p(x)}(h(x)),h(\phi^p(x)))$, which belongs to G^s_ψ because of Definition 3.5(i), so that $\psi^q(h(x))\in Y^s(h(x))$. By using [29, Lemma 5.3], the element $\lim_{k\to\infty}\psi^{qk}(h(x))$ exists in Y and is a periodic point with period q.

(iii) By Definition 3.5(i) with Lemma 2.6, we have

$$\lim_{k\to\infty}\psi^{qk}\big(h(\phi^n(x))\big)=\lim_{k\to\infty}\psi^{qk}\big(\psi^{c_1^n(x)}h(x)\big)\big)=\psi^{c_1^n(x)}\big(\lim_{k\to\infty}\psi^{qk}\big(h(x)\big)\big),$$

so that the equality (4.7) holds.

(iv) For each $n\in\mathbb{Z}$, we have $qn=c_1^p(x)n=c_1^{pn}(x)$ by (i), so that the equality

$$\lim_{k\to\infty}\psi^{qk}\big(\psi^{qn}(h(x))\big)=\lim_{k\to\infty}\psi^{qk}\big(h(\phi^{pn}(x))\big)=\lim_{k\to\infty}\psi^{qk}\big(h(x)\big)$$

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holds because of Definition 3.5(i). It then follows that

$$\begin{aligned}\lim_{n\to\infty}\psi^{qn}(\eta_h(x)) &= \lim_{n\to\infty}\psi^{qn}\big(\lim_{k\to\infty}\psi^{qk}(h(x))\big)\\ &= \lim_{n\to\infty}\big(\lim_{k\to\infty}\psi^{qn+qk}(h(x))\big)\\ &= \lim_{n\to\infty}\big(\lim_{k\to\infty}\psi^{qk}(\psi^{qn}(h(x)))\big)\\ &= \lim_{n\to\infty}\big(\lim_{k\to\infty}\psi^{qk}(h(x))\big)\\ &= \lim_{k\to\infty}\psi^{qk}(h(x)) = \eta_h(x),\end{aligned}$$

and also for $j = 1,\dots,q-1$,

$$\begin{aligned}\lim_{n\to\infty}\psi^{qn+j}(\eta_h(x)) &= \psi^j\big(\lim_{n\to\infty}\psi^{qn}(\eta_h(x))\big)\\ &= \psi^j\big(\lim_{n\to\infty}\psi^{qn}(h(x))\big)\\ &= \lim_{n\to\infty}\psi^{qn+j}(h(x)).\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} d\big(\psi^n(\eta_h(x)),\psi^n(h(x))\big) = 0$$

where the above $d(\cdot,\cdot)$ is the metric on Y, so that we obtain that $\eta_h(x)\in Y^s(h(x))$.

(v) Assume that p is the least period of x. We will show that $q = c_1^p(x)$ is the least period of $\eta_h(x)$. Let q_0 be the least period of $\eta_h(x)$, such that $q = q_0\cdot m$ for some $m\in\mathbb{N}$ and $\psi^{q_0}(\eta_h(x)) = \eta_h(x)$. Hence, we have

$$\lim_{k\to\infty}\psi^{qk+j}\big(\psi^{q_0}(h(x))\big) = \lim_{k\to\infty}\psi^{qk+j}(h(x)),\quad j = 0,1,\dots,q-1,$$

so that

$$\lim_{n\to\infty}\psi^n(\psi^{q_0}(h(x))) = \lim_{n\to\infty}\psi^n(h(x)).$$

By Lemma 2.6, we have that $(\psi^{q_0}(h(x)),h(x))\in G_\psi^a$, and hence q_0 is a stable period of $h(x)$. As q is the least stable period of $h(x)$ by Lemma 4.4 for stably period points and $q = q_0\cdot m$, we get $m = 1$; that is, q is the least period of $\eta_h(x)$.

(vi) We will prove $c_2^q(\eta_h(x)) = p$, where $q = c_1^p(x)$. As the function c_2^q is continuous, we have

$$c_2^q(\eta_h(x)) = \lim_{k\to\infty} c_2^q\big(\psi^{qk}(h(x))\big).$$

By the cocycle property (3.2) for c_2 and Definition 3.5(v), we have

$$\begin{aligned}c_2^q\big(\psi^{qk}(h(x))\big) &= c_2^{q+qk}(h(x)) - c_2^{qk}(h(x))\\ &= c_2^{c_1^{(k+1)p}(x)}(h(x)) - c_2^{c_1^{kp}(x)}(h(x))\\ &= \big((k+1)p - d_2(\psi^{c_1^{(k+1)p}(x)}(h(x)),h(\phi^{(k+1)p}(x)))\big)\\ &\quad-\big(kp - d_2(\psi^{c_1^{kp}(x)}(h(x)),h(\phi^{kp}(x)))\big)\\ &= p - d_2\big(\psi^{(k+1)q}(h(x)),h(x)\big) + d_2(\psi^{kp}(h(x)),h(x)\big).\end{aligned}$$

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We then have

$$\lim_{n\to\infty} \psi^n\big(\psi^{qk}(h(x))\big) = \lim_{n\to\infty} \psi^n\big(h(\phi^{qk}(x))\big) = \lim_{n\to\infty} \psi^n(h(x))$$

This implies that $\psi^{qk}(h(x)) \in Y^s(h(x))$ for all $k \in \mathbb{Z}$. As $\eta_h(x) \in Y^s(h(x))$ by (iv), we have $\psi^{qk}(h(x)) \in Y^s(\eta_h(x))$ for all $k \in \mathbb{Z}$, so that there exists $k_0 \in N$ such that for all $k \geq k_0$ and $l \in \mathbb{N}$

$$d(\psi^{ql}(\psi^{qk}(h(x))), \psi^{ql}(\eta_h(x))) < \epsilon_0.$$

Hence for $j = 1, \dots, q-1$, there exists $k_j \in \mathbb{N}$ such that for all $k \geq k_j$ and $l \in \mathbb{N}$

$$d\big(\psi^{ql+j}(\psi^{qk}(h(x))), \psi^{ql+j}\big(\eta_h(x)\big)\big) < \epsilon_0.$$

We then find $K \in \mathbb{N}$ such that for all $k \geq K$ and $n \in \mathbb{N}$,

$$d\big(\psi^n(\psi^{qk}(h(x))), \psi^n(\eta_h(x))\big) < \epsilon_0.$$

This implies that $\psi^{qk}(h(x)) \in Y^s(\eta_h(x), \epsilon_0)$ for all $k \geq K$. Since

$$\lim_{k\to\infty} \psi^{(k+1)q}(h(x)) = \lim_{k\to\infty} \psi^{qk}(h(x)) = \eta_h(x),$$

by the continuity of d_2, we see that

$$\lim_{k\to\infty} d_2\big(\psi^{(k+1)q}(h(x)), h(x)\big) = \lim_{k\to\infty} d_2\big(\psi^{kp}(h(x)), h(x)\big) = d_2\big(\eta_h(x), h(x)\big),$$

thus proving $\lim_{k\to\infty} c_2^q(\psi^{qk}(h(x))) = p$. ∎

For a q-periodic point y in Y, we put

$$\eta_{h^{-1}}(y) = \lim_{m\to\infty} \phi^{c_2^q(y)\cdot m}(h^{-1}(y)).$$

The above limit exists in X by a similar manner to Lemma 4.6(ii), and $\eta_{h^{-1}}(y)$ is $c_2^q(y)$-periodic point in X.

Lemma 4.7 *For a periodic point x in X, we have*

$$\eta_{h^{-1}}(\eta_h(x)) = \phi^{-d_2(\eta_h(x),h(x))}(x). \tag{4.8}$$

Hence $\eta_{h^{-1}}(\eta_h(x))$ belongs to the periodic orbit of x under ϕ.

Proof Suppose that $\phi^p(x) = x$. Take the constants $0 < \epsilon_1 < \epsilon_0$ and $0 < \lambda_0 < 1$ for the Smale space (X, ϕ) as in Definition 2.1 and right after Definition 2.1. By using Definition 3.5(ii), we know that Lemma 4.6(iv) implies that $(\eta_h(x), h(x)) \in G_\psi^s$, so that

$$\big(\phi^{d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x))), x\big) \in G_\phi^s$$

because of Definition 3.5 (iv). Hence, for $\epsilon > 0$ with $0 < \epsilon < \epsilon_1$, there exists $n_0 \in \mathbb{N}$ such that

$$d\Big(\phi^n\big(\phi^{n_0}\big(\phi^{d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x)))\big)\big), \phi^n(\phi^{n_0}(x))\Big) < \epsilon \quad \text{for } n = 0, 1, 2, \dots$$

where the above $d(\cdot, \cdot)$ is the metric on X, and hence

$$\phi^{n_0}\big(\phi^{d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x)))\big) \in X^s(\phi^{n_0}(x), \epsilon).$$

For any $l \in \mathbb{N}$, we have by (2.1)

$$
\begin{aligned}
d\Big(\phi^l\big(\phi^{n_0+d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x)))\big), \phi^l(\phi^{n_0}(x))\Big) \\
\le \lambda_0^l d\big(\phi^{n_0+d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x)))), \phi^{n_0}(x)\big) \\
\le \lambda_0^l \cdot \epsilon,
\end{aligned}
$$

so that

$$
\lim_{n\to\infty} \phi^{pn}\big(\phi^{n_0+d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x)))\big) = \lim_{n\to\infty} \phi^{pn}(\phi^{n_0}(x)) = \phi^{n_0}(x).
$$

Since

$$
\begin{aligned}
&\lim_{n\to\infty} \phi^{pn}(\phi^{n_0+d_2(\eta_h(x),h(x))}(h^{-1}(\eta_h(x))) \\
&= \phi^{n_0+d_2(\eta_h(x),h(x))}\big(\lim_{n\to\infty} \phi^{pn}(h^{-1}(\eta_h(x)))\big) \\
&= \phi^{n_0+d_2(\eta_h(x),h(x))}\big(\eta_{h^{-1}}(\eta_h(x))\big),
\end{aligned}
$$

the equality

$$
\phi^{n_0+d_2(\eta_h(x),h(x))}\big(\eta_{h^{-1}}(\eta_h(x))\big) = \phi^{n_0}(x)
$$

holds, thus proving (4.8). ∎

We thus reach the following proposition.

Proposition 4.8 *Suppose that* $(X,\phi) \underset{\text{SCOE}}{\sim} (Y,\psi)$. *Then there exists a bijective map* $\xi_h \colon \mathrm{P}_{\mathrm{orb}}(X) \to \mathrm{P}_{\mathrm{orb}}(Y)$ *such that*

$$
|\xi_h(\gamma)| = |c_1^{|\gamma|}(x)| \quad \textit{for } \gamma \in \mathrm{P}_{\mathrm{orb}}(X) \textit{ with } \gamma = \{\phi^n(x) \mid n \in \mathbb{Z}\}.
$$

Proof For $\gamma = \{\phi^n(x) \mid n \in \mathbb{Z}\} \in \mathrm{P}_{\mathrm{orb}}(X)$, put $p = |\gamma|$ the positive least period of x. Define

$$
\xi_h(\gamma) = \big\{ \psi^n(\eta_h(x)) \mid n \in \mathbb{Z}\big\}.
$$

Since $\eta_h(x)$ is a periodic point in Y with its least period $c_1^p(x)$, $\xi_h(\gamma)$ is a periodic orbit in Y such that $|\xi_h(\gamma)| = |c_1^p(x)|$. We note the corresponding statement for h^{-1} to Lemma 4.6 (iii) holds, so that we have the equality

$$
\eta_{h^{-1}}\big(\psi^n(y)\big) = \phi^{c_2^n(y)}\big(\eta_{h^{-1}}(y)\big), \qquad n \in \mathbb{Z} \tag{4.9}
$$

for a periodic point $y \in Y$. By (4.7) and (4.9), we have

$$
\eta_{h^{-1}}\big(\psi^n(\eta_h(x))\big) = \phi^{c_2^n(\eta_h(x))}\big(\eta_{h^{-1}}(\eta_h(x))\big) = \phi^{c_2^n(\eta_h(x))-d_2(\eta_h(x)),h(x))}(x)
$$

Hence, $\eta_{h^{-1}}(\psi^n(\eta_h(x)))$ belongs to γ, so that we have $\xi_{h^{-1}}(\xi_h(\gamma)) = \gamma$. Similarly, we have $\xi_h(\xi_{h^{-1}}(\gamma')) = \gamma'$ for $\gamma' \in \mathrm{P}_{\mathrm{orb}}(Y)$. We thus conclude that the map $\xi_h \colon \mathrm{P}_{\mathrm{orb}}(X) \to \mathrm{P}_{\mathrm{orb}}(Y)$ is bijective and satisfies the desired property. ∎

The zeta function $\zeta_\phi(t)$ for the dynamical system (X,ϕ) is defined by

$$
\zeta_\phi(t) := \exp\Big\{ \sum_{n=1}^{\infty} \frac{t^n}{n} |\mathrm{Per}_n(X)|\Big\} \qquad (\textit{cf.}\ [2,13,24,35]),
$$

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where $|\mathrm{Per}_n(X)|$ means the cardinal number of the finite set $\mathrm{Per}_n(X)$. Suppose that $(X,\phi)\underset{\mathrm{ACOE}}{\sim}(Y,\psi)$. By Proposition 4.8, there is a bijective map $\xi_h\colon \mathrm{P_{orb}}(X)\to\mathrm{P_{orb}}(Y)$ such that

$$|\xi_h(\gamma)| = |c_1^{|\gamma|}(x)| \quad \text{for } \gamma\in\mathrm{P_{orb}}(X) \text{ with } \gamma=\{\phi^n(x)\mid n\in\mathbb{Z}\}.$$

We set the two kinds of dynamical zeta functions

$$\zeta_{\xi_h}(t) := \prod_{\gamma\in\mathrm{P_{orb}}(X)} (1-t^{|\xi_h(\gamma)|})^{-1},$$

$$\zeta_{\phi,c_1}(s) := \exp\Big\{\sum_{n=1}^{\infty}\frac{1}{n}\sum_{x\in\mathrm{Per}_n(X)}\exp\big(-s|c_1^n(x)|\big)\Big\} \qquad (\textit{cf.}\ [24,35]).$$

By putting $t=e^{-s}$, we see that

$$\zeta_{\xi_h}(t)=\zeta_{\phi,c_1}(s)$$

by general theory of dynamical zeta function (*cf.* [24,35]).

Theorem 4.9 *Suppose that $(X,\phi)\underset{\mathrm{SCOE}}{\sim}(Y,\psi)$. Let $h\colon X\to Y$ be a homeomorphism that gives rise to a stably continuous orbit equivalence between them. Then we have*

$$\zeta_\phi(t)=\zeta_{\xi_{h^{-1}}}(t) \quad \textit{and} \quad \zeta_\psi(t)=\zeta_{\xi_h}(t).$$

Proof There exists a bijection $\xi_h\colon \mathrm{P_{orb}}(X)\to\mathrm{P_{orb}}(Y)$ such that $|\xi_h(\gamma)|=|c_1^{|\gamma|}(x)|$ for $\gamma\in\mathrm{P_{orb}}(X)$ with $\gamma=\{\phi^n(x)\mid n\in\mathbb{Z}\}$. As ξ_h is bijective with $|\xi_h(\gamma)|=|c_1^{|\gamma|}(x)|$, it is direct to see that $\zeta_\psi(t)=\zeta_{\xi_h}(t)$, and similarly $\zeta_\phi(t)=\zeta_{\xi_{h^{-1}}}(t)$. ∎

We remark that a similar statement for UCOE holds.

5 Asymptotic Ruelle Algebras $\mathcal{R}_\phi^a$ with Dual Actions

Let us recall the construction of the groupoid C^*-algebras from étale groupoids ([30]). Let G be an étale groupoid with range map $r\colon G\to G^\circ$ and source map $s\colon G\to G^\circ$ from G to the unit space G° of G. In [30], "r-discrete" was used instead of "étale".

The (reduced) groupoid C^*-algebra $C_r^*(G)$ for an étale groupoid G is defined in the following way ([30]). Let $C_c(G)$ be the set of all continuous functions on G with compact support that has a natural product structure of $*$-algebra given by

$$(f*g)(u)=\sum_{r(t)=r(u)} f(t)g(t^{-1}u)=\sum_{u=t_1t_2} f(t_1)g(t_2),$$

$$f^*(u)=\overline{f(u^{-1})}, \qquad f,g\in C_c(G), \quad u\in G.$$

Let $C_0(G^\circ)$ be the C^*-algebra of all continuous functions on the unit space G° that vanish at infinity. The algebra $C_c(G)$ is a $C_0(G^\circ)$-right module, endowed with a $C_0(G^\circ)$-valued inner product given by

$$(\xi f)(t)=\xi(t)f(s(t)), \qquad \xi\in C_c(G),\ f\in C_0(G^\circ),\ t\in G,$$

$$\langle\xi,\eta\rangle(x)=\sum_{x=s(t)}\overline{\xi(t)}\eta(t), \qquad \xi,\eta\in C_c(G),\ x\in G^\circ.$$

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Let us denote by $l^2(G)$ the completion of the inner product $C_0(G^\circ)$-module $C_c(G)$. It is a Hilbert C^*-right module over the commutative C^*-algebra $C_0(G^\circ)$. We denote by $B(l^2(G))$ the C^*-algebra of all bounded adjointable $C_0(G^\circ)$-module maps on $l^2(G)$. Let π be the $*$-homomorphism of $C_c(G)$ into $B(l^2(G))$ defined by $\pi(f)\xi = f * \xi$ for $f, \xi \in C_c(G)$. Then the closure of $\pi(C_c(G))$ in $B(l^2(G))$ is called the (reduced) C^*-algebra of the groupoid G, which we denote by $C_r^*(G)$. If we endow $C_c(G)$ with the universal C^*-norm, its completion is called the the (full) C^*-algebra of the groupoid G, which we denote by $C^*(G)$. By a general theory of groupoid C^*-algebras, $C_r^*(G)$ is canonically isomorphic to $C^*(G)$ if the groupoid is amenable ([30]). An étale groupoid G is said to be essentially principal if the interior of $G' = \{\gamma \in G \mid s(\gamma) = r(\gamma)\}$ is G° ([31, Definition 3.1]). By Renault [30, Proposition 4.7], [31, Proposition 4.2], $C_0(G^\circ)$ is maximal abelian in $C_r^*(G)$ if and only if G is essentially principal.

Definition 5.1 A Smale space (X, ϕ) is said to be *asymptotically essentially free* if the interior of the set of n-asymptotic periodic points $\{x \in X \mid (\phi^n(x), x) \in G_\phi^a\}$ is empty for every $n \in \mathbb{Z}$ with $n \neq 0$.

