ON FORMULAE OF MACBEATH AND HUSSEIN

by W. W. STOTHERS

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1. Introduction. In his thesis [**1**], Hussein considered regular permutations of order 2 and 3 in S_n whose product is an n-cycle. For such a pair, we must have

$$n = 6(2g-1) \tag{1}$$

for some $g \geqslant 1$. Such a permutation pair corresponds to a free cycloidal subgroup of the classical modular group (see, e.g., [**3**]). Previously the free subgroups and the cycloidal subgroups of fixed genus had been enumerated ([**4**], [**5**]).

Hussein showed that $C(g)$, the *number* of such subgroups, is given by

$$C(g) = \frac{n!}{2^{n/2}3^{n/3}(n/2)!\,(n/3)} \sum_{r=1}^{n-1} (-1)^r a_r b_r / d_r, \tag{2}$$

where d_r is the binomial coefficient ${}^{n-1}C_r$, and a_r and b_r are given by

$$(x+1)(x^2-1)^{-1+n/2} = \sum_{r=1}^{n-1} a_r x^{n-r-1}, \tag{3}$$

$$(x^2+x+1)(x^3-1)^{-1+n/3} = \sum_{r=0}^{n-1} b_r x^{n-r-1}. \tag{4}$$

Macbeath [**2**] noted that, for the first few values of g, $C(g)$ was given by the formula

$$\frac{(6g-3)!}{g!\,(3g-2)!\,2^{2g-1}3^g}. \tag{5}$$

Hussein checked this for $g \leqslant 17$. The first few values are

$$\begin{aligned} C(1) &= 1 \\ C(2) &= 105 \\ C(3) &= 50050 \\ C(4) &= 56581525 \\ C(5) &= 117123756750. \end{aligned} \tag{6}$$

We show that formula (5) is correct for all g.

Most of what follows was obtained by interactive use of an algebra system (muMATH). Although the final proof *could* be undertaken by hand, it would be misleading to disguise the rôle of the machine in the enterprise.

2. Hussein's formula. We note that Hussein's a_r, b_r, c_r are functions of n as well as of r. We will rewrite the sum in terms of functions of a single variable.

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Six simple calculations show that, for the cases $r \equiv 0,1,2,3,4,5 \pmod 6$,

$$(-1)^r a_r b_r / d_r = (-1)^{r+[r/2]+[r/3]} \alpha(r)\alpha(n-1-r) \tag{7}$$

where

$$\alpha(r) = r!/([r/2]!)([r/3]!). \tag{8}$$

For $s = 0, 1, \ldots, 5$, put

$$\beta(s, k) = \alpha(6k+s). \tag{9}$$

Now the *sum* in (2) can be split into sums over residue classes modulo 6.

$$\begin{aligned} S(g) &= \sum_{r=0}^{n-1} (-1)^{r+[r/2]+[r/3]} \alpha(r)\alpha(n-1-r) \\ &= \sum_{k=0}^{2g-2} (-1)^k \sum_{s=0}^{5} \varepsilon(s)\beta(s, k)\beta(5-s, 2g-2-k) \end{aligned} \tag{10}$$

where $\varepsilon(0) = \varepsilon(5) = 1$, $\varepsilon(1) = \varepsilon(2) = \varepsilon(3) = \varepsilon(4) = -1$. Further, the term $\beta(s, k)\beta(5-s, 2g-2-k)$ is unchanged if we replace s by $5-s$ and k by $(2g-2-k)$, so

$$S(g) = 2 \sum_{k=0}^{2g-2} (-1)^k \sum_{s=0}^{2} \varepsilon(s)\beta(s, k)\beta(5-s, 2g-2-k). \tag{11}$$

