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Epilogue: quantum field theory

Where does this description of matter and radiation need to go next? The answer is that it needs to include interactions between different physical fields and between different excitations in the same field.

In order to pursue this course, one needs to extend the simple linear response analysis of classical field theory to a non-linear response analysis. In the presence of interactions, linear response is only a first-order approximation to the response of the field to a source. Interactions turn the fields themselves into sources (sources for the fields themselves and others to scatter from). Non-linear response theory requires quantum field theory, because the products of fields, which arise in interactions, bring up the issue of the ordering of fields, which only the second quantization can resolve. It means the possibility of creation and annihilation of particle pairs and negative probabilities, which the anti-particle concept and the vacuum concept repair the consistency.

An area which has not been touched upon in this book is that of Grassman variables [136], which describe fermionic interactions. These arose fleetingly in connection with TCP invariance (see section 10.5). In interacting theories, one needs to account for their anti-commuting properties.

A full discussion of quantum field theory, and all of its computational algorithms, is far beyond the scope of this book. The purpose of this chapter is only to indicate briefly how the quantum theory of fields leads to more of the same. As usual, the most pleasing way to derive corrections to the classical theory within a dynamically complete framework is, of course, through an action principle. Schwinger's generalization of the action principle, for quantum field theory, provides the most economical and elegant transition from the classical to the quantum universe. It leads, amongst other things, to the so-called *effective action* for quantum field theory. This effective action demonstrates, most forcibly, the way in which quantum field theory completes the cause–effect response theory we have used in previous chapters.

15.1 Classical loose ends

In classical physics, few problems may be described by single-particle equations. In many-particle theories, one normally invokes the continuum hypothesis and turns to effective equations in order to study bulk quantities. The same strategy is more difficult in the quantum theory, since an effective theory of the quantum nature is less obvious. Quantum field theory is a theory of multiple quanta, or discrete excitations within a field. More than that, it is a theory of multiple *identical* quanta. This is helpful in overcoming some of the problems with classical field theory.

Quantum mechanics deals primarily with one-particle systems. In quantum mechanics, many-particle systems must be handled as discrete N-body problems. Identical particles must be handled by cumbersome Slater determinants, or explicit symmetrization. In quantum field theory one replaces this with a continuum theory of field operators, subject to algebraic constraints.

It is this last point which leads one to modify quantum mechanics. Instead of trying to symmetrize over wavefunctions for identical particles, one uses the normalization properties of the wavefunction to generate the required multiplicity. The identical nature of the particles then follows, subject to certain restrictions on the spacetime symmetries of the fields. In particular, it is necessary to specify the topology of the field with respect to interchanges of particles. The Pauli principle, in particular, places a strong constraint on the spacetime properties of the field.

Finally, the existence of negative energy states requires additional assumptions to avoid the problem of decay. The anti-particle concept and the vacuum concept (the existence of a lowest possible state) have a formal expression in quantum field theory, but have to be put in by hand in a classical theory.

15.2 Quantum action principle

Schwinger has introduced an action principle for quantum mechanics [115, 120] which turns out to be equivalent to path integral formulations [47]. In quantum field theory one is interested in computing transition or scattering amplitudes of the form

$$\langle s_2|s_1\rangle, \tag{15.1}$$

where the states denoted by s_1 and s_2 are assumed to be complete at each arbitrary time, so that

$$\langle s_1|s_2\rangle = \int \langle s_1, t_1|\alpha, t\rangle \, \mathrm{d}\alpha \, \langle \alpha, t|s_2, t_2\rangle. \tag{15.2}$$

Schwinger's quantum action principle states that

$$\delta\langle s_2|s_1\rangle = \frac{\mathrm{i}}{\hbar}\langle s_2|\delta S_{12}|s_1\rangle \tag{15.3}$$

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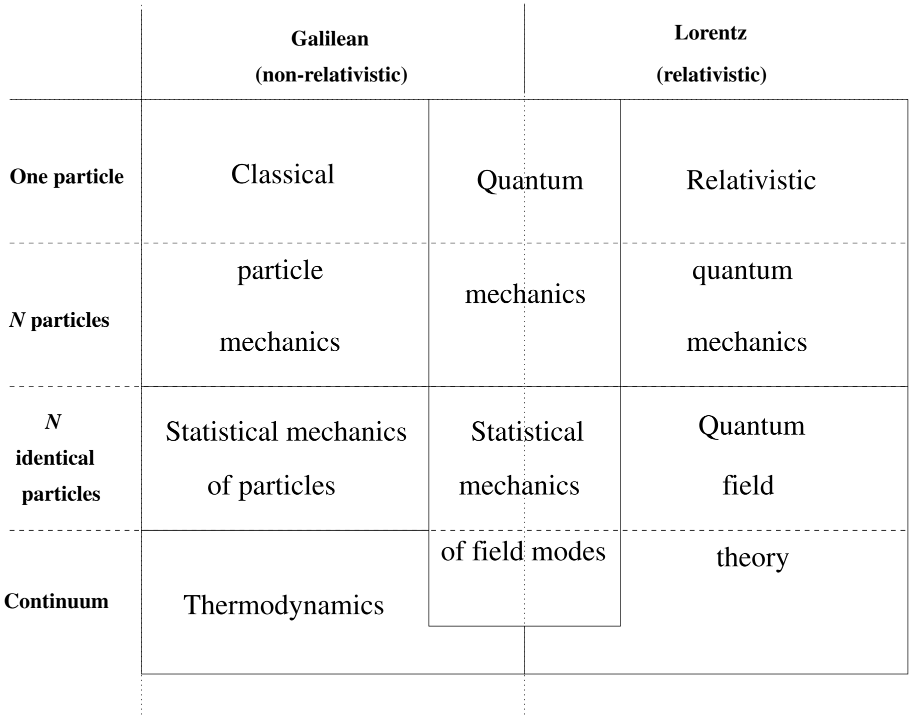