We always assume that the space X is infinite. Recall that a Smale space (X, ϕ) is said to be irreducible if for every ordered pair of open sets $U, V \subset X$, there exists $K \in \mathbb{N}$ such that $\phi^K(U) \cap V \neq \varnothing$. It is equivalent to the condition that for every ordered pair of open sets $U, V \subset X$, there exists $K \in \mathbb{N}$ such that $\phi^{-K}(U) \cap V \neq \varnothing$. The referee kindly showed to the author the following lemma with its proof below. The author deeply thanks the referee.

Lemma 5.2 *If a Smale space (X, ϕ) is irreducible and X is infinite, then (X, ϕ) is asymptotically essentially free.*

Proof Let $U_n, n \in \mathbb{N}$ be a countable base of open sets of the topology of X. Since (X, ϕ) is irreducible, the set $\bigcup_{n=0}^\infty \phi^{-n}(U_m)$ is dense in X for every $m \in \mathbb{N}$. By Baire's category theorem, $\bigcap_{m=1}^\infty \bigcup_{n=0}^\infty \phi^{-n}(U_m)$ is dense in X. The set $\bigcap_{m=1}^\infty \bigcup_{n=0}^\infty \phi^{-n}(U_m)$ coincides with the set of points whose forward orbit is dense in X. Now suppose that for a fixed $n \neq 0$, the interior of the set of n-asymptotic periodic points $\{x \in X \mid (\phi^n(x), x) \in G_\phi^a\}$ contains a non-empty open set U. There exists a point $x \in U$ such that the forward orbit of x is dense in X. Since $(\phi^n(x), x) \in G_\phi^a$, we have

$$\lim_{m\to\infty} d\big(\phi^m(\phi^n(x)), \phi^m(x)\big) = 0,$$

so that $\phi^n(x) \in X^s(x)$. By [29, Lemma 5.3], there exists $\lim_{k\to\infty} \phi^{kn}(x)$, denoted by y, in the set of n-periodic points $\mathrm{Per}_n(X)$. We note that although [29, Lemma 5.3] is considering only mixing Smale spaces, the assertion [29, Lemma 5.3] holds in the irreducible Smale space with X being infinite. Since X is infinite, one can find a point $z \notin \{y, \phi(y), \dots, \phi^{n-1}(y)\}$. Put $\epsilon = \frac{1}{4}\mathrm{Min}\{d(z, \phi^i(y)) \mid i = 0, 1, \dots, n-1\}$. Let us denote by $N_\epsilon(z)$ the ϵ-neighborhood of z of open ball. We put $V = \bigcup_{i=0}^{n-1} N_\epsilon(\phi^i(y))$, so that we have $V \cap N_\epsilon(z) = \varnothing$. Since X is compact, there exists $\delta > 0$ such that for all $w_1, w_2 \in X$, $d(w_1, w_2) < \delta$ implies $d(\phi^j(w_1), \phi^j(w_2)) < \epsilon$ for all $j = 0, 1, \dots, n-1$.

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In particular, for $j = 0$, we have $\delta < \epsilon$. Since $\lim_{k\to\infty} \phi^{kn}(x) = y$, there exists $K \in \mathbb{N}$ such that $d(\phi^{kn}(x), y) < \delta$ for all $k \geq K$. Hence, we have

$$d\big(\phi^j(\phi^{kn}(x)), \phi^j(y)\big) < \epsilon \quad \text{for all } j = 0, 1, \ldots, n-1, \quad k \geq K,$$

so that $\phi^{kn+j}(x) \in N_\epsilon(\phi^j(y))$ for all $j = 0, 1, \ldots, n-1$, $k \geq K$. Hence we have $\phi^m(x) \in V$ for all $m \geq K \cdot n$. This contradicts the condition that the forward orbit of x is dense in X. We thus conclude that the interior of the set $\{x \in X \mid (\phi^n(x), x) \in G_\phi^a\}$ is empty. ∎

Lemma 5.3 *A Smale space (X, ϕ) is asymptotically essentially free if and only if the groupoid $G_\phi^a \rtimes \mathbb{Z}$ is essentially principal.*

Proof As we have

$$\begin{aligned}(G_\phi^a \rtimes \mathbb{Z})' &= \bigcup_{n\in\mathbb{Z}} \big\{(x, n, z) \in G_\phi^a \rtimes \mathbb{Z} \mid x = z\big\}\\ &= \bigcup_{n\in\mathbb{Z}} \big\{(x, n, x) \in X \times \mathbb{Z} \times X \mid (\phi^n(x), x) \in G_\phi^a\big\},\end{aligned}$$

the interior $\mathrm{int}((G_\phi^a \rtimes \mathbb{Z})')$ of $(G_\phi^a \rtimes \mathbb{Z})'$ is

$$\mathrm{int}\big((G_\phi^a \rtimes \mathbb{Z})'\big) = \bigcup_{n\in\mathbb{Z}} \mathrm{int}\big(\{(x, n, x) \in X \times \mathbb{Z} \times X \mid (\phi^n(x), x) \in G_\phi^a\}\big).$$

For $n = 0$, we see that

$$\mathrm{int}\big(\{(x, 0, x) \in X \times \mathbb{Z} \times X \mid (x, x) \in G_\phi^a\}\big) = (G_\phi^a \rtimes \mathbb{Z})^\circ = X.$$

Hence , $\mathrm{int}((G_\phi^a \rtimes \mathbb{Z})') = (G_\phi^a \rtimes \mathbb{Z})^\circ$ if and only if

$$\mathrm{int}(\{(x, n, x) \in X \times \mathbb{Z} \times X \mid (\phi^n(x), x) \in G_\phi^a\})$$

is empty for all $n \in \mathbb{Z}$ except $n = 0$. This implies that (X, ϕ) is asymptotically essentially free if and only if $G_\phi^a \rtimes \mathbb{Z}$ is essentially principal. ∎

The following proposition as well as Lemma 5.5 is well known to experts through [25, Theorem 3.1]. The proof is also direct from Renault's result [30, Proposition 4.6].

Proposition 5.4 *If a Smale space (X, ϕ) is irreducible, then the groupoid C^*-algebra $C_r^*(G_\phi^a \rtimes \mathbb{Z})$ is simple.*

Lemma 5.5 (*cf.* [29, Theorem 1.1]) *The groupoids G_ϕ^a and $G_\phi^a \rtimes \mathbb{Z}$ are both amenable.*

By Lemma 5.5, the full groupoid C^*-algebras $C^*(G_\phi^a)$, $C^*(G_\phi^a \rtimes \mathbb{Z})$ and the reduced groupoid C^*-algebras $C_r^*(G_\phi^a)$, $C_r^*(G_\phi^a \rtimes \mathbb{Z})$ are canonically isomorphic respectively. We do not distinguish them and write them $C^*(G_\phi^a)$, $C^*(G_\phi^a \rtimes \mathbb{Z})$, respectively.

For an irreducible Smale space (X, ϕ), the asymptotic Ruelle algebra $\mathcal{R}_\phi^a$ is defined as the groupoid C^*-algebras $C^*(G_\phi^a \rtimes \mathbb{Z})$ of the étale groupoid $G_\phi^a \rtimes \mathbb{Z}$. The algebra was written R_a in Putnam's paper [25]. In this paper, we denote it by $\mathcal{R}_\phi^a$. As in [25, 29], the groupoid $G_\phi^a \rtimes \mathbb{Z}$ is the semidirect product of the groupoid G_ϕ^a by the integer group $\mathbb{Z}$, one knows that the algebra $\mathcal{R}_\phi^a$ is naturally isomorphic to the crossed product C^*-algebra $C^*(G_\phi^a) \rtimes \mathbb{Z}$ of the groupoid C^*-algebra $C^*(G_\phi^a)$ by $\mathbb{Z}$.

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In the construction of the groupoid C^*-algebra $C^*(G_\phi^a \rtimes \mathbb{Z})$, we first define the unitary group U_t^ϕ for $t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ on $l^2(G_\phi^a \rtimes \mathbb{Z})$ by setting

$$[U_t^\phi \xi](x,n,z) = \exp(2\pi\sqrt{-1}nt)\xi(x,n,z) \tag{5.1}$$

for $\xi \in l^2(G_\phi^a \rtimes \mathbb{Z})$, $(x,n,z) \in G_\phi^a \rtimes \mathbb{Z}$. The automorphisms $\mathrm{Ad}(U_t^\phi)$, $t \in \mathbb{T}$ on $B(l^2(G_\phi^a \rtimes \mathbb{Z}))$ leave $\mathcal{R}_\phi^a$ globally invariant, and yield an action of $\mathbb{T}$ on $\mathcal{R}_\phi^a$. Let us denote by ρ_t^ϕ the action $\mathrm{Ad}(U_t^\phi)$, $t \in \mathbb{T}$ on $\mathcal{R}_\phi^a$. It is direct to see that the action is exactly corresponds to the dual action of the crossed product $C^*(G_\phi^a) \rtimes \mathbb{Z}$.

A continuous function $f\colon G_\phi^a \rtimes \mathbb{Z} \to \mathbb{Z}$ is called a continuous homomorphism if it satisfies

$$f(\gamma_1\gamma_2) = f(\gamma_1) + f(\gamma_2) \qquad \text{for } \gamma_1,\gamma_2 \in G_\phi^a \rtimes \mathbb{Z}.$$

It defines a one-parameter unitary group $U_t(f)$, $t \in \mathbb{T}$ on $l^2(G_\phi^a \rtimes \mathbb{Z})$ by setting

$$[U_t(f)\xi](x,n,z) = \exp(2\pi\sqrt{-1}f(x,n,z)t)\xi(x,n,z)$$

for $\xi \in l^2(G_\phi^a \rtimes \mathbb{Z})$, $(x,n,z) \in G_\phi^a \rtimes \mathbb{Z}$. In particular, for the continuous homomorphism $d_\phi(x,n,z) = n$, we have $U_t(d_\phi) = U_t^\phi$ by (5.1).

Now suppose that $(X,\phi) \underset{\mathrm{ACOE}}{\sim} (Y,\psi)$. Let $\varphi_h\colon G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ be the isomorphism of the étale groupoids and let $h\colon X \to Y$ be the homeomorphism that gives rise to the asymptotic continuous orbit equivalence between them. We define two unitaries

$$V_h\colon l^2(G_\psi^a \rtimes \mathbb{Z}) \longrightarrow l^2(G_\phi^a \rtimes \mathbb{Z}) \quad \text{and} \quad V_{h^{-1}}\colon l^2(G_\phi^a \rtimes \mathbb{Z}) \longrightarrow l^2(G_\psi^a \rtimes \mathbb{Z}),$$

by setting

$$\begin{aligned} [V_h\zeta](x,n,z) &= \zeta\big(\varphi_h(x,n,z)\big), \qquad \zeta \in l^2(G_\psi^a \rtimes \mathbb{Z}),\ (x,n,z) \in G_\phi^a \rtimes \mathbb{Z},\\ [V_{h^{-1}}\xi](y,m,w) &= \xi\big(\varphi_{h^{-1}}(y,m,w)\big), \quad \xi \in l^2(G_\phi^a \rtimes \mathbb{Z}),\ (y,m,w) \in G_\psi^a \rtimes \mathbb{Z}. \end{aligned} \tag{5.2}$$

Since the unit space $(G_\phi^a \rtimes \mathbb{Z})^\circ$ is identified with the original space X through the correspondence $(x,0,x) \in (G_\phi^a \rtimes \mathbb{Z})^\circ \to x \in X$ and $(G_\phi^a \rtimes \mathbb{Z})^\circ$ is clopen in $G_\phi^a \rtimes \mathbb{Z}$, we regard $C(X)$ as a subalgebra of $\mathcal{R}_\phi^a$. Similarly, $C(Y)$ is regarded as a subalgebra of $\mathcal{R}_\psi^a$.

Proposition 5.6 *Suppose that $(X,\phi) \underset{\mathrm{ACOE}}{\sim} (Y,\psi)$, and keep the above notation. Let $\varphi_h\colon G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ be the isomorphism of the étale groupoids giving rise to $(X,\phi) \underset{\mathrm{ACOE}}{\sim} (Y,\psi)$. Let $f\colon G_\phi^a \rtimes \mathbb{Z} \to \mathbb{Z}$ and $g\colon G_\psi^a \rtimes \mathbb{Z} \to \mathbb{Z}$ be continuous homomorphisms satisfying $f = g \circ \varphi_h$. Then there exists an isomorphism $\Phi\colon \mathcal{R}_\phi^a \to \mathcal{R}_\psi^a$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ and*

$$\Phi \circ \mathrm{Ad}\big(U_t(f)\big) = \mathrm{Ad}\big(U_t(g)\big) \circ \Phi, \qquad \text{for } t \in \mathbb{T}. \tag{5.3}$$

Proof We set $\Phi = \mathrm{Ad}(V_{h^{-1}})$. It satisfies $\Phi(C_c(G_\phi^a \rtimes \mathbb{Z})) = C_c(G_\psi^a \rtimes \mathbb{Z})$, and hence $\Phi(\mathcal{R}_\phi^a) = \mathcal{R}_\psi^a$, and $\Phi(C(X)) = C(Y)$. For $\zeta \in l^2(G_\psi^a \rtimes \mathbb{Z})$, $(y,m,w) \in G_\psi^a \rtimes \mathbb{Z}$ and

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$a \in C_c(G_\phi^a \rtimes \mathbb{Z})$, we have the following equalities:

$$\begin{aligned}
&\big[(\Phi \circ \mathrm{Ad}(U_t(f)))(a)\zeta\big](y,m,w)\\
&\quad= [V_{h^{-1}} U_t(f) a U_t(f)^* V_h \zeta](y,m,w)\\
&\quad= [U_t(f) a U_t(f)^* V_h \zeta](\varphi_h^{-1}(y,m,w))\\
&\quad= \exp(2\pi\sqrt{-1} f(\varphi_h^{-1}(y,m,w))t)[a * (U_t(f)^* V_h \zeta)](\varphi_h^{-1}(y,m,w))\\
&\quad= \exp\big(2\pi\sqrt{-1} f(\varphi_h^{-1}(y,m,w))t\big)\\
&\qquad\cdot\Big(\sum_{r(\gamma)=r(\varphi_h^{-1}(y,m,w))} a(\gamma)(U_t(f)^* V_h\zeta)(\gamma^{-1}\varphi_h^{-1}(y,m,w))\Big)\\
&\quad= \exp\big(2\pi\sqrt{-1} f(\varphi_h^{-1}(y,m,w))t\big)\\
&\qquad\cdot\Big(\sum_{r(\gamma)=h^{-1}(y)} a(\gamma)\exp(-2\pi\sqrt{-1} f(\gamma^{-1}\varphi_h^{-1}(y,m,w))t)\\
&\qquad\quad\cdot(V_h\zeta)(\gamma^{-1}\varphi_h^{-1}(y,m,w))\Big)\\
&\quad= \exp\big(2\pi\sqrt{-1} f(\varphi_h^{-1}(y,m,w))t\big)\\
&\qquad\cdot\Big(\sum_{r(\gamma)=h^{-1}(y)} a(\gamma)\exp\big(-2\pi\sqrt{-1}(f(\gamma^{-1}) + f(\varphi_h^{-1}(y,m,w)))t\big)\\
&\qquad\quad\cdot(V_h\zeta)(\gamma^{-1}\varphi_h^{-1}(y,m,w))\Big)\\
&\quad= \sum_{r(\gamma)=h^{-1}(y)} a(\gamma)\exp\big(-2\pi\sqrt{-1} f(\gamma^{-1})t\big)\zeta\big(\varphi_h(\gamma^{-1})\cdot(y,m,w)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\big[(\mathrm{Ad}(U_t(g))\circ\Phi)(a)\zeta\big](y,m,w)\\
&\quad= [U_t(g) V_{h^{-1}} a V_h U_t(g)^*\zeta](y,m,w)\\
&\quad= \exp(2\pi\sqrt{-1} g(y,m,w)t)[V_{h^{-1}} a V_h U_t(g)^*\zeta](y,m,w)\\
&\quad= \exp(2\pi\sqrt{-1} g(y,m,w)t)[a V_h U_t(g)^*\zeta](\varphi_h^{-1}(y,m,w))\\
&\quad= \exp(2\pi\sqrt{-1} g(y,m,w)t)\\
&\qquad\cdot\Big(\sum_{r(\gamma)=r(\varphi_h^{-1}(y,m,w))} a(\gamma)(V_h U_t(g)^*\zeta)\big(\gamma^{-1}\varphi_h^{-1}(y,m,w)\big)\Big)\\
&\quad= \exp(2\pi\sqrt{-1} g(y,m,w)t)\\
&\qquad\cdot\Big(\sum_{r(\gamma)=h^{-1}(y)} a(\gamma)(U_t(g)^*\zeta)\big(\varphi_h(\gamma^{-1}\varphi_h^{-1}(y,m,w))\big)\Big)\\
&\quad= \exp(2\pi\sqrt{-1} g(y,m,w)t)\\
&\qquad\cdot\Big(\sum_{r(\gamma)=h^{-1}(y)} a(\gamma)\exp\big(-2\pi\sqrt{-1} g(\varphi_h(\gamma^{-1})\cdot(y,m,w))t\big)\\
&\qquad\quad\cdot\zeta\big(\varphi_h(\gamma^{-1})\cdot(y,m,w)\big)\Big)
\end{aligned}$$

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$$= \sum_{r(\gamma)=h^{-1}(y)} a(\gamma) \exp\big(-2\pi\sqrt{-1} g(\varphi_h(\gamma^{-1}))t\big)\,\zeta\big(\varphi_h(\gamma^{-1})\cdot(y,m,w)\big).$$

By assumption, we see that $f(\gamma^{-1}) = g(\varphi_h(\gamma^{-1}))$, so that we obtain $\Phi\circ \mathrm{Ad}(U_t(f)) = \mathrm{Ad}(U_t(g))\circ\Phi$. ■

We assume that $(X,\phi) \underset{\mathrm{ACOE}}{\sim} (Y,\psi)$. Let $h\colon X \to Y$ be a homeomorphism that gives rise to the asymptotic continuous orbit equivalence between them. Take the continuous functions $c_1\colon X\to\mathbb{Z}$, $c_2\colon Y\to\mathbb{Z}$ and two-cocycle functions $d_1\colon G^a_\phi \to \mathbb{Z}$, $d_2\colon G^a_\psi\to\mathbb{Z}$ satisfying Definition 3.2 of asymptotic continuous orbit equivalence. We set two functions

$$\text{(5.4)}\qquad \begin{aligned} c_\phi(x,n,z) &= c_1^n(x) + d_1(\phi^n(x),z), && (x,n,z)\in G^a_\phi \rtimes\mathbb{Z},\\ c_\psi(y,m,w) &= c_2^m(y) + d_2(\psi^m(y),w), && (y,m,w)\in G^a_\psi\rtimes\mathbb{Z}.\end{aligned}$$

They satisfy

$$\begin{aligned} c_\phi(\gamma_1\gamma_2) &= c_\phi(\gamma_1)+c_\phi(\gamma_2) && \text{for } \gamma_1,\gamma_2\in G^a_\phi\rtimes\mathbb{Z},\\ c_\psi(\gamma_1'\gamma_2') &= c_\psi(\gamma_1')+c_\psi(\gamma_2') && \text{for } \gamma_1',\gamma_2'\in G^a_\psi\rtimes\mathbb{Z},\end{aligned}$$

and hence they are continuous homomorphisms satisfying

$$\begin{aligned} \varphi_h(x,n,z) &= \big(h(x), c_\phi(x,n,z), h(z)\big), && (x,n,z)\in G^a_\phi\rtimes\mathbb{Z},\\ \varphi_{h^{-1}}(y,m,w) &= \big(h^{-1}(y), c_\psi(y,m,w), h^{-1}(w)\big), && (y,m,w)\in G^a_\psi\rtimes\mathbb{Z}.\end{aligned}$$

We note that the following identities hold:

$$\text{(5.5)}\qquad \begin{aligned} d_\psi\big(\varphi_h(x,n,z)\big) &= c_\phi(x,n,z), & d_\phi\big(\varphi_h^{-1}(y,m,w)\big) &= c_\psi(y,m,w),\\ c_\psi\big(\varphi_h(x,n,z)\big) &= d_\phi(x,n,z) = n, & c_\phi\big(\varphi_h^{-1}(y,m,w)\big) &= d_\psi(y,m,w) = m.\end{aligned}$$

Theorem 5.7 *Suppose that Smale spaces (X,ϕ) and (Y,ψ) are irreducible. Then the following assertions are equivalent:*

- (i) *(X,ϕ) and (Y,ψ) are asymptotically continuous orbit equivalent.*
- (ii) *There exists an isomorphism $\mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ and*

  $$\Phi\circ\rho^\phi_t = \mathrm{Ad}(U_t(c_\psi))\circ\Phi,\qquad \Phi\circ\mathrm{Ad}(U_t(c_\phi)) = \rho^\psi_t\circ\Phi\qquad \textit{for } t\in\mathbb{T}$$

  for some continuous homomorphisms $c_\phi\colon G^a_\phi\rtimes\mathbb{Z}\to\mathbb{Z}$ and $c_\psi\colon G^a_\psi\rtimes\mathbb{Z}\to\mathbb{Z}$.