Now consider (8) for each residue class modulo 6 to get

$$\left.\begin{aligned} &\beta(1, k) = (6k+1)\beta(0, k) \\ &\beta(2, k) = 2\beta(1, k) \\ &\beta(3, k) = 3\beta(2, k) = 6\beta(1, k) \\ &\beta(4, k) = 2\beta(3, k) = 12\beta(1, k) \\ &\beta(5, k) = (6k+5)\beta(4, k) = 12(6k+5)\beta(1, k). \end{aligned}\right\} \tag{12}$$

Then (11) can be rewritten as

$$S(g) = 72 \sum_{k=0}^{2g-2} (-1)^k (4g-6k-3)\beta(0, k)\beta(1, 2g-2-k). \tag{13}$$

Of course, after (12), the $\beta(1, -)$ in (13) could be replaced but this does not seem to help. It is our aim to establish recurrence relations for $S(g)$.

3. Recurrence relations and power series. Since

$$\beta(0, k) = (6k)!/(2k)!\,(3k)!, \tag{14}$$

calculation of $\beta(0, k+1)/\beta(0, k)$ gives

$$(k+1)\beta(0, k+1) = 12(6k+1)(6k+5)\beta(0, k). \tag{15}$$

Similarly,

$$(k+1)\beta(1, k+1) = 12(6k+7)(6k+5)\beta(1, k). \tag{16}$$

We now express all our information using *formal* power series [the series involved converge only at $z=0$, so no analytic significance should be understood]. Put

$$f(z)=\sum_{s=0}^{\infty}(-1)^s\beta(0,s)z^s, \tag{17}$$

$$g(z)=\sum_{s=0}^{\infty}(-1)^s\beta(1,s)z^s. \tag{18}$$

To include the sum on the right of (13), we put

$$\gamma(k)=\sum_{s=0}^{k}(-1)^s(2k+1-6s)\beta(0,s)\beta(1,k-s) \tag{19}$$

and define $h(z)$ by

$$h(z)=\sum_{k=0}^{\infty}\gamma(k)z^k. \tag{20}$$

To recover the values for *even* arguments, we put

$$\begin{aligned} t(z)&=h(z)+h(-z)\\ &=2\sum_{g=1}^{\infty}\gamma(2g-2)z^{2g-2}. \end{aligned} \tag{21}$$

We now define S by

$$S(g)=\frac{(2g-1)(6g-3)!}{g!\,(3g-2)}2^{4g-4}3^{3g-4}, \tag{22}$$

and put

$$u(z)=\sum_{g=1}^{\infty}S(g)z^{2g-2}. \tag{23}$$

Macbeath's assertion now becomes (in view of (2), (10), (13), (19), (20), (21))

$$u(z)=t(z). \tag{24}$$

Note that, considering $S(g+1)/S(g)$,

$$(2g-1)(g+1)S(g+1)=2^7 3^4(2g+1)^2(6g-1)(6g+1)S(g). \tag{25}$$

4. Relations among derivatives From (12) and (17)–(20),

$$s(z)=2zg'(z)f(z)-4zg(z)f'(z)+f(z)g(z). \tag{26}$$

Also, from (12), (15), (17) and (18),

$$6zf'(z)=g(-z)-f(z). \tag{27}$$

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Similarly, from (12), (16), (17) and (18),

$$432z^2g'(z) = -360zg(z) + g(z) - f(-z). \tag{28}$$

Now $t(z)$ and $u(z)$ are even functions, so we define

$$\left.\begin{aligned} F(z) &= f(z)f(-z) \\ G(z) &= g(z)g(-z) \\ M(z) &= f(z)g(z) - f(-z)g(-z) \\ P(z) &= f(z)g(z) + f(-z)g(-z). \end{aligned}\right\} \tag{29}$$

Then, from (26), (27), (28), (29),

$$216zt(z) = M(z) - 288zG(z). \tag{30}$$

Also, from (26), . . . , (29),

$$\begin{aligned} 6zF'(z) &= P(z) - 2F(z) \\ 432z^2G'(z) &= -720zG(z) + M(z) \\ 432z^2M'(z) &= P(z) - 2F(z) - 432zM(z) \\ 432z^2P'(z) &= 144zG(z) - 432zP(z) + M(z). \end{aligned} \tag{31}$$