Fig. 15.1. Overlap between classical and quantum and statistical theories.

where S is the action operator, obtained by replacing the classical fields ϕ by field operators ϕ. The form of the action is otherwise the same as in the classical theory (which is the great bonus of this formulation). Specifically,

$$S_{12} = \int_{\sigma_1}^{\sigma_2} (\mathrm{d}x)\ \mathcal{L}. \tag{15.4}$$

Since operators do not necessarily commute, one must adopt a specific operator-ordering prescription which makes the action self-adjoint

$$S^\dagger = S. \tag{15.5}$$

The action should also be symmetrical with respect to time reversals, as in the classical theory. Central to quantum field theory is the idea of 'unitarity'. This ensures the reversibility of physical laws and the conservation of energy. In view of the property expressed by eqn. (15.2), successive variations of the amplitude with respect to a source,

$$S \to S - \int (\mathrm{d}x)\ J\phi, \tag{15.6}$$

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lead automatically to a time ordering of the operators:

$$\begin{aligned}
\frac{\delta}{\delta J(x)}\langle s_2|s_1\rangle &= \frac{\mathrm{i}}{\hbar}\langle s_2|\phi(x)|s_1\rangle \\
&= \int \frac{\mathrm{i}}{\hbar}\langle s_2|\alpha, x\rangle \mathrm{d}\alpha \langle \alpha, x|\phi(x)|s_1\rangle \\
\frac{\delta^2}{\delta J(x')\delta J(x)}\langle s_2|s_1\rangle &= \frac{\delta}{\delta J(x')}\langle s_2|x\rangle \frac{\mathrm{i}}{\hbar}\langle x|\phi(x)|s_1\rangle \\
&= \int \left(\frac{\mathrm{i}}{\hbar}\right)^2 \langle s_2|\phi(x')|\alpha, x\rangle \mathrm{d}\alpha \langle \alpha, x|\phi(x)|s_1\rangle \\
&= \left(\frac{\mathrm{i}}{\hbar}\right)^2 \langle s_2|T\phi(x')\phi(x)|s_1\rangle,
\end{aligned} \tag{15.7}$$

where the T represents time ordering of the field operators. The classical limit of the action principle is taken by letting $\hbar \to 0$, from which we obtain $\delta S = 0$. Thus, only the operator equations of motion survive. The amplitude is suppressed, and thus so are the states. This makes the operator nature of the equations of motion unimportant.

15.2.1 Operator variations

The objects of variation, the fields, are now operators in this formulation, so we need to know what the variation of an operator means. As usual, this can be derived from the differential generating structure of the action principle.

It is useful to distinguish between two kinds of variation: variations which lead to unitary transformations of the field on a spacelike hyper-surface, and variations which are dynamical, or are orthogonal to, such a hyper-surface. Suppose we consider an infinitesimal change in the state $|s_1\rangle$, with generator G,

$$|s_2\rangle \to |s_2\rangle + \delta|s_2\rangle = (1 - \mathrm{i}G)|s_2\rangle, \tag{15.8}$$

where G is a generator of infinitesimal unitary transformations $U = \mathrm{e}^{\mathrm{i}G}$, such that

$$U^\dagger U = 1. \tag{15.9}$$

Note that the transformation is the first term in an expansion of $\mathrm{e}^{-\mathrm{i}G}$. Then we have

$$\begin{aligned}
\delta|a\rangle &= -\mathrm{i}G|a\rangle \\
\delta\langle a| &= \langle a|\mathrm{i}G.
\end{aligned} \tag{15.10}$$

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So if F is any unitary operator, then the change in its value under this unitary variation can be thought of as being due to the change in the states, as a result of the unitary generator G:

$$\langle a|F'|b\rangle = \langle a'|(1+\mathrm{i}G)F(1-iG)|b'\rangle + \cdots, \tag{15.11}$$

which is the first terms in the infinitesimal expansion of

$$\langle a|F'|b\rangle = \langle a'|\mathrm{e}^{\mathrm{i}G}F\mathrm{e}^{-\mathrm{i}G}|b'\rangle. \tag{15.12}$$