Proof (i) ⇒ (ii): Take $f = d_\phi, g = c_\psi$ in equality (5.3). We then have $\mathrm{Ad}(U_t(d_\phi)) = \rho^\phi_t$ and $c_\psi\circ\varphi_h = d_\phi$ by (5.5). Hence by (5.3), we obtain

$$\text{(5.6)}\qquad \Phi\circ\rho^\phi_t = \mathrm{Ad}(U_t(c_\psi))\circ\Phi,\qquad t\in\mathbb{T}.$$

Take $f = c_\phi, g = d_\psi$ in equality (5.3). We then have $\mathrm{Ad}(U_t(d_\psi)) = \rho^\psi_t$ and $c_\phi\circ(\varphi_h)^{-1} = d_\psi$ by (5.5). Hence by (5.3), we obtain

$$\text{(5.7)}\qquad \Phi\circ\mathrm{Ad}(U_t(c_\phi)) = \rho^\psi_t\circ\Phi,\qquad t\in\mathbb{T}.$$

(ii) ⇒ (i): Since the Smale spaces (X,ϕ) and (Y,ψ) are both asymptotically essentially free, the étale groupoids $G_\phi^a \rtimes \mathbb{Z}$ and $G_\psi^a \rtimes \mathbb{Z}$ are both essentially principal by Lemma 5.3. By Renault [31, Proposition 4.11], an isomorphism $\mathcal{R}_\phi^a \to \mathcal{R}_\psi^a$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ yields an isomorphism of the underlying étale groupoids $G_\phi^a \rtimes \mathbb{Z}$ and $G_\psi^a \rtimes \mathbb{Z}$. Hence by Theorem 3.4, we see the implication (ii) ⇒ (i). ■

Remark 5.8 Similar discussions to Theorem 5.7 for topological Markov shifts with continuous orbit equivalence are seen in several papers (*cf.* [5, 6, 14–17, 21], etc.).

6 Asymptotic Conjugacy

In this section, we will introduce a notion of asymptotic conjugacy between Smale spaces and describe the asymptotic conjugacy in terms of the Ruelle algebras with its dual actions. Smale spaces (X,ϕ) in this section are assumed to be irreducible and the space X to be infinite.

Definition 6.1 Smale spaces (X,ϕ) and (Y,ψ) are said to be *asymptotically conjugate* if they are asymptotically continuously orbit equivalent such that we can take their cocycle functions such as $c_1 \equiv 1$, $c_2 \equiv 1$ and $d_1 \equiv 0$, $d_2 \equiv 0$ in Definition 3.2.

In this situation, we write $(X,\phi) \underset{a}{\cong} (Y,\psi)$. Namely, we have (X,ϕ) and (Y,ψ) are said to be asymptotically conjugate if and only if there exists a homeomorphism $h\colon X \to Y$ that satisfies the following four conditions:

(a) There exists a continuous function $k_{1,n}\colon X \to \mathbb{Z}_+$ for each $n \in \mathbb{Z}$ such that

$$\big(\psi^{k_{1,n}(x)+n}(h(x)), \psi^{k_{1,n}(x)}(h(\phi^n(x)))\big) \in G_\psi^{s,0},$$
$$\big(\psi^{-k_{1,n}(x)+n}(h(x)), \psi^{-k_{1,n}(x)}(h(\phi^n(x)))\big) \in G_\psi^{u,0}.$$

(b) There exists a continuous function $k_{2,n}\colon Y \to \mathbb{Z}_+$ for each $n \in \mathbb{Z}$ such that

$$\big(\phi^{k_{2,n}(y)+n}(h^{-1}(y)), \phi^{k_{2,n}(y)}(h^{-1}(\psi^n(y)))\big) \in G_\phi^{s,0},$$
$$\big(\phi^{-k_{2,n}(y)+n}(h^{-1}(y)), \phi^{-k_{2,n}(y)}(h^{-1}(\psi^n(y)))\big) \in G_\phi^{u,0}.$$

(c) There exists a continuous function $m_1\colon G_\phi^a \to \mathbb{Z}_+$ such that

$$\big(\psi^{m_1(x,z)}(h(x)), \psi^{m_1(x,z)}(h(z))\big) \in G_\psi^{s,0} \qquad \text{for } (x,z) \in G_\phi^a,$$
$$\big(\psi^{-m_1(x,z)}(h(x)), \psi^{-m_1(x,z)}(h(z))\big) \in G_\psi^{u,0} \qquad \text{for } (x,z) \in G_\phi^a.$$

(d) There exists a continuous function $m_2\colon G_\psi^a \to \mathbb{Z}_+$ such that

$$\big(\phi^{m_2(y,w)}(h^{-1}(y)), \phi^{m_2(y,w)}(h^{-1}(w))\big) \in G_\phi^{s,0} \qquad \text{for } (y,w) \in G_\psi^a,$$
$$\big(\phi^{-m_2(y,w)}(h^{-1}(y)), \phi^{-m_2(y,w)}(h^{-1}(w))\big) \in G_\phi^{u,0} \qquad \text{for } (y,w) \in G_\psi^a.$$

Recall that the Ruelle algebra $\mathcal{R}_\phi^a$ is defined as the groupoid C^*-algebra $C^*(G_\phi^a \rtimes \mathbb{Z})$ of the étale groupoid $G_\phi^a \rtimes \mathbb{Z}$. It is naturally isomorphic to the crossed product $C^*(G_\phi^a) \rtimes$

$\mathbb{Z}$ of the C^*-algebra $C^*(G_\phi^a)$ by the automorphism ϕ^* on $C^*(G_\phi^a)$ induced by the formula

$$\phi^*(f)(x,z) = f\big(\phi(x),\phi(z)\big) \qquad \text{for } f \in C_c(G_\phi^a),\ (x,z) \in G_\phi^a.$$

Define the unitary U_ϕ on $l^2(G_\phi^a \rtimes \mathbb{Z})$ by setting

$$(6.1)\quad (U_\phi \xi)(x,n,z) = \xi(\phi(x), n-1, z) \qquad \text{for } \xi \in l^2(G_\phi^a \rtimes \mathbb{Z}),\ (x,n,z) \in G_\phi^a \rtimes \mathbb{Z}.$$

It is direct to see that

$$U_\phi f U_\phi^* = \phi^*(f) \qquad \text{for } f \in C_c(G_\phi^a),$$

where

$$f(x,m,z) = \begin{cases} f(x,z) & \text{if } m = 0, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } (x,z) \in G_\phi^a.$$

Now we assume that (X,ϕ) is irreducible, so that the C^*-algebra $\mathcal{R}_\phi^a$ is simple by Proposition 5.4. Hence we know that $\mathcal{R}_\phi^a$ is isomorphic to the C^*-algebra $C^*(C^*(G_\phi^a), U_\phi)$ generated by the its subalgebra $C^*(G_\phi^a)$ and the unitary U_ϕ. The following lemma follows directly from J. Renault's result [31, Proposition 4.11].

Lemma 6.2 *Let (X,ϕ) and (Y,ψ) be irreducible Smale spaces. The following assertions are equivalent:*

- (i) *There exists an isomorphism $\varphi\colon G_\phi^a \rtimes \mathbb{Z} \to G_\psi^a \rtimes \mathbb{Z}$ of étale groupoids such that $\varphi(G_\phi^a) = G_\psi^a$ and $\varphi(G_\phi^{a,0}) = G_\psi^{a,0}$;*
- (ii) *there exists an isomorphism $\Phi\colon \mathcal{R}_\phi^a \to \mathcal{R}_\psi^a$ of C^*-algebras such that $\Phi(C^*(G_\phi^a)) = C^*(G_\psi^a)$ and $\Phi(C(X)) = C(Y)$.*

Proof By Lemma 2.5, the spaces $G_\phi^{a,0}, G_\psi^{a,0}$ are identified with X, Y respectively as topological spaces. They are also identified with the unit spaces $(G_\phi^a \rtimes \mathbb{Z})^\circ, (G_\psi^a \rtimes \mathbb{Z})^\circ$, respectively. Since (X,ϕ) and (Y,ψ) are irreducible and hence asymptotically essentially free, the étale groupoids $G_\phi^a \rtimes \mathbb{Z}$ and $G_\psi^a \rtimes \mathbb{Z}$ are both essentially principal by Lemma 5.3. The implication (i) ⇒ (ii) is direct. By Renault [31, Proposition 4.11], an isomorphism $\mathcal{R}_\phi^a \to \mathcal{R}_\psi^a$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ yields an isomorphism φ of the underlying étale groupoids $G_\phi^a \rtimes \mathbb{Z}$ and $G_\psi^a \rtimes \mathbb{Z}$. By the construction of the isomorphism φ of the étale groupoids, we see that $\varphi(G_\phi^a) = G_\psi^a$ by the additional condition $\Phi(C^*(G_\phi^a)) = C^*(G_\psi^a)$, thus proving the implication (ii) ⇒ (i). ■

Proposition 6.3 *Let (X,ϕ) and (Y,ψ) be irreducible Smale spaces. Suppose that there exists an isomorphism $\Phi\colon \mathcal{R}_\phi^a \to \mathcal{R}_\psi^a$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ and*

$$\Phi \circ \rho_t^\phi = \rho_t^\psi \circ \Phi, \qquad t \in \mathbb{T}.$$

Then there exists a homeomorphism $h\colon X \to Y$ that gives rise to an asymptotic continuous orbit equivalence between (X,ϕ) and (Y,ψ) such that its cocycle functions satisfy

$$c_1 \equiv 1, \qquad c_2 \equiv 1, \qquad d_1 \equiv 0, \qquad d_2 \equiv 0.$$

Namely, (X,ϕ) and (Y,ψ) are asymptotically conjugate.

Proof Suppose that there exists an isomorphism $\Phi: \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ of C^*-algebras such that $\Phi(C(X)) = C(Y)$ and $\Phi \circ \rho^\phi_t = \rho^\psi_t \circ \Phi$, $t \in \mathbb{T}$. We will first show that $d_1 \equiv 0$, $d_2 \equiv 0$. Since the fixed point algebra $(\mathcal{R}^a_\phi)^{\rho^\phi}$ of $\mathcal{R}^a_\phi$ under ρ^ϕ is canonically isomorphic to the groupoid C^*-subalgebra $C^*(G^a_\phi)$, the isomorphism $\Phi: \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ satisfies $\Phi(C^*(G^a_\phi)) = C^*(G^a_\psi)$. By Lemma 6.2, we then find a homeomorphism $h: X \to Y$ and a groupoid isomorphism $\varphi_h: G^a_\phi \rtimes \mathbb{Z} \to G^a_\psi \rtimes \mathbb{Z}$ such that $\varphi_h(G^a_\phi) = G^a_\psi$, $\varphi_h|_{G^{a,0}_\phi} = h$ and $\Phi(f) = f \circ h^{-1}$ for $f \in C(X)$. For $(x,z) \in G^a_\phi$, we have

$$\varphi_h(x,0,z) = \big(h(x), c_\phi(x,0,z), h(z)\big) = \big(h(x), d_1(x,z), h(z)\big).$$

As $\varphi_h(x,0,z) \in G^a_\psi$, we know that $d_1(x,z) = 0$, and $d_2(y,w) = 0$ for $(y,w) \in G^a_\psi$.

We will next show that $c_1 \equiv 1$, $c_2 \equiv 1$. Since the isomorphism $\Phi: \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ satisfies $\Phi(C(X)) = C(Y)$, the groupoid isomorphism $\varphi_h: G^a_\phi \rtimes \mathbb{Z} \to G^a_\psi \rtimes \mathbb{Z}$ with homeomorphism $h: X \to Y$ yields an asymptotic continuous orbit equivalence between them. They also satisfy the equalities

$$\text{(6.2)} \quad \varphi_h(x,n,z) = \big(h(x), c_\phi(x,n,z), h(z)\big), \qquad (x,n,z) \in G^a_\phi \rtimes \mathbb{Z},$$
$$\varphi_{h^{-1}}(y,m,w) = \big(h^{-1}(y), c_\psi(y,m,w), h^{-1}(w)\big), \qquad (y,m,w) \in G^a_\psi \rtimes \mathbb{Z}.$$

Let V_h be the unitary defined in (5.2). As in the proof of Proposition 5.6, by putting $\Phi_h = \mathrm{Ad}(V_{h^{-1}})$, we have an isomorphism $\Phi_h: \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ such that $\Phi_h(C(X)) = C(Y)$ and

$$\Phi_h \circ \rho^\phi_t = \mathrm{Ad}(U_t(c_\psi)) \circ \Phi_h, \qquad \Phi_h \circ \mathrm{Ad}(U_t(c_\phi)) = \rho^\psi_t \circ \Phi_h.$$

Let U_ϕ be the unitary defined in (6.1), which corresponds to the implementing unitary of the positive generator of the group representation of $\mathbb{Z}$ in the crossed product $C^*(G^a_\phi) \rtimes \mathbb{Z}$. It satisfies the equality $U_\phi f U^*_\phi = f \circ \phi$ for $f \in C(X)$. For $f \in C(X)$, as $\Phi(f) = \Phi_h(f)$, we see that

$$\Phi(U_\phi)\Phi_h(f)\Phi(U^*_\phi) = \Phi(f \circ \phi) = \Phi_h(f \circ \phi) = \Phi_h(U_\phi)\Phi_h(f)\Phi_h(U^*_\phi),$$

so that

$$\Phi_h^{-1}(\Phi(U_\phi)) f \Phi_h^{-1}(\Phi(U^*_\phi)) = U_\phi f U^*_\phi.$$

Hence, we have

$$U^*_\phi \Phi_h^{-1}(\Phi(U_\phi)) f = f U^*_\phi \Phi_h^{-1}(\Phi(U_\phi)) \quad \text{for all } f \in C(X).$$

Since (X,ϕ) is irreducible and hence asymptotically essentially free, the groupoid $G^a_\phi \rtimes \mathbb{Z}$ is essentially principal by Lemma 5.3. By [30, Proposition 4.7] or [31, Proposition 4.2], $C(X) = C((G^a_\phi \rtimes \mathbb{Z})^\circ)$ is a maximal abelian C^*-subalgebra of $\mathcal{R}^a_\phi$. Hence, there exists a unitary $f_0 \in C(X)$ such that $U^*_\phi \Phi_h^{-1}(\Phi(U_\phi)) = f_0$, so that

$$\text{(6.3)} \qquad \Phi(U_\phi) = \Phi_h(U_\phi f_0).$$

Since $\Phi \circ \rho^\phi_t = \rho^\psi_t \circ \Phi$ and $\Phi_h \circ \mathrm{Ad}(U_t(c_\phi)) = \rho^\psi_t \circ \Phi_h$, we get the following by equality (6.3):

$$\text{(6.4)} \qquad \Phi \circ \rho^\phi_t(U_\phi) = \big(\Phi_h \circ \mathrm{Ad}(U_t(c_\phi))\big)(U_\phi f_0).$$

As $\rho_t^{\phi}(U_\phi) = \exp(2\pi\sqrt{-1}t)U_\phi$, the equality (6.4) becomes

$$\exp(2\pi\sqrt{-1}t)\Phi(U_\phi) = \Phi_h\big(U_t(c_\phi)U_\phi f_0 U_t(c_\phi)^*\big). \tag{6.5}$$

As $f_0 \in C(X)$ and $U_t(c_\phi)^* = U_t(-c_\phi)$, we have the following for $\xi \in l^2(G_\phi^a \rtimes \mathbb{Z})$ and $(x, n, z) \in G_\phi^a \rtimes \mathbb{Z}$:

$$\begin{aligned}[U_t(-c_\phi)f_0\xi](x,n,z) &= \exp\big(2\pi\sqrt{-1}(-c_\phi(x,n,z))t\big)[f_0\xi](x,n,z)\\ &= f_0(x)\exp\big(2\pi\sqrt{-1}(-c_\phi(x,n,z))t\big)\xi(x,n,z)\\ &= \big[f_0(x)U_t(-c_\phi)\xi\big](x,n,z),\end{aligned}$$

so that $U_t(c_\phi)^* f_0 = f_0 U_t(c_\phi)^*$. Hence, equation (6.5) implies

$$\exp(2\pi\sqrt{-1}t)\Phi(U_\phi) = \Phi_h\big(U_t(c_\phi)U_\phi U_t(c_\phi)^* f_0\big),$$

which becomes by (6.3),

$$\exp(2\pi\sqrt{-1}t)\Phi_h(U_\phi) = \Phi_h\big(U_t(c_\phi)U_\phi U_t(c_\phi)^*\big),$$

so that

$$\exp(2\pi\sqrt{-1}t)U_\phi = U_t(c_\phi)U_\phi U_t(c_\phi)^*. \tag{6.6}$$

For $\xi \in l^2(G_\phi^a \rtimes \mathbb{Z})$ and $(x, n, z) \in G_\phi^a \rtimes \mathbb{Z}$, we have the equalities

$$\begin{aligned}&\big[U_t(c_\phi)U_\phi U_t(c_\phi)^*\xi\big](x,n,z)\\ &\quad= \exp\big(2\pi\sqrt{-1}c_\phi(x,n,z)t\big)[U_\phi U_t(-c_\phi)\xi](x,n,z)\\ &\quad= \exp\big(2\pi\sqrt{-1}c_\phi(x,n,z)t\big)[U_t(-c_\phi)\xi](\phi(x),n-1,z)\\ &\quad= \exp\big(2\pi\sqrt{-1}(c_\phi(x,n,z) - c_\phi(\phi(x),n-1,z))t\big)\xi(\phi(x),n-1,z).\end{aligned}$$

On the other hand,

$$[\exp(2\pi\sqrt{-1}t)U_\phi\xi](x,n,z) = \exp(2\pi\sqrt{-1}t)\xi(\phi(x),n-1,z).$$

By (6.6), we have

$$c_\phi(x,n,z) - c_\phi\big(\phi(x),n-1,z\big) = 1.$$

By (5.4), we see that

$$\begin{aligned}&c_\phi(x,n,z) - c_\phi\big(\phi(x),n-1,z\big)\\ &\quad= \big\{c_1^n(x) + d_1(\phi^n(x),z)\big\} - \big\{c_1^{n-1}(\phi(x)) + d_1(\phi^{n-1}(\phi(x)),z)\big\}\\ &\quad= c_1(x).\end{aligned}$$

Therefore, we have $c_1(x) = 1$ for all $x \in X$, and $c_2(y) = 1$ for all $y \in Y$ similarly. ∎

Recall that $d_\phi(x, n, z) = n$ for $(x, n, z) \in G_\phi^a \rtimes \mathbb{Z}$ defines a continuous homomorphism $d_\phi\colon G_\phi^a \rtimes \mathbb{Z} \to \mathbb{Z}$.