Observe that the first and third of these yield

$$432z(zM(z))' = 6zF'(z). \tag{32}$$

Hence, on comparing constant terms,

$$72zM(z) = F(z) - 1. \tag{33}$$

Also, using (30) and (31) we can express *any* derivative of $t(z)$ as a linear combination of $F(z)$, $G(z)$, $M(z)$ and $P(z)$ with coefficients rational functions of z. For example,

$$\begin{aligned} t^{(3)}(z) = {} & \left[\frac{1}{7776z^4}F(z) + \frac{1}{279936z^4}\right] - \frac{160}{27z^2}G(z) \\ & + \left[\frac{1}{40310784z^5} + \frac{41}{972z^3}\right]M(z) - \frac{7}{93312z^4}P(z). \end{aligned} \tag{34}$$

Using (33) and expressions for $t^{(0)}(z), \ldots, t^{(3)}(z)$, we *could* eliminate F, G, M, P to get a fourth order linear differential equation for $t(z)$ (with polynomial coefficients). [In fact this was done, but the output was of little use since we have an easier alternative. Without Macbeath's guess, however, this would have yielded a recurrence relation of considerable length for the $\gamma(2g-2)$.]

Recall that the $S(g)$ (a simple multiple of the conjectured $C(g)$) satisfies the recurrence relation (25). Then we can construct a differential equation for its generating function $u(z)$ (see (23)).

We consider the coefficient of z^{2g} ($g \geq 1$) and the constant terms of each side to see

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that

$$z^2u^{(2)}(z)+2zu^{(1)}(z)-2u(z)+4$$
$$=12^4z^2(9z^4u^{(4)}(z)+144z^3u^{(3)}(z)+665u^{(2)}(z)+965zu^{(1)}(z)+315u(z)). \quad (35)$$

Now to prove $t(z)=u(z)$ we need only check that the function $t(z)$ satisfies (35) and that the first four coefficients agree. Since the derivatives of $t(z)$ can be calculated in terms of F, G, M, P, this is a simple (machine) verification. The comparison of the coefficients was essentially done by Hussein and the results are implicit in (6).

5. Postscript. We observe that a recurrence for the $C(g)$ could have been obtained without Macbeath's conjectured value but, as indicated in §4, this would have been much more complicated than that implicit in (25).

Finally, we make an observation which might stimulate some further (experimental or theoretical) work. If we use the relation between subgroups and permutation pairs in [**3**, Lemma 1.2], then the number of regular permutations of order 2 and 3 in S_n whose product is an n-cycle is (from (5))

$$\frac{(12g-7)!\,(6g-3)!}{g!\,(3g-2)!\,2^{2g-1}3g}. \quad (36)$$

On the other hand, the number of regular permutations of order 2 in S_n is

$$\frac{(12g-6)!}{2^{6g-3}(6g-3)!} \quad (37)$$

and the number of order 3 is

$$\frac{(12g-6)!}{3^{4g-2}(4g-2)!}. \quad (38)$$

Since $n=6(2g-1)$ is congruent to 2 modulo 4, the product of such permutations must be odd. Now, we have $\frac{1}{2}(12g-6)!$ odd permutations, of which $(12g-7)!$ are n-cycles. Hence the "expected" number of pairs giving rise to an n-cycle is (from (37) and (38)),

$$\frac{(12g-7)!\,(12g-6)!}{(6g-3)!\,(4g-2)!\,2^{6g-2}3^{4g-3}}. \quad (39)$$

A tedious calculation using Stirling's formula shows that, asymptotically, (36) and (39) agree. It would be interesting to investigate whether this occurs for other cycle patterns.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF GLASGOW
GLASGOW G12 8QW

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