Eqn. (15.11) can be likened to what one would expect to be the definition of variation in the operator

$$\langle a|F'|b\rangle = \langle a'|F+\delta F|b'\rangle, \tag{15.13}$$

in order to define the infinitesimal variation of an operator,

$$\delta F = -\mathrm{i}\,[F, G]\,. \tag{15.14}$$

15.2.2 Example: operator equations of motion

The same result can also be obtained from Hamilton's equations for dynamical changes. Consider variations in time. The generator of time translations is the Hamiltonian

$$\delta_t F = \left(\frac{\partial F}{\partial t} - \frac{\mathrm{d}F}{\mathrm{d}t}\right)\delta t, \tag{15.15}$$

since

$$\delta F = F(t+\delta t) - F(t). \tag{15.16}$$

(The numerical value of t is not affected by the operator transformation.) Hence, using our definition, we obtain

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial t} - \mathrm{i}[F, H], \tag{15.17}$$

which is the time development equation for the operator F.

15.3 Path integral formulation

Feynman's famous path integral formulation of quantum field theory can be thought of as an integral solution to the differential Schwinger action principle in eqn. (15.3). To see this, consider supplementing the action S with a source term [22, 125, 128]

$$S \to S - \int (\mathrm{d}x)\; J\phi. \tag{15.18}$$

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The operator equations of motion are now

$$\frac{\delta S}{\delta \phi} = J \equiv E[\phi], \tag{15.19}$$

and we define $E[\phi]$ as the operator one obtains from the first functional derivative of the action operator. From the Schwinger action principle, we have,

$$\frac{\delta^n}{\delta J^n}\langle 0|0\rangle = \left(\frac{-\mathrm{i}}{\hbar}\right)^n \langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle\Big|_{J=0}, \tag{15.20}$$

where the T indicates time ordering implicit in the action integral. This may be summarized (including causal operator ordering) by

$$\langle 0|0\rangle_J = \left\langle 0\left|T\exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}V_t J\phi\right)\right|0\right\rangle_J. \tag{15.21}$$

We may now use this to write the operator $E[\phi]$ in terms of functional derivatives with respect to the source,

$$E[\delta_J]\langle 0|0\rangle_J = J\langle 0|0\rangle_J. \tag{15.22}$$

This is a functional differential equation for the amplitude $\langle 0|0\rangle_J$. We can attempt to solve it by substituting a trial solution

$$\langle 0|0\rangle_J = \int \mathrm{d}\phi F[\phi]\exp\left(\frac{-\mathrm{i}}{\hbar}\int \mathrm{d}V_t J\phi\right). \tag{15.23}$$

Substituting in, and using $J = \mathrm{i}\hbar\frac{\delta}{\delta\phi}$,

$$\begin{aligned} 0 &= \int \mathrm{d}\phi\,\{E[\delta_J] - J\}\,F[\phi]\exp\left(\mathrm{i}\int \mathrm{d}V_t J\phi\right) \\ &= \int \mathrm{d}\phi\Big\{E[\delta_J]\exp\left(\frac{-\mathrm{i}}{\hbar}\int \mathrm{d}V_t J\phi\right) \\ &\quad - \mathrm{i}\hbar F[\phi]\frac{\delta}{\delta\phi}\exp\left(-\frac{\mathrm{i}}{\hbar}\int \mathrm{d}V_t J\phi\right)\Big\}. \end{aligned} \tag{15.24}$$

Integrating by parts with respect, moving the derivative $\frac{\delta}{\delta\phi}$, yields

$$\begin{aligned} 0 = \int \mathrm{d}\phi\,\left\{E[\phi]F[\phi] + \mathrm{i}\hbar\frac{\delta F}{\delta\phi}\right\}T\exp\left(\mathrm{i}\int \mathrm{d}V_t J\phi\right) \\ -\mathrm{i}\left[F[\phi]T\exp\left(\mathrm{i}\int \mathrm{d}V_t J\phi\right)\right]_{-\infty}^{+\infty}. \end{aligned} \tag{15.25}$$

Assuming that the surface term vanishes independently gives

$$E[\phi]F[\phi] = -\mathrm{i}\hbar\frac{\delta F}{\delta\phi}, \tag{15.26}$$

and thus

$$F[\phi] = C\exp\left(\frac{\mathrm{i}}{\hbar}S[\phi]\right). \tag{15.27}$$

Thus, the transformation function for vacua, in the presence of a source, may be taken to be

$$\langle 0|0\rangle_J = \int \mathrm{d}\phi \exp\left(\frac{\mathrm{i}}{\hbar}(S[\phi] - \int \mathrm{d}V_t J\phi)\right). \tag{15.28}$$

15.4 Postscript

For all of its limitations, classical covariant field theory is a remarkable stepping stone, both sturdy and refined, building on the core principles of symmetry and causality. Its second quantized extension has proven to be the most successful strategy devised for understanding fundamental physical law. These days, classical field theory tends to merit only an honourable mention, as a foundation for other more enticing topics, yet the theoretical toolbox of covariant classical field theory underpins fundamental physics with a purity, elegance and unity which are virtually unparalleled in science. By dwelling on the classical aspects of the subject, this book scratches the surface of this pivotal subject and celebrates more than a century of fascinating discovery.

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