Theorem 6.4 *Let (X, ϕ) and (Y, ψ) be irreducible Smale spaces. Then the following assertions are equivalent:*

(i) *(X, ϕ) and (Y, ψ) are asymptotically conjugate: $(X, \phi) \underset{a}{\cong} (Y, \psi)$.*

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(ii) *There exists an isomorphism* $\varphi\colon G^a_\phi \rtimes \mathbb{Z} \to G^a_\psi \rtimes \mathbb{Z}$ *of étale groupoids such that* $d_\psi \circ \varphi = d_\phi$.

(iii) *There exists an isomorphism* $\Phi\colon \mathcal{R}^a_\phi \to \mathcal{R}^a_\psi$ *of* C^**-algebras such that*

$$\Phi(C(X)) = C(Y) \quad \textit{and} \quad \Phi \circ \rho^\phi_t = \rho^\psi_t \circ \Phi, \qquad t \in \mathbb{T}.$$

Proof The implication (iii) ⇒ (i) follows from Proposition 6.3. In the proof of Proposition 6.3, we showed that there exists an isomorphism of groupoids $\varphi_h\colon G^a_\phi \rtimes \mathbb{Z} \to G^a_\psi \rtimes \mathbb{Z}$ such that $c_1 \equiv 1$, $c_2 \equiv 1$ and $d_1 \equiv 0$, $d_2 \equiv 0$. Hence we have

$$c_\phi(x,n,z) = c_1^n(x) + d_1(\phi^n(x), z) = n \quad \text{for } (x,n,z) \in G^a_\phi \rtimes \mathbb{Z}$$

and $c_\psi(y,m,w) = m$, similarly. This implies that $c_\phi = d_\phi$ and $c_\psi = d_\psi$. By (6.2), we obtain $d_\psi \circ \varphi = d_\phi$. This argument shows that the implications (iii) ⇒ (ii) ⇒ (i) hold.

We will show the implication (i) ⇒ (iii). Suppose that (X,ϕ) and (Y,ψ) are asymptotically conjugate. Take a homeomorphism $h\colon X \to Y$, which gives rise to the asymptotic conjugacy. In the proof of (i) ⇒ (ii) of Theorem 5.7, we know that $c_\phi = d_\phi$ and $c_\psi = d_\psi$, because $c_\phi(x,n,z) = n$ for $(x,n,z) \in G^a_\phi \rtimes \mathbb{Z}$ and $c_\psi(y,m,w) = m$ for $(y,m,w) \in G^a_\psi \rtimes \mathbb{Z}$, similarly, which come from the conditions $c_1 \equiv 1$, $c_2 \equiv 1$, $d_1 \equiv 0$, $d_2 \equiv 0$. Hence, we have

$$\mathrm{Ad}\big(U_t(c_\phi)\big) = \mathrm{Ad}\big(U_t(d_\phi)\big) = \rho^\phi_t \quad \text{and} \quad \mathrm{Ad}\big(U_t(c_\psi)\big) = \mathrm{Ad}\big(U_t(d_\psi)\big) = \rho^\psi_t.$$

We thus obtain the equality $\Phi_h \circ \rho^\phi_t = \rho^\psi_t \circ \Phi_h$ by (5.6) or (5.7). ∎

7 Extended Ruelle Algebras $\mathcal{R}^{s,u}_\phi$

In this section, we will introduce an extended Ruelle algebra $\mathcal{R}^{s,u}_\phi$ from a certain amenable étale groupoid of a Smale space (X,ϕ). The introduced C^*-algebra contains the asymptotic Ruelle algebra $\mathcal{R}^a_\phi$ as a fixed point subalgebra under some circle action. The extended Ruelle algebras will be useful in the following sections to investigate the asymptotic Ruelle algebra $\mathcal{R}^a_\phi$ for topological Markov shifts from the view points of Cuntz–Krieger algebras.

We first introduce the following groupoid $G^{s,u}_\phi \rtimes \mathbb{Z}^2$ for a Smale space (X,ϕ) that will be proved to be étale and amenable:

$$G^{s,u}_\phi \rtimes \mathbb{Z}^2 = \{(x,p,q,y) \in X \times \mathbb{Z} \times \mathbb{Z} \times X \mid (\phi^p(x), y) \in G^s_\phi,\ (\phi^q(x), y) \in G^u_\phi\}.$$

The following lemma is straightforward.

Lemma 7.1 *For* $(x,p,q,y), (x',p',q',y') \in G^{s,u}_\phi \rtimes \mathbb{Z}^2$*, we have*

(i) $(x, p+p', q+q', y') \in G^{s,u}_\phi \rtimes \mathbb{Z}^2$ *if* $y = x'$;

(ii) $(y, -p, -q, x) \in G^{s,u}_\phi \rtimes \mathbb{Z}^2$.

Two elements $(x,p,q,y),(x',p',q',y') \in G_\phi^{s,u} \rtimes \mathbb{Z}^2$ are composable if and only if $y = x'$. The multiplication and the inverse in $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ are given by

$$(x,p,q,y)\cdot(x',p',q',y') = (x,p+p',q+q',y') \quad \text{if } y = x',$$
$$(x,p,q,y)^{-1} = (y,-p,-q,x).$$

We write the unit space $(G_\phi^{s,u} \rtimes \mathbb{Z}^2)^\circ$ of $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ as

$$(G_\phi^{s,u} \rtimes \mathbb{Z}^2)^\circ = \{(x,0,0,x) \mid x \in X\},$$

which is identified with X. Define the range map, source map

$$r,s\colon G_\phi^{s,u} \rtimes \mathbb{Z}^2 \longrightarrow (G_\phi^{s,u} \rtimes \mathbb{Z}^2)^\circ$$

by

$$r(x,p,q,y) = (x,0,0,x), \qquad s(x,p,q,y) = (y,0,0,y).$$

For $p,q \in \mathbb{Z}$ and $n = 0,1,\dots$, we set

$$G_\phi^{s,u,n}(p,q) = \{(x,y) \in X \times X \mid (\phi^p(x),y) \in G_\phi^{s,n}, (\phi^q(x),y) \in G_\phi^{u,n}\},$$
$$G_\phi^{s,u}(p,q) = \{(x,y) \in X \times X \mid (\phi^p(x),y) \in G_\phi^{s}, (\phi^q(x),y) \in G_\phi^{u}\}.$$

For each n, the set $G_\phi^{s,u,n}(p,q)$ is endowed with the relative topology from $X \times X$. Since $G_\phi^{*,n} \subset G_\phi^{*,n+1}$ for $* = s,u$ and $n = 0,1,\dots$, we have

$$G_\phi^{s,u,n}(p,q) \subset G_\phi^{s,u,n+1}(p,q) \quad \text{and} \quad G_\phi^{s,u}(p,q) = \bigcup_{n=0}^{\infty} G_\phi^{s,u,n}(p,q). \tag{7.1}$$

We can endow $G_\phi^{s,u}(p,q)$ with inductive limit topology from the inductive system (7.1) of the topological spaces $\{G_\phi^{s,u,n}(p,q)\}_{n\in\mathbb{Z}_+}$. Since we can identify $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ with the disjoint union $\sqcup_{(p,q)\in\mathbb{Z}^2} G_\phi^{s,u}(p,q)$, the groupoid $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ has the topology defined from the topology of the disjoint union $\sqcup_{(p,q)\in\mathbb{Z}^2} G_\phi^{s,u}(p,q)$. We then have the following proposition.

Proposition 7.2 *$G_\phi^{s,u} \rtimes \mathbb{Z}^2$ is an étale groupoid.*

Proof We will show that the range map $r : (x,p,q,y) \in G_\phi^{s,u} \rtimes \mathbb{Z}^2 \to (x,0,0,x) \in G_\phi^{s,u} \rtimes \mathbb{Z}^2$ is a local homeomorphism. Take an arbitrary point $(x,p,q,y) \in G_\phi^{s,u} \rtimes \mathbb{Z}^2$. Since $G_\phi^{s,u} \rtimes \mathbb{Z}^2 = \sqcup_{(p,q)\in\mathbb{Z}^2} G_\phi^{s,u}(p,q)$ and $G_\phi^{s,u}(p,q) = \bigcup_{n=0}^{\infty} G_\phi^{s,u,n}(p,q)$, we assume that (x,y) belongs to $G_\phi^{s,u,N}(p,q)$ for some $N \in \mathbb{Z}_+$, so that

$$(\phi^p(x),y) \in G_\phi^{s,N}, \qquad (\phi^q(x),y) \in G_\phi^{u,N}$$

which imply that

$$(\phi^{N+p}(x),\phi^N(y)) \in G_\phi^{s,0}, \qquad (\phi^{-(N-q)}(x),\phi^{-N}(y)) \in G_\phi^{u,0} \tag{7.2}$$

and

$$d(\phi^{N+p+n}(x),\phi^{N+n}(y)) < \epsilon_0, \qquad d(\phi^{-(N-q+n)}(x),\phi^{-(N+n)}(y)) < \epsilon_0$$

for all $n = 0,1,2,\dots$. Take $z \in X$ such that $d(\phi^N(y),\phi^{N+p}(z)) < \epsilon_0$ and $d(x,z)$ is small enough so that $(\phi^N(y),\phi^{N+p}(z)) \in \Delta_{\epsilon_0}$. Hence, the point $[\phi^N(y),\phi^{N+p}(z)]$

defines an element of X, and we have an element $\phi^{-N}([\phi^N(y), \phi^{N+p}(z)])$ in X. Since we can assume that $[\phi^N(y), \phi^{N+p}(z)] \in X^u(\phi^N(y), \epsilon_0)$, we have

$$\begin{aligned} d\big(y, \phi^{-N}([\phi^N(y), \phi^{N+p}(z)])\big) &= d\big(\phi^{-N}(\phi^N(y)), \phi^{-N}([\phi^N(y), \phi^{N+p}(z)])\big) \\ &< \lambda_0^N d\big(\phi^N(y), [\phi^N(y), \phi^{N+p}(z)]\big) \\ &< \lambda_0^N \epsilon_0. \end{aligned}$$

Similarly we have an element

$$\big[\phi^{-(N-q)}(z), \phi^{-N}(y)\big] \in X \quad \text{and} \quad \phi^N\big([\phi^{-(N-q)}(z), \phi^{-N}(y)]\big) \in X$$

such that

$$d\big(y, \phi^N([\phi^{-(N-q)}(z), \phi^{-N}(y)])\big) < \lambda_0^N \epsilon_0.$$

We can also assume that $\lambda_0^N < \frac{1}{2}$ by taking N large enough, so that

$$\begin{aligned} & d\big(\phi^N([\phi^{-(N-q)}(z), \phi^{-N}(y)]), \phi^{-N}([\phi^N(y), \phi^{N+p}(z)])\big) \\ & \quad < d\big(y, \phi^N([\phi^{-(N-q)}(z), \phi^{-N}(y)])\big) + d\big(y, \phi^{-N}([\phi^N(y), \phi^{N+p}(z)])\big) \\ & \quad < 2\lambda_0^N \epsilon_0 < \epsilon_0. \end{aligned}$$

Hence we have

$$\Big(\phi^N\big([\phi^{-(N-q)}(z), \phi^{-N}(y)]\big), \phi^{-N}\big([\phi^N(y), \phi^{N+p}(z)]\big)\Big) \in \Delta_{\epsilon_0},$$

so that the element

$$\gamma(z) := \big[\phi^N\big([\phi^{-(N-q)}(z), \phi^{-N}(y)]\big), \phi^{-N}\big([\phi^N(y), \phi^{N+p}(z)]\big)\big] \tag{7.3}$$

is defined in X. The map γ is defined on a small neighborhood of x and gives rise to a continuous map on the neighborhood. The conditions in (7.2) imply

$$[\phi^N(y), \phi^{N+p}(x)] = \phi^N(y), \qquad [\phi^{-(N-q)}(x), \phi^{-N}(y)] = \phi^{-N}(y).$$

Hence, for $z = x$ in (7.3), we have

$$\begin{aligned} \gamma(x) &= \big[\phi^N\big([\phi^{-(N-q)}(x), \phi^{-N}(y)]\big), \phi^{-N}\big([\phi^N(y), \phi^{N+p}(x)]\big)\big] \\ &= \big[\phi^N\big(\phi^{-N}(y)\big), \phi^{-N}\big(\phi^N(y)\big)\big] \\ &= [y, y] = y. \end{aligned}$$

We will next show that γ is injective. Suppose that $\gamma(z) = \gamma(z')$ for z, z' in a small neighborhood of x. Since

$$\begin{aligned} \phi^{-N}(\gamma(z)) &= \big[[\phi^{-(N-q)}(z), \phi^{-N}(y)], \phi^{-2N}([\phi^N(y), \phi^{N+p}(z)])\big] \\ &= \big[\phi^{-(N-q)}(z), \phi^{-2N}([\phi^N(y), \phi^{N+p}(z)])\big] \end{aligned} \tag{7.4}$$

and similarly

$$\phi^{-N}(\gamma(z')) = \big[\phi^{-(N-q)}(z'), \phi^{-2N}([\phi^N(y), \phi^{N+p}(z')])\big]. \tag{7.5}$$

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We then have by (7.4), (7.5)

$$\begin{aligned}\phi^{-(N-q)}(z) &= \big[\,[\phi^{-(N-q)}(z), \phi^{-2N}([\phi^N(y), \phi^{N+p}(z)])], \phi^{-(N-q)}(z)\big]\\ &= [\phi^{-N}(\gamma(z)), \phi^{-(N-q)}(z)]\\ &= [\phi^{-N}(\gamma(z')), \phi^{-(N-q)}(z)]\\ &= \big[\,[\phi^{-(N-q)}(z'), \phi^{-2N}([\phi^N(y), \phi^{N+p}(z')])], \phi^{-(N-q)}(z)\big]\\ &= [\phi^{-(N-q)}(z'), \phi^{-(N-q)}(z)],\end{aligned}$$

so that

$$\phi^{-(N-q)}(z) = \big[\,\phi^{-(N-q)}(z'), \phi^{-(N-q)}(z)\big]. \tag{7.6}$$

Since z, z' are in a small neighborhood of x, we can assume that

$$d\big(\,\phi^{-(N-q)}(z'), \phi^{-(N-q)}(z)\big) < \epsilon_0. \tag{7.7}$$

By (7.6) and (7.7), we know

$$\phi^{-(N-q)}(z') \in X^u\big(\,\phi^{-(N-q)}(z), \epsilon_0\big),$$

so that

$$d\big(\,\phi^{-(N-q+n)}(z'), \phi^{-(N-q+n)}(z)\big) < \epsilon_0 \quad \text{for all } n = 0, 1, 2, \dots. \tag{7.8}$$

As z, z' are in a small neighborhood of x, we can assume that

$$d\big(\,\phi^{-n}(x), \phi^{-n}(z)\big) < \frac{\epsilon_0}{2}, \quad d\big(\,\phi^{-n}(x), \phi^{-n}(z')\big) < \frac{\epsilon_0}{2} \quad \text{for all } n = 0, 1, \dots, N-q.$$

Hence, we have

$$d\big(\,\phi^{-n}(z'), \phi^{-n}(z)\big) < \epsilon_0 \quad \text{for all } n = 0, 1, \dots N-q. \tag{7.9}$$

By (7.8) and (7.9), we obtain

$$d\big(\,\phi^{-n}(z'), \phi^{-n}(z)\big) < \epsilon_0 \quad \text{for all } n = 0, 1, 2, \dots$$

and hence $z' \in X^u(z, \epsilon_0)$.

We similarly observe that

$$d(\phi^n(z'), \phi^n(z)) < \epsilon_0 \quad \text{for all } n = 0, 1, 2, \dots$$

and hence $z' \in X^s(z, \epsilon_0)$, so that

$$z' \in X^u(z, \epsilon_0) \cap X^s(z, \epsilon_0)$$

and $z' = [z, z] = z$. This shows that γ is injective on a small neighborhood of x and locally a homeomorphism by the definition of γ. As a consequence, the groupoid $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ is étale. ∎

Lemma 7.3 *The étale groupoid $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ is amenable.*

Proof Consider the groupoid homomorphism $\eta\colon (x, p, q, y) \in G_\phi^{s,u} \rtimes \mathbb{Z}^2 \to (p, q) \in \mathbb{Z}^2$. The kernel is $G_\phi^s \cap G_\phi^u = G_\phi^a$, which is amenable by Lemma 5.5. Hence by [1, Proposition 5.1.2], we conclude that $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ is amenable. ∎

Definition 7.4 A Smale space (X,ϕ) is said to be *(s,u)-essentially free* if the interior of the set $\{x\in X \mid (\phi^p(x),x)\in G^s_\phi, (\phi^q(x),x)\in G^u_\phi\}$ is empty for each $(p,q)\in\mathbb{Z}\times\mathbb{Z}$ with $(p,q)\neq(0,0)$.

The following lemma, which was kindly suggested by the referee, is proved in a similar way to Lemma 5.2

Lemma 7.5 *If (X,ϕ) is irreducible and X is infinite, then (X,ϕ) is (s,u)-essentially free.*

Proof Suppose that the set

$$\operatorname{int}\left\{x\in X \mid (\phi^p(x),x)\in G^s_\phi, (\phi^q(x),x)\in G^u_\phi\right\}$$

contains a non-empty open set U for a fixed $(p,q)\in\mathbb{Z}\times\mathbb{Z}$ with $(p,q)\neq(0,0)$. We can assume that $p\neq 0$. Since

$$\operatorname{int}\left\{x\in X \mid (\phi^p(x),x)\in G^s_\phi, (\phi^q(x),x)\in G^u_\phi\right\}\subset \operatorname{int}\left\{x\in X \mid (\phi^p(x),x)\in G^s_\phi\right\},$$

we have a non-empty open set U such that

$$U\subset \operatorname{int}\left\{x\in X \mid (\phi^p(x),x)\in G^s_\phi\right\}$$

for a fixed $p\neq 0$. By the same argument as the proof of Lemma 5.2, we have a contradiction, thus proving (X,ϕ) is (s,u)-essentially free. ∎

Lemma 7.6 *A Smale space (X,ϕ) is (s,u)-essentially free if and only if the étale groupoid $G^{s,u}_\phi\rtimes\mathbb{Z}^2$ is essentially principal.*

Proof As we have

$$\begin{aligned}(G^{s,u}_\phi\rtimes\mathbb{Z}^2)' &= \bigcup_{p,q\in\mathbb{Z}}\left\{(x,p,q,y)\in G^{s,u}_\phi\rtimes\mathbb{Z}^2 \mid x=y\right\}\\ &= \bigcup_{p,q\in\mathbb{Z}}\left\{(x,p,q,x)\in X\times\mathbb{Z}\times\mathbb{Z}\times X \mid (\phi^p(x),x)\in G^s_\phi, (\phi^q(x),x)\in G^u_\phi\right\},\end{aligned}$$

the interior $\operatorname{int}((G^{s,u}_\phi\rtimes\mathbb{Z}^2)')$ of $G^{s,u}_\phi\rtimes\mathbb{Z}^2$ is

$$\operatorname{int}\left((G^{s,u}_\phi\rtimes\mathbb{Z}^2)'\right) = \bigcup_{p,q\in\mathbb{Z}}\operatorname{int}\left(\{(x,p,q,x)\in X\times\mathbb{Z}\times\mathbb{Z}\times X \mid (\phi^p(x),x)\in G^s_\phi, (\phi^q(x),x)\in G^u_\phi\}\right).$$

For $p=q=0$, we see that

$$\operatorname{int}\left(\{(x,0,0,x)\in X\times\mathbb{Z}\times\mathbb{Z}\times X \mid (x,x)\in G^s_\phi, (x,x)\in G^u_\phi\}\right) = (G^{s,u}_\phi\rtimes\mathbb{Z}^2)^\circ = X.$$

Hence, $\operatorname{int}((G^{s,u}_\phi\rtimes\mathbb{Z}^2)') = (G^{s,u}_\phi\rtimes\mathbb{Z}^2)^\circ$ if and only if the interior of

$$\left\{(x,p,q,x)\in X\times\mathbb{Z}\times\mathbb{Z}\times X \mid (\phi^p(x),x)\in G^s_\phi, (\phi^q(x),x)\in G^u_\phi\right\}$$

is empty for all $p,q\in\mathbb{Z}$ except $p=q=0$. This implies that (X,ϕ) is (s,u)-essentially free if and only if $G^{s,u}_\phi\rtimes\mathbb{Z}^2$ is essentially principal. ∎

Definition 7.7 The groupoid C^*-algebra $C^*(G_\phi^{s,u} \rtimes \mathbb{Z}^2)$ of the étale amenable groupoid $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ for a Smale space (X, ϕ) is called the *extended asymptotic Ruelle algebra* or simply the *extended Ruelle algebra* and written $\mathcal{R}_\phi^{s,u}$.

Since $G_\phi^{s,u} \rtimes \mathbb{Z}^2$ is amenable, the C^*-algebra $\mathcal{R}_\phi^{s,u}$ is identified with the reduced groupoid C^*-algebra $C_r^*(G_\phi^{s,u} \rtimes \mathbb{Z}^2)$ on $l^2(G_\phi^{s,u} \rtimes \mathbb{Z}^2)$ in a canonical way.

Similarly to Proposition 5.4, we obtain the following.

Proposition 7.8 *If a Smale space (X, ϕ) is irreducible and X is infinite, then the C^*-algebra $\mathcal{R}_\phi^{s,u}$ is simple.*

We note that the above proposition also follows from [29, Theorem 1.4] through Proposition 7.10, which will be shown later.

Let U_{z_1,z_2}, $(z_1, z_2) \in \mathbb{T}^2 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_i| = 1\}$ be an action of $\mathbb{T}^2$ to the unitary group of $B(l^2(G_\phi^{s,u} \rtimes \mathbb{Z}^2))$ defined by

$$(U_{z_1,z_2}\xi)(x, p, q, y) = z_1^p z_2^{-q} \xi(x, p, q, y)$$

for $\xi \in l^2(G_\phi^{s,u} \rtimes \mathbb{Z}^2), (x, p, q, y) \in G_\phi^{s,u} \rtimes \mathbb{Z}^2$. It is easy to see that the automorphisms $\mathrm{Ad}(U_{z_1,z_2})$ of $B(l^2(G_\phi^{s,u} \rtimes \mathbb{Z}^2))$ for $(z_1, z_2) \in \mathbb{T}^2$ leave $\mathcal{R}_\phi^{s,u}$ globally invariant. They give rise to an action of $\mathbb{T}^2$ on $\mathcal{R}_\phi^{s,u}$, denoted by $\rho_\phi^{s,u}$. Let us denote by $\delta_z^\phi = \rho_{\phi,(z,z)}^{s,u}$, $z \in \mathbb{T}$ the action of $\mathbb{T}$, called the *diagonal action*. Recall that the asymptotic Ruelle algebra $\mathcal{R}_\phi^a$ is defined by the groupoid C^*-algebra $C^*(G_\phi^a \rtimes \mathbb{Z})$ of the étale groupoid $G_\phi^a \rtimes \mathbb{Z}$. We then have the following theorem.

Theorem 7.9 *Assume that a Smale space (X, ϕ) is irreducible and X is infinite. Then the fixed point algebra $(\mathcal{R}_\phi^{s,u})^{\delta^\phi}$ of $\mathcal{R}_\phi^{s,u}$ under the diagonal action δ^ϕ is isomorphic to the asymptotic Ruelle algebra $\mathcal{R}_\phi^a$.*

Proof The étale groupoid $G_\phi^a \rtimes \mathbb{Z}$ is identified with the subgroupoid

$$\{(x, p, p, y) \in X \times \mathbb{Z} \times \mathbb{Z} \times X \mid (\phi^p(x), y) \in G_\phi^s,\ (\phi^p(x), y) \in G_\phi^u\} \subset G_\phi^{s,u} \rtimes \mathbb{Z}^2$$

of $G_\phi^{s,u} \rtimes \mathbb{Z}^2$, which is written $(G_\phi^{s,u} \rtimes \mathbb{Z}^2)^D$. Since $(G_\phi^{s,u} \rtimes \mathbb{Z}^2)^D$ is clopen in $G_\phi^{s,u} \rtimes \mathbb{Z}^2$, we have a natural inclusion relation $C_c((G_\phi^{s,u} \rtimes \mathbb{Z}^2)^D) \subset C_c(G_\phi^{s,u} \rtimes \mathbb{Z}^2)$ of the algebras. For $f \in C_c((G_\phi^{s,u} \rtimes \mathbb{Z}^2)^D)$, we put

$$\mathcal{E}_\phi(f)(x, p, q, y) = \begin{cases} f(x, p, p, y) & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

Then $\mathcal{E}_\phi$ defines a continuous linear map from $C_c(G_\phi^{s,u} \rtimes \mathbb{Z}^2)$ to $C_c((G_\phi^{s,u} \rtimes \mathbb{Z}^2)^D)$ and extends to $\mathcal{R}_\phi^{s,u}$ by the formula

$$\mathcal{E}_\phi(f) = \int_{\mathbb{T}} \delta_z^\phi(f) dz \quad \text{for } f \in \mathcal{R}_\phi^{s,u},$$

so that we have a conditional expectation from $\mathcal{R}_\phi^{s,u}$ onto $\mathcal{R}_\phi^a$. It is routine to check that $\mathcal{E}_\phi(\mathcal{R}_\phi^{s,u})$ is the fixed point algebra $(\mathcal{R}_\phi^{s,u})^{\delta^\phi}$ of $\mathcal{R}_\phi^{s,u}$ under the diagonal action δ^ϕ. ∎

The author would like to thank the referee who kindly suggested the following proposition.

Proposition 7.10 *The extended Ruelle algebra $\mathcal{R}_\phi^{s,u}$ is stably isomorphic to the tensor product $\mathcal{R}_\phi^s \otimes \mathcal{R}_\phi^u$ of the stable Ruelle algebra $\mathcal{R}_\phi^s$ and the unstable Ruelle algebra $\mathcal{R}_\phi^u$.*

Proof It is easy to see that the correspondence

$$(x,p,z)\times(y,q,w) \in (G_\phi^s \rtimes \mathbb{Z})\times(G_\phi^u \rtimes \mathbb{Z}) \longrightarrow \big((x,z)\times(y,w),(p,q)\big) \in (G_\phi^s \times G_\phi^u)\rtimes\mathbb{Z}^2$$

yields an isomorphism of étale groupoids between $(G_\phi^s \rtimes \mathbb{Z})\times(G_\phi^u \rtimes \mathbb{Z})$ and $(G_\phi^s \times G_\phi^u)\rtimes\mathbb{Z}^2$. Hence we have

$$\mathcal{R}_\phi^s \otimes \mathcal{R}_\phi^u = C^*\big((G_\phi^s \rtimes \mathbb{Z})\times(G_\phi^u \rtimes \mathbb{Z})\big) \cong C^*\big((G_\phi^s \times G_\phi^u)\rtimes\mathbb{Z}^2\big).$$

As in the proof of [25, Theorem 3.1], the diagonal $\Delta = \{((x,z)\times(x,z),(p,q)) \in (G_\phi^s \times G_\phi^u)\rtimes\mathbb{Z}^2\}$ is an abstract transversal in the sense of Muhly, Renault, and Williams [23]. Since the reduction of $(G_\phi^s \times G_\phi^u)\rtimes\mathbb{Z}^2$ to Δ is clearly isomorphic to $G_\phi^{s,u}\rtimes\mathbb{Z}^2$ as étale groupoids, we see by [23, Theorem 2.8] that $C^*(G_\phi^{s,u}\rtimes\mathbb{Z}^2)$ is stably isomorphic to $C^*((G_\phi^s \times G_\phi^u)\rtimes\mathbb{Z}^2)$, so that the extended Ruelle algebra $\mathcal{R}_\phi^{s,u}$ is stably isomorphic to the tensor product $\mathcal{R}_\phi^s \otimes \mathcal{R}_\phi^u$. ■

8 Asymptotic Continuous Orbit Equivalence in Topological Markov Shifts

In the first part of this section, we will deal with topological Markov shifts, which are often called *shifts of finite type*, as examples of Smale spaces. They have been studied by Ruelle, Putnam and Putnam-Spielberg, etc. from the view point of Smale spaces. The following description follows from Putnam's lecture note [26, Section 1].

Let $A = [A(i,j)]_{i,j=1}^N$ be an $N \times N$ matrix with entries $A(i,j)$ in $\{0,1\}$ for $i,j = 1,\dots,N$ such that none of its rows or columns is zero. We assume that $N \geq 2$ and the matrix A is irreducible and not any permutation matrix. Let us denote by $\overline{X}_A$ the shift space of the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$, which is defined by

$$\overline{X}_A = \big\{(x_n)_{n\in\mathbb{Z}} \in \{1,\dots,N\}^{\mathbb{Z}} \mid A(x_n,x_{n+1}) = 1 \text{ for all } n \in \mathbb{Z}\big\}$$

with shift transformation $\overline{\sigma}_A$ defined by $\overline{\sigma}_A((x_n)_{n\in\mathbb{Z}}) = (x_{n+1})_{n\in\mathbb{Z}}$. We note that the assumption that A is irreducible and not any permutation matrix implies that the shift space $\overline{X}_A$ is infinite and hence homeomorphic to a Cantor discontinuum.

Take and fix an arbitrary real number λ_0 with $0 < \lambda_0 < 1$. The space $\overline{X}_A$ is endowed with the metric d defined by

$$d((x_n)_{n\in\mathbb{Z}},(y_n)_{n\in\mathbb{Z}}) = \begin{cases} 0 & \text{if } (x_n)_{n\in\mathbb{Z}} = (y_n)_{n\in\mathbb{Z}}, \\ 1 & \text{if } x_0 \neq y_0, \\ (\lambda_0)^{k+1} & \text{if } k = \operatorname{Max}\{|n| \mid x_i = y_i \text{ for all } i; |i| \leq n\}. \end{cases}$$

With this metric d, the space $\overline{X}_A$ is a compact Hausdorff space such that the topological dynamical system $(\overline{X}_A, \overline{\sigma}_A)$ is called the *two-sided topological Markov shift defined by A*. For $k \in \mathbb{Z}_+$, we set

$$B_k(\overline{X}_A) = \left\{ (x_n)_{n=1}^k \in \{1, \dots, N\}^k \mid A(x_n, x_{n+1}) = 1, n = 1, \dots, k-1 \right\}$$

and $B_*(\overline{X}_A) = \bigcup_{k=0}^{\infty} B_k(\overline{X}_A)$, where $B_0(\overline{X}_A)$ denotes the empty word $\varnothing$. Each member of $B_k(\overline{X}_A)$ is called an *admissible word of length k*.

We will view the topological Markov shift as a Smale space in the following way. Take $\epsilon_0 = 1$, so that we have $(x, y) \in \Delta_{\epsilon_0}$ if and only if $x_0 = y_0$. Hence, the bracket $[x, y] = ([x, y]_n)_{n \in \mathbb{Z}} \in \overline{X}_A$ for $(x, y) \in \Delta_{\epsilon_0}$ can be defined by

$$[x, y]_n = \begin{cases} x_n & \text{if } n \leq 0, \\ y_n & \text{if } n \geq 0. \end{cases}$$

Since $x_0 = y_0$, $([x, y]_n)_{n \in \mathbb{Z}}$ defines an element of $\overline{X}_A$. We then have

$$\overline{X}_A^s(x, \epsilon_0) = \{ y \in \overline{X}_A \mid y_n = x_n \text{ for } n = 0, 1, 2, \dots \},$$
$$\overline{X}_A^u(x, \epsilon_0) = \{ y \in \overline{X}_A \mid y_n = x_n \text{ for } n = 0, -1, -2, \dots \}.$$

As in Putnam's lecture note [26, Section 1], the two-sided topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ with the metric d becomes a Smale space for $\epsilon_0 = 1$ and λ_0 itself.

For $n = 0, 1, 2, \dots$, we write

$$G_A^{s,n} = G_{\overline{\sigma}_A}^{s,n}, \qquad G_A^{u,n} = G_{\overline{\sigma}_A}^{u,n}, \qquad G_A^{a,n} = G_{\overline{\sigma}_A}^{a,n}.$$

Since

$$G_A^{s,0} = \left\{ (x, y) \in \overline{X}_A \times \overline{X}_A \mid y_i = x_i \text{ for all } i = 0, 1, 2, \dots \right\},$$
$$G_A^{u,0} = \left\{ (x, y) \in \overline{X}_A \times \overline{X}_A \mid y_i = x_i \text{ for all } i = 0, -1, -2, \dots \right\},$$
$$G_A^{a,0} = G_A^{s,0} \cap G_A^{u,0} = \left\{ (x, y) \in \overline{X}_A \times \overline{X}_A \mid x = y \right\},$$

we know for $n = 0, 1, 2, \dots$,

$$G_A^{s,n} = \left\{ (x, y) \in \overline{X}_A \times \overline{X}_A \mid y_i = x_i \text{ for all } i = n, n+1, n+2, \dots \right\},$$
$$G_A^{u,n} = \left\{ (x, y) \in \overline{X}_A \times \overline{X}_A \mid y_i = x_i \text{ for all } i = -n, -n-1, -n-2, \dots \right\},$$
$$G_A^{a,n} = G_A^{s,n} \cap G_A^{u,n} = \left\{ (x, y) \in \overline{X}_A \times \overline{X}_A \mid y_i = x_i \text{ for all } |i| = n, n+1, n+2, \dots \right\}.$$

All of them are given the relative topology of $\overline{X}_A \times \overline{X}_A$. Each of them defines an equivalence relation on $\overline{X}_A$. We set

$$G_A^s = \bigcup_{n=0}^{\infty} G_A^{s,n}, \qquad G_A^u = \bigcup_{n=0}^{\infty} G_A^{u,n}, \qquad G_A^a = \bigcup_{n=0}^{\infty} G_A^{a,n},$$

and they are endowed with the inductive limit topology, respectively. Putnam studied these three equivalence relations G_A^s, G_A^u, and G_A^a on $\overline{X}_A$ by regarding them as topological groupoids. He studied the associated groupoid C^*-algebras $C^*(G_A^s)$, $C^*(G_A^u)$, and $C^*(G_A^a)$ which have been denoted by $S(\overline{X}_A, \overline{\sigma}_A)$, $U(\overline{X}_A, \overline{\sigma}_A)$, and $A(\overline{X}_A, \overline{\sigma}_A)$, respectively. He pointed out that they are all stably AF-algebras. He investigated their semi-direct products as groupoids

$$G_A^s \rtimes \mathbb{Z} = \left\{ (x, n, y) \in \overline{X}_A \times \mathbb{Z} \times \overline{X}_A \mid (\overline{\sigma}_A^n(x), y) \in G_A^s \right\},$$
$$G_A^u \rtimes \mathbb{Z} = \left\{ (x, n, y) \in \overline{X}_A \times \mathbb{Z} \times \overline{X}_A \mid (\overline{\sigma}_A^n(x), y) \in G_A^u \right\},$$
$$G_A^a \rtimes \mathbb{Z} = \left\{ (x, n, y) \in \overline{X}_A \times \mathbb{Z} \times \overline{X}_A \mid (\overline{\sigma}_A^n(x), y) \in G_A^a \right\}.$$

Putnam has also deeply studied the associated groupoid C^*-algebras $C^*(G_A^s \rtimes \mathbb{Z})$, $C^*(G_A^u \rtimes \mathbb{Z})$, and $C^*(G_A^a \rtimes \mathbb{Z})$ which have been written R_s, R_u, and R_a, respectively in his papers. In this paper, we denote them by $\mathcal{R}_A^s$, $\mathcal{R}_A^u$, and $\mathcal{R}_A^a$, respectively, to emphasize the matrix A. We note that the irreducibility of the Smale space $(\overline{X}_A, \overline{\sigma}_A)$ corresponds to the irreducibility of the matrix A, and the condition that $\overline{X}_A$ is infinite corresponds to the property that the matrix A is not any permutation matrix.

In the second part of this section, we study asymptotic continuous orbit equivalence defined for Smale spaces in Section 3 focusing on topological Markov shifts.

Let $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ be topological Markov shifts. We will regard them as Smale spaces and consider conditions under which they become asymptotic continuous orbit equivalence.

Lemma 8.1 *Conditions* (i) *and* (ii) *in Remark 3.3 are equivalent to the following conditions* (i) *and* (ii), *respectively.*

(i) *There exists a continuous function $k_1 \colon \overline{X}_A \to \mathbb{Z}_+$ such that*

$$\left(\overline{\sigma}_B^{k_1(x)+c_1(x)}(h(x)), \overline{\sigma}_B^{k_1(x)}\left(h(\overline{\sigma}_A(x))\right)\right) \in G_B^{s,0}, \tag{8.1}$$
$$\left(\overline{\sigma}_B^{-k_1(x)+c_1(x)}(h(x)), \overline{\sigma}_B^{-k_1(x)}\left(h(\overline{\sigma}_A(x))\right)\right) \in G_B^{u,0}.$$

(ii) *There exists a continuous function $k_2 \colon \overline{X}_B \to \mathbb{Z}_+$ such that*

$$\left(\overline{\sigma}_A^{k_2(y)+c_2(y)}\left(h^{-1}(y)\right), \overline{\sigma}_A^{k_2(y)}\left(h^{-1}(\overline{\sigma}_B(y))\right)\right) \in G_A^{s,0},$$
$$\left(\overline{\sigma}_A^{-k_2(y)+c_2(y)}\left(h^{-1}(y)\right), \overline{\sigma}_A^{-k_2(y)}\left(h^{-1}\left(\overline{\sigma}_B(y)\right)\right)\right) \in G_A^{u,0}.$$

Proof (i) We will prove that equality (8.1) implies (3.4) by putting $k_{1,n}(x) = k_1^n(x)$. Suppose that there exists a continuous function $k_1 \colon \overline{X}_A \to \mathbb{Z}_+$ satisfying equality (8.1). Since

$$G_B^{s,0} = \left\{ (x, y) \in \overline{X}_B \times \overline{X}_B \mid y_i = x_i \text{ for all } i = 0, 1, 2, \ldots \right\},$$

$G_B^{s,0}$ is an equivalence relation in $\overline{X}_B \times \overline{X}_B$. In equality (8.1), we have

$$\overline{\sigma}_B^{k_1(x)}\left(h(\overline{\sigma}_A(x))\right) \in \overline{X}_B\left(\overline{\sigma}_B^{k_1(x)+c_1(x)}(h(x)), \epsilon_0\right),$$

so that by Lemma 2.3, for any $m \in \mathbb{N}$,

$$\overline{\sigma}_B^m\left(\overline{\sigma}_B^{k_1(x)}(h(\overline{\sigma}_A(x)))\right) \in \overline{X}_B\left(\overline{\sigma}_B^m(\overline{\sigma}_B^{k_1(x)+c_1(x)}(h(x))), \epsilon_0\right), \tag{8.2}$$

and hence

$$\left(\overline{\sigma}_B^{m+k_1(x)+c_1(x)}(h(x)), \overline{\sigma}_B^{m+k_1(x)}\left(h(\overline{\sigma}_A(x))\right)\right) \in G_B^{s,0}.$$

Take $m = k_1(\overline{\sigma}_A(x)) + c_1(\overline{\sigma}_A(x))$ so that we have

$$\Big(\overline{\sigma}_B^{k_1(\overline{\sigma}_A(x))+c_1(\overline{\sigma}_A(x))+k_1(x)+c_1(x)}(h(x)),\\ \overline{\sigma}_B^{k_1(\overline{\sigma}_A(x))+c_1(\overline{\sigma}_A(x))+k_1(x)}\big(h(\overline{\sigma}_A(x))\big)\Big) \in G_B^{s,0},$$

that is

$$\Big(\overline{\sigma}_B^{k_1^2(x)+c_1^2(x)}(h(x)), \overline{\sigma}_B^{c_1(\overline{\sigma}_A(x))+k_1^2(x)}\big(h(\overline{\sigma}_A(x))\big)\Big) \in G_B^{s,0}. \tag{8.3}$$

By replacing x with $\overline{\sigma}_A(x)$ in the equality (8.1) and (8.2), we have

$$\Big(\overline{\sigma}_B^{k_1(\overline{\sigma}_A(x))+c_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A(x))\big), \overline{\sigma}_B^{k_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A^2(x))\big)\Big) \in G_B^{s,0},$$

$$\overline{\sigma}_B^m\Big(\overline{\sigma}_B^{k_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A^2(x))\big)\Big) \in \overline{X}_B\Big(\overline{\sigma}_B^m\Big(\overline{\sigma}_B^{k_1(\overline{\sigma}_A(x))+c_1(\overline{\sigma}_A(x))}(h(\overline{\sigma}_A(x)))\Big), \epsilon_0\Big),$$

so that

$$\Big(\overline{\sigma}_B^{m+k_1(\overline{\sigma}_A(x))+c_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A(x))\big), \overline{\sigma}_B^{m+k_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A^2(x))\big)\Big) \in G_B^{s,0}.$$

Take $m = k_1(x)$ so that we have

$$\Big(\overline{\sigma}_B^{k_1(x)+k_1(\overline{\sigma}_A(x))+c_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A(x))\big), \overline{\sigma}_B^{k_1(x)+k_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A^2(x))\big)\Big) \in G_B^{s,0},$$

that is

$$\Big(\overline{\sigma}_B^{k_1^2(x)+c_1(\overline{\sigma}_A(x))}\big(h(\overline{\sigma}_A(x))\big), \overline{\sigma}_B^{k_1^2(x)}\big(h(\overline{\sigma}_A^2(x))\big)\Big) \in G_B^{s,0}. \tag{8.4}$$

By (8.3) and (8.4), we have

$$\Big(\overline{\sigma}_B^{k_1^2(x)+c_1^2(x)}(h(x)), \overline{\sigma}_B^{k_1^2(x)}\big(h(\overline{\sigma}_A^2(x))\big)\Big) \in G_B^{s,0}.$$

This proves (3.4) for $n = 2$. We can prove (3.4) inductively for general n in a similar fashion, and we can see (i). Assertion (ii) is shown in a similar way to (i). ■

For $x = (x_n)_{n\in\mathbb{Z}} \in \overline{X}_A$, we put

$$x_- = (x_{-n})_{n=0}^{\infty}, \qquad x_+ = (x_n)_{n=0}^{\infty}.$$

Hence we have $(x,z) \in G_A^{s,0}$ (resp. $(x,z) \in G_A^{u,0}$) if and only if $x_+ = z_+$ (resp. $x_- = z_-$).

By Remark 3.3 with Lemma 8.1, we can reformulate asymptotic continuous orbit equivalence in topological Markov shifts in the following way.

Proposition 8.2 *Topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are asymptotically continuous orbit equivalent if and only if there exist a homeomorphism $h\colon \overline{X}_A \to \overline{X}_B$, continuous functions $c_1\colon \overline{X}_A \to \mathbb{Z}$, $c_2\colon \overline{X}_B \to \mathbb{Z}$, and two-cocycle functions $d_1\colon G_A^a \to \mathbb{Z}$, $d_2\colon G_B^a \to \mathbb{Z}$, such that*

(1) $c_1^m(x) + d_1(\overline{\sigma}_A^m(x), \overline{\sigma}_A^m(z)) = c_1^m(z) + d_1(x,z), \quad (x,z) \in G_A^a,\ m \in \mathbb{Z}$,
(2) $c_2^m(y) + d_2(\overline{\sigma}_B^m(y), \overline{\sigma}_B^m(w)) = c_2^m(w) + d_2(y,w), \quad (y,w) \in G_B^a,\ m \in \mathbb{Z}$,

and

(i) *there exists a continuous function* $k_1\colon \overline{X}_A \to \mathbb{Z}_+$ *such that*

$$\overline{\sigma}_B^{k_1(x)+c_1(x)}(h(x))_+ = \overline{\sigma}_B^{k_1(x)}(h(\overline{\sigma}_A(x)))_+,$$
$$\overline{\sigma}_B^{-k_1(x)+c_1(x)}(h(x))_- = \overline{\sigma}_B^{-k_1(x)}(h(\overline{\sigma}_A(x)))_-;$$

(ii) *there exists a continuous function* $k_2\colon \overline{X}_B \to \mathbb{Z}_+$ *such that*

$$\overline{\sigma}_A^{k_2(y)+c_2(y)}(h^{-1}(y))_+ = \overline{\sigma}_A^{k_2(y)}(h^{-1}(\overline{\sigma}_B(y)))_+,$$
$$\overline{\sigma}_A^{-k_2(y)+c_2(y)}(h^{-1}(y))_- = \overline{\sigma}_A^{-k_2(y)}(h^{-1}(\overline{\sigma}_B(y)))_-;$$

(iii) *there exists a continuous function* $m_1\colon G_A^a \to \mathbb{Z}_+$ *such that*

$$\overline{\sigma}_B^{m_1(x,z)+d_1(x,z)}(h(x))_+ = \overline{\sigma}_B^{m_1(x,z)}(h(z))_+ \quad \text{for } (x,z) \in G_A^a,$$
$$\overline{\sigma}_B^{-m_1(x,z)+d_1(x,z)}(h(x))_- = \overline{\sigma}_B^{-m_1(x,z)}(h(z))_- \quad \text{for } (x,z) \in G_A^a;$$

(iv) *there exists a continuous function* $m_2\colon G_B^a \to \mathbb{Z}_+$ *such that*

$$\overline{\sigma}_A^{m_2(y,w)+d_2(y,w)}(h^{-1}(y))_+ = \overline{\sigma}_A^{m_2(y,w)}(h^{-1}(w))_+ \quad \text{for } (y,w) \in G_B^a,$$
$$\overline{\sigma}_A^{-m_2(y,w)+d_2(y,w)}(h^{-1}(y))_- = \overline{\sigma}_A^{-m_2(y,w)}(h^{-1}(w))_- \quad \text{for } (y,w) \in G_B^a;$$

(v) $c_2^{c_1^n(x)}(h(x)) + d_2(\overline{\sigma}_B^{c_1^n(x)}(h(x)), h(\overline{\sigma}_A^n(x))) = n, \qquad x \in \overline{X}_A, n \in \mathbb{Z};$

(vi) $c_1^{c_2^n(y)}(h^{-1}(y)) + d_1(\overline{\sigma}_A^{c_2^n(y)}(h^{-1}(y)), h^{-1}(\overline{\sigma}_B^n(y))) = n, \qquad y \in \overline{X}_B n \in \mathbb{Z};$

(vii) $c_2^{d_1(x,z)}(h(x)) + d_2(\overline{\sigma}_B^{d_1(x,z)}(h(x)), h(z)) = 0, \qquad (x,z) \in G_A^a;$

(viii) $c_1^{d_2(y,w)}(h^{-1}(y)) + d_1(\overline{\sigma}_A^{d_2(y,w)}(h^{-1}(y)), h^{-1}(w)) = 0, \qquad (y,w) \in G_B^a.$

9 Approach from Cuntz–Krieger Algebras

Let $A = [A(i,j)]_{i,j=1}^N$ be an irreducible square matrix with entries in $\{0,1\}$. We assume that A is not any permutation matrix. Let $\{S_i \mid i = 1,\dots,N\}$ be the canonical generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_A$ defined by the matrix A, and similarly let $\{T_j \mid j = 1,\dots,N\}$ be the canonical generating partial isometries of the Cuntz–Krieger algebra $\mathcal{O}_{A^t}$ defined by the transposed matrix A^t of A ([7]). They are the universal unique C^*-algebras subject to the following operator relations, respectively

$$\sum_{j=1}^N S_j S_j^* = 1, \qquad S_i^* S_i = \sum_{j=1}^N A(i,j) S_j S_j^*, \qquad i = 1,\dots,N,$$
$$\sum_{j=1}^N T_j T_j^* = 1, \qquad T_i^* T_i = \sum_{j=1}^N A^t(i,j) T_j T_j^*, \qquad i = 1,\dots,N.$$

In the algebra $\mathcal{O}_A$, the automorphisms $\rho_t^A \in \mathrm{Aut}(\mathcal{O}_A), t \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by $\rho_t^A(S_i) = e^{2\pi\sqrt{-1}t} S_i, i = 1,\dots,N$ yield an action of $\mathbb{T}$ on $\mathcal{O}_A$ is called the *gauge action.* It is well known that the fixed point algebra $(\mathcal{O}_A)^{\rho^A}$ of $\mathcal{O}_A$ under the gauge action ρ^A is an AF-algebra written $\mathcal{F}_A$, whose maximal abelian C^*-subalgebra consisting of diagonal elements is written $\mathcal{D}_A$. For an admissible word $\mu = (\mu_1,\dots,\mu_m) \in B_m(\overline{X}_A)$, we

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denote by S_μ the partial isometry $S_{\mu_1}\cdots S_{\mu_m}$. The C^*-algebra $\mathcal{F}_A$ is generated by partial isometries of the form $S_\mu S_\nu^*$ for $\mu,\nu \in B_m(\overline{X}_A)$, $m=1,2,\dots$, and the C^*-algebra $\mathcal{D}_A$ is generated by projections of the form $S_\mu S_\mu^*$ for $\mu \in B_*(\overline{X}_A)$. Let X_A be the shift space of the right one-sided topological Markov shift (X_A,σ_A), which is defined by the compact Hausdorff space

$$X_A = \left\{ (x_n)_{n\in\mathbb{N}} \in \{1,\dots,N\}^{\mathbb{N}} \mid A(x_n,x_{n+1}) = 1,\ n\in\mathbb{N} \right\}$$

with shift transformation $\sigma_A((x_n)_{n\in\mathbb{N}}) = (x_{n+1})_{n\in\mathbb{N}}$. As in [7, Section 7], the C^*-algebra $\mathcal{D}_A$ is canonically isomorphic to the commutative C^*-algebra $C(X_A)$ of all continuous functions on X_A.

We similarly write the partial isometry $T_{\overline{\xi}} = T_{\xi_k}\cdots T_{\xi_1}$ for $\overline{\xi} = (\xi_k,\dots,\xi_1) \in B_k(\overline{X}_{A^t})$ and the C^*-subalgebras $\mathcal{F}_{A^t},\mathcal{D}_{A^t}$ of $\mathcal{O}_{A^t}$ for the transposed matrix A^t, respectively.

Let us consider the tensor product C^*-algebra $\mathcal{O}_{A^t}\otimes\mathcal{O}_A$. In the algebra $\mathcal{O}_{A^t}\otimes\mathcal{O}_A$, we define the projections

$$E_A = \sum_{j=1}^N T_jT_j^* \otimes S_j^*S_j, \qquad E_{A^t} = \sum_{j=1}^N T_j^*T_j \otimes S_jS_j^*.$$

The projection E_A appeared in Kaminker–Putnam [9, Section 4] in the study of K-theoretic duality between $\mathcal{O}_A$ and $\mathcal{O}_{A^t}$.

Lemma 9.1 (*cf.* [9, Section 4])

$$E_A = E_{A^t}. \tag{9.1}$$

Proof We have

$$E_A = \sum_{i=1}^N T_iT_i^*\otimes S_i^*S_i = \sum_{i=1}^N T_iT_i^*\otimes\Big(\sum_{j=1}^N A(i,j)S_jS_j^*\Big) = \sum_{i=1}^N\sum_{j=1}^N A(i,j)T_iT_i^*\otimes S_jS_j^*,$$

$$E_{A^t} = \sum_{j=1}^N T_j^*T_j\otimes S_jS_j^* = \sum_{j=1}^N\Big(\sum_{i=1}^N A^t(j,i)T_iT_i^*\Big)\otimes S_jS_j^* = \sum_{i=1}^N\sum_{j=1}^N A^t(j,i)T_iT_i^*\otimes S_jS_j^*,$$

thus proving (9.1). ∎

Definition 9.2 (The extended Ruelle algebra for topological Markov shift) We define the C^*-algebra $\mathcal{R}_A^{s,u}$ by

$$\mathcal{R}_A^{s,u} = E_A(\mathcal{O}_{A^t}\otimes\mathcal{O}_A)E_A$$

as a C^*-subalgebra of the tensor product C^*-algebra $\mathcal{O}_{A^t}\otimes\mathcal{O}_A$.

We also define C^*-subalgebras

$$\mathfrak{D}_A^{s,u} = E_A(\mathcal{D}_{A^t}\otimes\mathcal{D}_A)E_A, \qquad \mathfrak{F}_A^{s,u} = E_A(\mathcal{F}_{A^t}\otimes\mathcal{F}_A)E_A.$$

Therefore, we have C^*-subalgebras of $\mathcal{R}_A^{s,u}$

$$\mathfrak{D}_A^{s,u} \subset \mathfrak{F}_A^{s,u} \subset \mathcal{R}_A^{s,u}.$$

For an admissible word $\xi = (\xi_1,\dots,\xi_k) \in B_k(\overline{X}_A)$, we denote by $\overline{\xi}$ the admissible word $(\xi_k,\dots,\xi_1)$ in $\overline{X}_{A^t}$, obtained by reversing the symbols of the word $(\xi_1,\dots,\xi_k)$.

Lemma 9.3 *For* $\mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\overline{X}_A)$, $\overline{\xi} = (\xi_k, \ldots, \xi_1)$, *and* $\overline{\eta} = (\eta_l, \ldots, \eta_1) \in B_*(\overline{X}_{A^t})$, *the following two conditions are equivalent:*

(i) $E_A(T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*) E_A = T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*$;

(ii) $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$.

Proof We have the following equalities:

$$\begin{aligned} E_A(T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*) &= \sum_{i=1}^N T_i^* T_i T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_i S_i^* S_\mu S_\nu^* \\ &= T_{\mu_1}^* T_{\mu_1} T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_{\mu_1} S_{\mu_1}^* S_\mu S_\nu^* \\ &= A^t(\mu_1, \xi_k) T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \\ &= A(\xi_k, \mu_1) T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*. \end{aligned}$$

Similarly, we have

$$(T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*) E_A = A(\eta_l, \nu_1) T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*.$$

Hence the equality $E_A(T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*) E_A = T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*$ holds if and only if $A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1$. ∎

Let us denote by $\mathcal{R}_A^\circ$ the $*$-subalgebra of $\mathcal{R}_A^{s,u}$ linearly spanned by the operators of the form

$$T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \quad \text{for} \quad A(\xi_k, \mu_1) = A(\eta_l, \nu_1) = 1,$$

where

$$\begin{aligned} &\mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\overline{X}_A), \\ &\overline{\xi} = (\xi_k, \ldots, \xi_1), \overline{\eta} = (\eta_l, \ldots, \eta_1) \in B_*(\overline{X}_{A^t}). \end{aligned}$$

Lemma 9.4 $\mathcal{R}_A^\circ$ *is dense in* $\mathcal{R}_A^{s,u}$.

Proof Let $\mathcal{P}_A$ be the $*$-algebra linearly spanned by the operators of the form $S_\mu S_\nu^*$ for $\mu, \nu \in B_*(\overline{X}_A)$. As in [7, Section 2], the algebra $\mathcal{P}_A$ becomes a dense $*$-subalgebra of $\mathcal{O}_A$. We denote by $\mathcal{P}_{A^t} \otimes \mathcal{P}_A$ the linear span of elements

$$T_{\overline{\xi}} T_{\overline{\eta}}^* \otimes S_\mu S_\nu^* \quad \text{for } \mu, \nu \in B_*(\overline{X}_A),\ \overline{\xi}, \overline{\eta} \in B_*(\overline{X}_{A^t}).$$

It becomes a dense $*$-subalgebra of the C^*-algebra of tensor products $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$. For any $Y \in \mathcal{R}_A^{s,u} \subset \mathcal{O}_{A^t} \otimes \mathcal{O}_A$, take $Y_n \in \mathcal{P}_{A^t} \otimes \mathcal{P}_A$ such that $\|Y - Y_n\| \to 0$ as $n \to \infty$. Since

$$\|Y - E_A Y_n E_A\| = \|E_A Y E_A - E_A Y_n E_A\| \le \|Y - Y_n\| \longrightarrow 0$$

as $n \to \infty$, and $E_A Y_n E_A$ belongs to $\mathcal{R}_A^\circ$, we conclude that $\mathcal{R}_A^\circ$ is dense in $\mathcal{R}_A^{s,u}$. ∎

Lemma 9.5 $\mathfrak{D}_A^{s,u}$ *is canonically isomorphic to* $C(\overline{X}_A)$.

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Proof For $\mu = (\mu_1, \ldots, \mu_m), \xi = (\xi_1, \ldots, \xi_k) \in B_*(\overline{X}_A)$ with $A(\xi_k, \mu_1) = 1$, denote by $\xi\mu$ the admissible word $(\xi_1, \ldots, \xi_k, \mu_1, \ldots, \mu_m) \in B_*(\overline{X}_A)$. Let $U_{\xi\mu}$ be the cylinder set of $\overline{X}_A$ defined by

$$U_{\xi\mu} = \{ (x_n)_{n\in\mathbb{Z}} \in \overline{X}_A \mid x_{-(k-1)} = \xi_1, \ldots, x_{-1} = \xi_{k-1}, x_0 = \xi_k, x_1 = \mu_1, \ldots, x_m = \mu_m \}.$$

Since $\mathfrak{D}_A^{s,u} = E_A(\mathcal{D}_{A^t} \otimes \mathcal{D}_A)E_A$ and

$$\mathcal{D}_A = C^*(S_\mu S_\mu^* \mid \mu \in B_*(X_A)), \qquad \mathcal{D}_{A^t} = C^*(T_{\bar{\xi}} T_{\bar{\xi}}^* \mid \xi \in B_*(X_{A^t})),$$

it is straightforward to see that the correspondence

$$T_{\bar{\xi}} T_{\bar{\xi}}^* \otimes S_\mu S_\mu^* \in \mathfrak{D}_A^{s,u} \longrightarrow \chi_{U_{\xi\mu}} \in C(\overline{X}_A)$$

yields an isomorphism between $\mathfrak{D}_A^{s,u}$ and $C(\overline{X}_A)$. ∎

Consider the automorphisms $\gamma^A_{(r,s)} = \rho_r^{A^t} \otimes \rho_s^A$, $(r,s) \in \mathbb{T}^2$ on $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$ for the gauge actions ρ^{A^t} on $\mathcal{O}_{A^t}$ and ρ^A on $\mathcal{O}_A$. Since $\gamma^A_{(r,s)}(E_A) = E_A$, we have an action γ^A of $\mathbb{T}^2$ on $\mathcal{R}_A^{s,u}$. The diagonal action δ_t^A, $t \in \mathbb{T}$ on $\mathcal{R}_A^{s,u}$ is defined by $\delta_t^A = \gamma^A_{(t,t)}$, $t \in \mathbb{T}$. On the other hand, the groupoid C^*-algebra $\mathcal{R}^{s,u}_{\overline{\sigma}_A} = C^*(G_A^{s,u} \rtimes \mathbb{Z}^2)$ of the étale amenable groupoid $G_A^{s,u} \rtimes \mathbb{Z}^2$ has an action $\rho^{s,u}_{\overline{\sigma}_A}$ of $\mathbb{T}^2$ defined in the paragraph right before Theorem 7.9. Its diagonal action $\delta^{\overline{\sigma}_A}$ of $\mathbb{T}$ on $\mathcal{R}^{s,u}_{\overline{\sigma}_A}$ is defined by $\delta_t^{\overline{\sigma}_A} = \rho^{s,u}_{\overline{\sigma}_A,(t,t)}$. Its fixed point algebra $(\mathcal{R}^{s,u}_{\overline{\sigma}_A})^{\delta^{\overline{\sigma}_A}}$ is isomorphic to the asymptotic Ruelle algebra $\mathcal{R}^a_{\overline{\sigma}_A}$ written $\mathcal{R}^a_A$. For the structure of the algebra $\mathcal{R}_A^{s,u}$, we have the following theorem.

Theorem 9.6 *Let A be an irreducible and non-permutation matrix with entries in $\{0,1\}$. Then the C^*-algebra $\mathcal{R}_A^{s,u}$ is a unital, simple, purely infinite, nuclear C^*-algebra isomorphic to the groupoid C^*-algebra $\mathcal{R}^{s,u}_{\overline{\sigma}_A}$ of the étale groupoid $G_A^{s,u} \rtimes \mathbb{Z}^2$. More precisely, there exists an isomorphism $\Phi\colon \mathcal{R}_A^{s,u} \to \mathcal{R}^{s,u}_{\overline{\sigma}_A}$ of C^*-algebras such that*

$$\Phi(\mathfrak{D}_A^{s,u}) = C(\overline{X}_A) \quad \text{and} \quad \Phi \circ \gamma^A_{(r,s)} = \rho^{s,u}_{\overline{\sigma}_A,(r,s)} \circ \Phi, \quad (r,s) \in \mathbb{T}^2. \tag{9.2}$$

In particular, we have $\Phi \circ \delta_t^A = \delta_t^{\overline{\sigma}_A} \circ \Phi$ for $t \in \mathbb{T}$.

Proof Since A is irreducible and not any permutation matrix, the Cuntz–Krieger algebras $\mathcal{O}_A, \mathcal{O}_{A^t}$ are both unital, simple, purely infinite and nuclear ([7, Theorem 2.14]). Hence, so is the algebra $E_A(\mathcal{O}_{A^t} \otimes \mathcal{O}_A)E_A = \mathcal{R}_A^{s,u}$. We will construct an isomorphism $\Phi\colon \mathcal{R}_A^{s,u} \to \mathcal{R}^{s,u}_{\overline{\sigma}_A}$ having the desired properties (9.2). As in [19, 30–32], the right one-sided topological Markov shift (X_A, σ_A) gives rise to an étale groupoid G_A, which is defined by

$$G_A = \{ ((x_i)_{i=1}^\infty, n, (y_j)_{j=1}^\infty) \in X_A \times \mathbb{Z} \times X_A \mid n = l - k, x_{i+k} = y_{i+l}, i = 1, 2, \ldots \}.$$

We have the groupoid G_{A^t} for the transposed matrix A^t in a similar way. It is well known that the groupoids G_A, G_{A^t} are amenable and étale such that their C^*-algebras $C^*(G_A), C^*(G_{A^t})$ are isomorphic to the Cuntz–Krieger algebras $\mathcal{O}_A, \mathcal{O}_{A^t}$, respectively. Let $G_{A^t} \times G_A$ be the direct product of the groupoids so that $C^*(G_{A^t} \times G_A)$ is isomorphic to the tensor product $C^*(G_{A^t}) \otimes C^*(G_A)$ of the groupoid C^*-algebras.

Hence we have a natural isomorphism $\Phi\colon \mathcal{O}_{A^t}\otimes\mathcal{O}_A \to C^*(G_{A^t}\times G_A)$. For elements

$$\big((x_i)_{i=1}^\infty, n, (y_i)_{i=1}^\infty\big)\in G_A \text{ with } n=l-k,\ x_{i+k}=y_{i+l} \text{ for } i\in\mathbb{N},$$
$$\big((x'_j)_{j=1}^\infty, n', (y'_j)_{j=1}^\infty\big)\in G_A \text{ with } n'=l'-k',\ x'_{j+k'}=y'_{j+l'} \text{ for } j\in\mathbb{N}$$

of the groupoid G_A, we assume that $A(x'_1,x_1)=A(y'_1,y_1)=1$. Put $x=(x_i)_{i=1}^\infty, y=(y_i)_{i=1}^\infty$ and $x'=(x'_j)_{j=1}^\infty, y'=(y'_j)_{j=1}^\infty$. We define a bi-infinite sequence $\pi(x',x)=(\pi(x',x)_i)_{i\in\mathbb{Z}}$ by setting

$$\pi(x',x)_i=\begin{cases} x_i & \text{if } i\ge 1,\\ x'_{-i+1} & \text{if } i\le 0.\end{cases}$$

Then $\pi(x',x)$, and similarly $\pi(y',y)$, belong to $\overline{X}_A$. Put $N=\operatorname{Max}\{l+1,l'\}$ and $p=-n, q=n'$. Since

$$\overline{\sigma}_A^p(\pi(x',x))_i=\pi(y',y)_i,\qquad i\ge N,$$
$$\overline{\sigma}_A^q(\pi(x',x))_i=\pi(y',y)_i,\qquad i\le -N,$$

we have

$$\big(\pi(x',x),p,q,\pi(y',y)\big)\in G_A^{s,u}\rtimes\mathbb{Z}^2.$$

Define the subgroupoid $G_{A^t}\times_A G_A$ of $G_{A^t}\times G_A$ by

$$G_{A^t}\times_A G_A=\big\{((x',n',y'),(x,n,y))\in G_{A^t}\times G_A \mid A(x'_1,x_1)=A(y'_1,y_1)=1\big\}.$$

It is easy to see that the correspondence

$$\big((x',n',y'),(x,n,y)\big)\in G_{A^t}\times_A G_A \longrightarrow \big(\pi(x',x),-n,n',\pi(y',y)\big)\in G_A^{s,u}\rtimes\mathbb{Z}^2$$

yields an isomorphism of étale groupoids, so that we can identify $G_{A^t}\times_A G_A$ and $G_A^{s,u}\rtimes\mathbb{Z}^2$ as étale groupoids through the above correspondence. Since $G_{A^t}\times_A G_A$ is a clopen subset of $G_{A^t}\times G_A$, the characteristic function $\chi_{G_{A^t}\times_A G_A}$ of $G_{A^t}\times_A G_A$ on $G_{A^t}\times G_A$ belongs to the C^*-algebra $C^*(G_{A^t}\times G_A)$, which is denoted by P_A. It then follows that the isomorphism $\Phi\colon\mathcal{O}_{A^t}\otimes\mathcal{O}_A\to C^*(G_{A^t}\times G_A)$ satisfies $\Phi(E_A)=P_A$. Hence the restriction of Φ to the subalgebra $E_A(\mathcal{O}_{A^t}\otimes\mathcal{O}_A)E_A$ gives rise to an isomorphism $E_A(\mathcal{O}_{A^t}\otimes\mathcal{O}_A)E_A\to P_AC^*(G_{A^t}\times G_A)P_A$, which is still denoted by Φ. As $P_AC^*(G_{A^t}\times G_A)P_A$ is identified with $C^*(G_A^{s,u}\rtimes\mathbb{Z}^2)$, we have an isomorphism $\Phi\colon\mathcal{R}_A^{s,u}\to\mathcal{R}_{\overline{\sigma}_A}^{s,u}$. It is also described in the following way. For $\mu=(\mu_1,\dots,\mu_m),\nu=(\nu_1,\dots,\nu_n)\in B_*(\overline{X}_A)$ and $\overline{\xi}=(\xi_k,\dots,\xi_1),\overline{\eta}=(\eta_l,\dots,\eta_1)\in B_*(\overline{X}_{A^t})$ with $A(\xi_k,\mu_1)=A(\eta_l,\nu_1)=1$, we know that

$$\big(\phi^{m-n}(x),y\big)\in G_A^{s,|m-n|},\qquad \big(\phi^{l-k}(x),y\big)\in G_A^{u,|l-k|}\quad \text{for } x\in U_{\xi\mu},\ y\in U_{\eta\nu}.$$

Let $\chi_{\xi\mu,\eta\nu}\in C_c(G_A^{s,u}\rtimes\mathbb{Z}^2)$ be the characteristic function of the clopen set

$$U_{\xi\mu,\eta\nu}=\Big\{(x,m-n,l-k,y)\in G_A^{s,u}\rtimes\mathbb{Z}^2 \mid x\in U_{\xi\mu}, y\in U_{\eta\nu},$$
$$\big(\overline{\sigma}_A^m(x),\overline{\sigma}_A^n(y)\big)\in G_A^{s,0},\big(\overline{\sigma}_A^{-k}(x),\overline{\sigma}_A^{-l}(y)\big)\in G_A^{u,0}\Big\}.$$

It is not difficult to see that the correspondence

$$T_{\overline{\xi}}T_{\overline{\eta}}^*\otimes S_\mu S_\nu^*\in\mathcal{R}_A^{s,u}\longrightarrow\chi_{\xi\mu,\eta\nu}\in C_c(G_A^{s,u}\rtimes\mathbb{Z}^2) \tag{9.3}$$

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gives rise to the isomorphism $\Phi\colon \mathcal{R}_A^{s,u} \to C^*(G_A^{s,u} \rtimes \mathbb{Z}^2)(= \mathcal{R}_{\overline{\sigma}_A}^{s,u})$. By (9.3), we easily see that Φ satisfies (9.2). ∎

Corollary 9.7 *The fixed point algebra $(\mathcal{R}_A^{s,u})^{\delta^A}$ of $\mathcal{R}_A^{s,u}$ under the diagonal gauge action δ^A is isomorphic to the asymptotic Ruelle algebra $\mathcal{R}_A^a$.*

Proof The fixed point algebra $(\mathcal{R}_{\overline{\sigma}_A}^{s,u})^{\delta^{\overline{\sigma}_A}}$ of $\mathcal{R}_{\overline{\sigma}_A}^{s,u}$ under $\delta^{\overline{\sigma}_A}$ is isomorphic to the asymptotic Ruelle algebra $\mathcal{R}_A^a$ by Theorem 7.9. Hence the assertion follows from Theorem 9.6. ∎

Put $U_i = T_i^* \otimes S_i$ in $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$ for $i = 1, \dots, N$. We set $U_A = \sum_{i=1}^N U_i$ in $\mathcal{O}_{A^t} \otimes \mathcal{O}_A$.

Lemma 9.8 *U_A is a unitary in $\mathcal{R}_A^{s,u}$, that is, $U_A U_A^* = U_A^* U_A = E_A$.*

Proof We have

$$E_A U_i = \Big(\sum_{j=1}^N T_j^* T_j \otimes S_j S_j^* \Big) U_i = \sum_{j=1}^N T_j^* T_j T_i^* \otimes S_j S_j^* S_i = T_i^* \otimes S_i$$

and similarly $U_i E_A = U_i$, so that we have $U_i \in \mathcal{R}_A^{s,u}$. Since we have $U_i U_i^* = T_i^* T_i \otimes S_i S_i^*$ and $U_i^* U_i = T_i T_i^* \otimes S_i^* S_i$, we see that

$$U_i U_i^* \cdot U_j U_j^* = U_i^* U_i \cdot U_j^* U_j = 0 \quad \text{if } i \neq j.$$

It then follows that

$$U_A^* U_A = \sum_{i=1}^N U_i^* U_i = \sum_{i=1}^N T_i T_i^* \otimes S_i^* S_i = E_A.$$

We have $U_A U_A^* = E_A$ similarly. ∎

Define the inner automorphism α_A of $\mathcal{R}_A^{s,u}$ by setting $\alpha_A = \mathrm{Ad}(U_A)$.

Proposition 9.9 *Let $\Phi\colon \mathcal{R}_A^{s,u} \to \mathcal{R}_{\overline{\sigma}_A}^{s,u}(= C^*(G_A^{s,u} \rtimes \mathbb{Z}^2))$ be the isomorphism defined in Theorem 9.6. Then the restriction $\Phi|_{\mathfrak{D}_A^{s,u}}\colon \mathfrak{D}_A^{s,u} \to C(\overline{X}_A)$ of Φ to the commutative C^*-subalgebra $\mathfrak{D}_A^{s,u}$ satisfies the relation:*

$$\Phi \circ \alpha_A = \overline{\sigma}_A^* \circ \Phi$$

where $\overline{\sigma}_A^*(f) = f \circ \overline{\sigma}_A$ for $f \in C(\overline{X}_A)$.

Proof For $\mu = (\mu_1, \dots, \mu_m), \xi = (\xi_1, \dots, \xi_k) \in B_*(\overline{X}_A)$ with $A(\xi_k, \mu_1) = 1$, we have

$$\begin{aligned} U_A(T_{\overline{\xi}} T_{\overline{\xi}}^* \otimes S_\mu S_\mu^*) U_A^* &= \sum_{i,j=1}^N T_i^* T_{\overline{\xi}} T_{\overline{\xi}}^* T_j \otimes S_i S_\mu S_\mu^* S_j^* \\ &= T_{\xi_1}^* T_{\xi_1} T_{\xi_{k-1}\cdots\xi_1} T_{\xi_{k-1}\cdots\xi_1}^* T_{\xi_1}^* T_{\xi_1} \otimes S_{\xi_k\mu} S_{\xi_k\mu}^* \\ &= T_{\xi_{k-1}\cdots\xi_1} T_{\xi_{k-1}\cdots\xi_1}^* \otimes S_{\xi_k\mu} S_{\xi_k\mu}^*. \end{aligned}$$

This shows that the equality $\Phi \circ \alpha_A = \overline{\sigma}_A^* \circ \Phi$ holds on $\mathfrak{D}_A^{s,u}$. ∎

We note that the unitary $\Phi(U_A)$ in $\mathcal{R}_{\overline{\sigma}_A}^{s,u}$ belongs to the asymptotic Ruelle algebra $\mathcal{R}_{\overline{\sigma}_A}^a$ and it is nothing but the unitary $U_{\overline{\sigma}_A}$ for $(X, \phi) = (\overline{X}_A, \overline{\sigma}_A)$ defined in (6.1).

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Remark 9.10 In [8, Proposition 6.7], C. G. Holton proved that if two primitive matrices A and B are shift equivalent (*cf.* [13]), then the asymptotic Ruelle algebras $\mathcal{R}_A^a$ and $\mathcal{R}_B^a$ are isomorphic by showing that the automorphism α_A induced by the original transformation $\overline{\sigma}_A$ on the AF-algebra $C^*(G_A^a)$ has the Rohlin property.

10 K-theory for the Asymptotic Ruelle Algebras for Full Shifts

In this final section, we will compute the K-groups and the trace values of the asymptotic Ruelle algebras $\mathcal{R}_A^a$ for some topological Markov shifts. In [25](*cf.* [11]), the K-theory formula for the asymptotic Ruelle algebras $\mathcal{R}_A^a$ for the topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ has been provided. In particular, ring and module structure of the K-groups were deeply studied in [11]. We will see the K-groups of the C^*-algebra $\mathcal{R}_A^a$ in a concrete way for full shifts by using the Putnam's formula in [25] which we will describe below. Let A be an $N \times N$ irreducible matrix with entries in $\{0,1\}$. Let us consider the abelian group $H(A)$ of the inductive limit

$$\mathbb{Z}^N \otimes \mathbb{Z}^N \xrightarrow{A^t \otimes A} \mathbb{Z}^N \otimes \mathbb{Z}^N \xrightarrow{A^t \otimes A} \cdots. \tag{10.1}$$

Under a natural identification between $\mathbb{Z}^N \otimes \mathbb{Z}^N$ and the $N \times N$ matrices $M_N(\mathbb{Z})$ over $\mathbb{Z}$, we set $H_k(A) = M_N(\mathbb{Z})$ for $k = 1, 2, \ldots$. Then the map $A^t \otimes A$ in (10.1) goes to the map $\iota_k \colon H_k(A) \to H_{k+1}(A)$ defined by $\iota_k([T,k]) = [ATA, k+1]$ for $[T,k] \in H_k(A)$ with $T \in M_N(\mathbb{Z})$. Define the homomorphism $\alpha_k \colon H_k(A) \to H_{k+1}(A)$ by $\alpha_k([T,k]) = [A^2 T, k+1]$ for $[T,k] \in H_k(A)$, which extends to an endomorphism $\alpha \colon H(A) \to H(A)$. Putnam showed the following K-theory formula by using the six-term exact sequence for K-theory of the C^*-algebra $\mathcal{R}_A^a$.

Proposition 10.1 (Putnam [25, p. 192])

$$K_0(\mathcal{R}_A^a) = \operatorname{Coker}\big(\operatorname{id} - \alpha \colon H(A) \longrightarrow H(A)\big),$$
$$K_1(\mathcal{R}_A^a) = \operatorname{Ker}\big(\operatorname{id} - \alpha \colon H(A) \longrightarrow H(A)\big).$$

We will compute the groups $K_*(\mathcal{R}_A^a)$ for the $N \times N$ matrix $A = \left[\begin{smallmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{smallmatrix}\right]$ with all entries being 1's, so that the topological Markov shift $(\overline{X}_A, \overline{\sigma}_A)$ is the full N-shift written $(\overline{X}_N, \overline{\sigma}_N)$. Let us denote by $\mathcal{R}_N^a$ the asymptotic Ruelle algebra $\mathcal{R}_A^a$ for the matrix A. For a natural number n, $\mathbb{Z}[\frac{1}{n}]$ means the subgroup $\{\frac{m}{n^k} \in \mathbb{R} \mid m, k \in \mathbb{Z}\}$ of the additive group $\mathbb{R}$. We provide the following lemma.

Lemma 10.2 *There exists an isomorphism $\xi \colon H(A) \to \mathbb{Z}[\frac{1}{N^2}]$ of abelian groups such that the diagram*

$$\begin{array}{ccc} H(A) & \xrightarrow{\alpha} & H(A) \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ \mathbb{Z}[\frac{1}{N^2}] & \xrightarrow{\operatorname{id}} & \mathbb{Z}[\frac{1}{N^2}] \end{array}$$

is commutative. Hence $\alpha = \operatorname{id}$ on $H(A)$.

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Proof For a matrix $T = [t_{ij}]_{i,j=1}^N \in M_N(\mathbb{Z})$, define $s_N(T) = \sum_{i,j=1}^N t_{ij}$. As $ATA = s_N(T)A$, the map $s_N\colon M_N(\mathbb{Z})(= H_k(A)) \to \mathbb{Z}$ defines a homomorphism such that $\iota_k([T,k]) = [s_N(T)A, k+1]$ for $T \in M_N(\mathbb{Z})$. For $[T,k],[S,k] \in H_k(A)$, $[T,k]$ and $[S,k]$ define the same element in $H(A)$ if and only if $s_N(T) = s_N(S)$. Define $\widetilde{s}_N\colon H_k(A) \to \mathbb{Z}$ by setting $\widetilde{s}_N([T,k]) = s_N(T)$ for $[T,k] \in H_k(A)$. Since

$$\widetilde{s}_N(\iota_k([T,k])) = s_N(s_N(T)A) = s_N(T)N^2 = N^2\widetilde{s}_N([T,k]),$$

we have the sequences of commutative diagrams:

$$\begin{array}{ccccccccc}
H_1(A) & \xrightarrow{\iota_1} & H_2(A) & \xrightarrow{\iota_2} & H_3(A) & \xrightarrow{\iota_3} & \cdots & \longrightarrow & H(A) \\
\downarrow{\scriptstyle \widetilde{s}_N} & & \downarrow{\scriptstyle \widetilde{s}_N} & & \downarrow{\scriptstyle \widetilde{s}_N} & & & & \downarrow \\
\mathbb{Z} & \xrightarrow{\times N^2} & \mathbb{Z} & \xrightarrow{\times N^2} & \mathbb{Z} & \xrightarrow{\times N^2} & \cdots & \longrightarrow & \mathbb{Z}[\frac{1}{N^2}]
\end{array}$$

and

$$\begin{array}{ccccccccc}
H_1(A) & \xrightarrow{\iota_1} & H_2(A) & \xrightarrow{\iota_2} & H_3(A) & \xrightarrow{\iota_3} & \cdots & \longrightarrow & H(A) \\
\downarrow{\scriptstyle \widetilde{s}_N} & & \downarrow{\scriptstyle \frac{1}{N^2}\widetilde{s}_N} & & \downarrow{\scriptstyle \frac{1}{N^4}\widetilde{s}_N} & & & & \downarrow \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} & \xrightarrow{\mathrm{id}} & \cdots & \longrightarrow & \mathbb{R}.
\end{array}$$

Hence, we can define an isomorphism $\xi\colon H(A) \to \mathbb{Z}[\frac{1}{N^2}] \subset \mathbb{R}$ by setting

$$\xi([T,k]) = \frac{1}{(N^2)^{k-1}}\widetilde{s}_N([T,k]) = \frac{1}{N^{2k-2}}s_N(T) \in \mathbb{Z}[\frac{1}{N^2}] \quad \text{for } [T,k] \in H_k(A).$$

Since $\alpha([T,k]) = [A^2T, k+1]$ and $s_N(A^2T) = N^2 s_N(T)$, we have

$$\xi\big(\alpha([T,k])\big) = \frac{1}{(N^2)^k}s_N(A^2T) = \frac{1}{(N^2)^{k-1}}s_N(T) = \xi([T,k]),$$

so that the isomorphism $\xi\colon H(A) \to \mathbb{Z}[\frac{1}{N^2}]$ satisfies $\xi \circ \alpha = \xi$, and hence we have $\alpha = \mathrm{id}$ on $H(A)$. ∎

As $\mathrm{id} - \alpha$ is the zero map on $H(A)$ with $\mathbb{Z}[\frac{1}{N^2}] = \mathbb{Z}[\frac{1}{N}]$ in $\mathbb{R}$, thus by the formula of Proposition 10.1, we have the following proposition.

Proposition 10.3 (*cf.* [11, Section 3.3]) $K_0(\mathcal{R}_N^a) \cong K_1(\mathcal{R}_N^a) \cong H(A) \cong \mathbb{Z}[\frac{1}{N}]$.

C. G. Holton proved that if an $N \times N$ matrix A is aperiodic, then the shift $\overline{\sigma}_N^*$ on the AF-algebra $C^*(G_A^a)$ has the Rohlin property [8, Theorem 6.1]. For the $N \times N$ matrix $A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$, the algebra $C^*(G_N^a)$, which is the C^*-algebra of the groupoid G_A^a is the UHF algebra of type N^∞, so that the crossed product $\mathcal{R}_N^a = C^*(G_N^a) \rtimes_{\overline{\sigma}_N^*} \mathbb{Z}$ is a simple A$\mathbb{T}$-algebra of real rank zero with a unique tracial state by [4, Theorem 1.1], [12, Theorem 1.3]. The unique tracial state on $\mathcal{R}_N^a$ is denoted by τ_N. It arises from the Parry measure on the full N-shift $(\overline{X}_N, \overline{\sigma}_N)$ (Putnam [25, Theorem 3.3]). We can determine the trace values of the K_0-group in the following way.

Lemma 10.4 $\tau_{N*}(K_0(\mathcal{R}_N^a)) = \mathbb{Z}[\frac{1}{N}]$ *in* $\mathbb{R}$.

Proof By Corollary 9.7, the algebra $\mathcal{R}_N^a$ is realized as the fixed point algebra of $\mathcal{R}_N^{s,u}$ under the diagonal gauge action. It is easy to see that $\mathcal{R}_N^a$ is generated by linear span of operators of the form $T_{\overline{\xi}}T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*$ for $\mu = (\mu_1, \dots, \mu_m), \nu = (\nu_1, \dots, \nu_n) \in B_*(\overline{X}_A)$, $\overline{\xi} = (\xi_k, \dots, \xi_1), \overline{\eta} = (\eta_l, \dots, \eta_1) \in B_*(\overline{X}_{A^t})$ such that $k + m = l + n$. Since the tracial state τ_N on $\mathcal{R}_N^a$ comes from the Parry measure on $\overline{X}_N$, we have

$$\tau_N(T_{\overline{\xi}}T_{\overline{\eta}}^* \otimes S_\mu S_\nu^*) = \begin{cases} \frac{1}{N^{k+m}} & \text{if } \overline{\xi} = \overline{\eta},\ \mu = \nu, \\ 0 & \text{otherwise.} \end{cases}$$

Through the six-term exact sequence

$$\begin{array}{ccccc} K_0(C^*(G_N^a)) & \xrightarrow{\ \mathrm{id}-\alpha\ } & K_0(C^*(G_N^a)) & \xrightarrow{\ \mathrm{id}\ } & K_0(\mathcal{R}_N^a) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{R}_N^a) & \xleftarrow{\ \mathrm{id}\ } & K_1(C^*(G_N^a)) & \xleftarrow{\ \mathrm{id}-\alpha\ } & K_1(C^*(G_N^a)) \end{array}$$

for the crossed product $\mathcal{R}_N^a = C^*(G_N^a) \rtimes \mathbb{Z}$ with the fact $K_1(C^*(G_N^a)) = 0$ and $\alpha = \mathrm{id}$, all elements of $K_0(\mathcal{R}_N^a)$ come from those of $K_0(C^*(G_N^a)) = H(A)$. We thus conclude that $\tau_{N*}(K_0(\mathcal{R}_N^a)) = \mathbb{Z}[\frac{1}{N}]$. ∎

For two natural numbers $1 < M, N \in \mathbb{N}$, let $M = p_1^{k_1} \cdots p_m^{k_m}$, $N = q_1^{l_1} \cdots q_n^{l_n}$ be the prime factorizations of M, N such that $p_1 < \cdots < p_m$, $q_1 < \cdots < q_n$ and $k_1, \dots, k_m, l_1, \dots, l_n \in \mathbb{N}$, respectively.

Proposition 10.5 *Keeping the above notation, the following assertions are equivalent.*

(i) *The Ruelle algebras* $\mathcal{R}_M^a$ *and* $\mathcal{R}_N^a$ *are isomorphic.*
(ii) $\mathbb{Z}[\frac{1}{M}] = \mathbb{Z}[\frac{1}{N}]$ *as subsets of values of* $\mathbb{R}$.
(iii) $\{p_1, \dots, p_m\} = \{q_1, \dots, q_n\}$, *that is,* $m = n$ *and* $p_1 = q_1, \dots, p_m = q_n$.

Proof (i) ⇒ (ii): Since the Ruelle algebras $\mathcal{R}_M^a$ and $\mathcal{R}_N^a$ have unique tracial state, respectively, the assertion follows from the preceding lemma.

(ii) ⇒ (i): The algebras $\mathcal{R}_M^a, \mathcal{R}_N^a$ are both A$\mathbb{T}$-algebras of real rank zero with unique tracial state. The condition $\mathbb{Z}[\frac{1}{M}] = \mathbb{Z}[\frac{1}{N}]$ implies that their K-theoretic dates

$$\big(K_0(\mathcal{R}_M^a), K_0(\mathcal{R}_M^a)_+, [1], K_1(\mathcal{R}_M^a)\big) = \big(K_0(\mathcal{R}_N^a), K_0(\mathcal{R}_N^a)_+, [1], K_1(\mathcal{R}_N^a)\big)$$

coincide because of Proposition 10.3 and Lemma 10.4. By a general classification theory of simple A$\mathbb{T}$-algebras of real rank zero, we conclude that the Ruelle algebras $\mathcal{R}_M^a$ and $\mathcal{R}_N^a$ are isomorphic.

The equivalence (ii) ⇔ (iii) is easy. ∎

We have the following corollary.

Corollary 10.6 *Let $M = p_1^{k_1} \cdots p_m^{k_m}$ and $N = q_1^{l_1} \cdots q_n^{l_n}$ be the prime factorizations of M, N as in the above proposition. If the sets $\{p_1, \dots, p_m\}$ and $\{q_1, \dots, q_n\}$ do not coincide with each other, then the two-sided full shifts $(\overline{X}_M, \overline{\sigma}_M)$ and $(\overline{X}_N, \overline{\sigma}_N)$ are not asymptotically continuous orbit equivalent.*

Proof Suppose that $\{p_1, \dots, p_m\} \neq \{q_1, \dots, q_n\}$. By the above proposition, the Ruelle algebras $\mathcal{R}_N^a, \mathcal{R}_M^a$ are not isomorphic. Since the isomorphism class of the Ruelle algebra is invariant under asymptotic continuous orbit equivalence by Theorem 5.7, we know that $(\overline{X}_M, \overline{\sigma}_M)$ and $(\overline{X}_N, \overline{\sigma}_N)$ are not asymptotically continuous orbit equivalent. ∎

11 Concluding Remarks

Before ending the paper, we refer to differences among asymptotic continuous orbit equivalence, asymptotic conjugacy and topological conjugacy of Smale spaces. It can be proved that topological conjugacy implies asymptotic conjugacy, which implies asymptotic continuous orbit equivalence. For an irreducible Smale space (X, ϕ), its inverse system (X, ϕ^{-1}) automatically becomes an irreducible Smale space by definition. We then see the following proposition.

Proposition 11.1 *An irreducible Smale space (X, ϕ) is asymptotically continuous orbit equivalent to its inverse (X, ϕ^{-1}).*

Proof In Definition 3.2, we set $Y = X, \psi = \phi^{-1}$ and take $h = \mathrm{id}$, $c_1 \equiv -1$, $c_2 \equiv -1$, $d_1 \equiv 0$, $d_2 \equiv 0$. We then see that $c_1^n(x) = -n$ for all $x \in X$ and $c_2^n(y) = -n$ for all $y \in Y$. It is direct to see that all conditions in Definition 3.2 hold for these c_1, c_2, d_1, d_2, so that (X, ϕ) is asymptotically continuous orbit equivalent to its inverse (X, ϕ^{-1}). ∎

We can easily explain the above situation in terms of C^*-algebras. We actually see that the identity map $\mathrm{id}\colon X \to X$ induces an isomorphism $\Phi\colon \mathcal{R}_\phi^a \to \mathcal{R}_{\phi^{-1}}^a$ of C^*-algebras such that

$$\Phi\big(C(X)\big) = C(X) \quad \text{and} \quad \Phi \circ \rho_t^{\phi} = \rho_{-t}^{\phi^{-1}} \circ \Phi,$$

because in Theorem 1.1(iii), we may have.

$$\mathrm{Ad}\left(U_t(c_{\phi^{-1}})\right) = \rho_{-t}^{\phi^{-1}}, \qquad \mathrm{Ad}\left(U_t(c_{\phi})\right) = \rho_t^{\phi}.$$

Corollary 11.2 *There exists a pair (X, ϕ) and (Y, ψ) of irreducible Smale spaces such that they are asymptotically continuous orbit equivalent but not topologically conjugate.*

Proof As in [13, Example 7.4.19], the matrix $A = \left[\begin{smallmatrix} 19 & 5 \\ 4 & 1 \end{smallmatrix}\right]$ is not shift equivalent to its transpose $A^t = \left[\begin{smallmatrix} 19 & 4 \\ 5 & 1 \end{smallmatrix}\right]$. Let (X, ϕ) and (Y, ψ) be the shifts of finite type defined by the matrices A and A^t, respectively. Since (Y, ψ) is naturally topologically conjugate to (X, ϕ^{-1}), the Smale spaces (X, ϕ) and (Y, ψ) are asymptotically continuous orbit equivalent by the preceding proposition. As shift equivalence relation of matrices is weaker than strong shift equivalence, by Williams' theorem [38] the shifts of finite type (X, ϕ) and (Y, ψ) are not topologically conjugate. ∎

In the recent paper [18], which is a continuation of this paper, the author shows that two-sided topological Markov shifts are topologically conjugate if and only if they are asymptotically conjugate. Hence the example in the proof of Corollary 11.2 shows us that there exists a pair (X, ϕ) and (Y, ψ) of irreducible Smale spaces such that they are asymptotically continuous orbit equivalent but not asymptotically conjugate. For a general irreducible Smale space, however, we do not know whether or not the asymptotic conjugacy implies topological conjugacy. This is an open question probably being affirmative.

We finally remark the following. We know that if two irreducible topological Markov shifts are asymptotically continuous orbit equivalent, then their asymptotic Ruelle algebras are isomorphic by Theorem 5.7, since these asymptotic Ruelle algebras $\mathcal{R}_A^a$ have unique tracial states τ_A coming from the Parry measures on the shift spaces. Hence, the trace values $\tau_{A*}(K_0(\mathcal{R}_A^a))$ are invariant under asymptotic continuous orbit equivalence. For two matrices

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

it is straightforward to see that $\tau_{A*}(K_0(\mathcal{R}_A^a)) \neq \tau_{B*}(K_0(\mathcal{R}_B^a))$ as subsets of $\mathbb{R}$, because $\tau_{B*}(K_0(\mathcal{R}_B^a))$ contains the trace values of the dimension group of the AF-algebra defined by the matrix B. Hence we know that the two-sided topological Markov shifts $(\overline{X}_A, \overline{\sigma}_A)$ and $(\overline{X}_B, \overline{\sigma}_B)$ are not asymptotically continuous orbit equivalent, whereas their one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are continuous orbit equivalent as in [14, Section 5].

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Department of Mathematics, Joetsu University of Education, Joetsu 943-8512, Japan
Email: kengo@juen.ac.jp