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LOCALLY PRIMITIVE GRAPHS OF PRIME-POWER ORDER

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Abstract

Let Γ be a finite connected undirected vertex transitive locally primitive graph of prime-power order. It is shown that either Γ is a normal Cayley graph of a 2-group, or Γ is a normal cover of a complete graph, a complete bipartite graph, or $\Sigma^{\times l}$, where $\Sigma = \mathbf{K}_{p^m}$ with p prime or Σ is the Schläfli graph (of order 27). In particular, either Γ is a Cayley graph, or Γ is a normal cover of a complete bipartite graph.

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1. Introduction

This is an application of Praeger's fundamental theory of symmetric graphs to the study of a class of locally primitive graphs.

Let Γ be a digraph with vertex set V. For $G \leq \mathrm{Aut}\,\Gamma$, a group of automorphisms, Γ is called *G-vertex transitive* if G is transitive on V. For a vertex v, let $\Gamma(v)$ be the set of vertices to which v is adjacent, and let $G_v = \{g \in G \mid v^g = v\}$. A G-vertex transitive digraph Γ is called *G-locally primitive* (or simply called locally primitive) if G_v acts primitively on $\Gamma(v)$ for all vertices v. As usual, the number of vertices of a digraph is called the *order*, and the size $|\Gamma(v)|$ is called the *out-valency* if Γ is regular. By $\Gamma^-(v)$ we mean the set of vertices that are adjacent to v. Then $|\Gamma(v) \cup \Gamma^-(v)|$ is called the *valency* of Γ for Γ regular. If, for any vertices u, v of Γ, u is adjacent to v if and only if v is adjacent to u, then Γ is called *undirected*. This paper aims to characterize undirected vertex transitive locally primitive graphs of prime-power order.

There are some typical examples of locally primitive graphs: the complete graphs $\mathbf{K}_n$, and the complete bipartite graphs $\mathbf{K}_{n,n}$. In particular, $\mathbf{K}_{p^m}$ with p prime and

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$\mathbf{K}_{2^m,2^m}$ are of prime-power order. More examples can be recursively constructed by direct product. Given digraphs Γ_i with vertex sets V_i for $1 \le i \le l$, their *direct product*, denoted by $\Gamma_1 \times \cdots \times \Gamma_l$, is the digraph Γ with the vertex set $V_1 \times \cdots \times V_l$ (Cartesian product) such that $(u_1, \ldots, u_l)$ is adjacent to $(v_1, \ldots, v_l)$ if u_i is adjacent in Γ_i to v_i for each i. In the special case where $\Gamma_1 = \cdots = \Gamma_l$, the direct product is simply denoted by $\Gamma_1^{\times l}$.

The direct product $\Gamma \times \mathbf{K}_2$ has vertex set $V \times \{1, 2\}$ such that $(u, 1)$ is adjacent to $(v, 2)$ if and only if u, v are adjacent in Γ. Hence $\Gamma \times \mathbf{K}_2$ is actually the so-called *standard double cover* of Γ. In particular, $\mathbf{K}_n \times \mathbf{K}_2 = \mathbf{K}_{n,n} - n\mathbf{K}_2$, the graph obtained by deleting a 1-factor from $\mathbf{K}_{n,n}$.

The *Schläfli graph* is the graph on isotropic lines in the U(4, 2) geometry, adjacent when disjoint; refer to [2] or 'http://www.win.tue.nl/∼aeb/graphs'. It is a strongly regular graph of valency 16, and its automorphism group is U(4, 2).2. Also, it is a locally primitive Cayley graph of $\mathbb{Z}_9{:}\mathbb{Z}_3$; see Lemma 2.6.

A digraph $\Gamma = (V, E)$ is called a *Cayley graph* of a group G if there is a nonempty set S of G such that $V = G$ and $E = \{\{g, sg\} \mid g \in G, s \in S\}$, which is denoted by Cay(G, S). Obviously, Cay(G, S) is undirected if and only if $S = S^{-1} := \{s^{-1} \mid s \in S\}$. It is known that a digraph Γ is a Cayley graph of a group G if and only if Aut Γ contains a subgroup that is isomorphic to G and regular on the vertex set; see [1, Proposition 16.3]. For convenience, this regular subgroup of Aut Γ is still denoted by G in this paper. If Aut Γ has a normal subgroup that is regular and isomorphic to G, then Γ is called a *normal Cayley graph* of G. Refer to [10, 15, 16] for various nice properties of normal Cayley graphs.

Assume that Γ is a G-vertex transitive digraph. Let N be a normal subgroup of G. Denote by V_N the set of N-orbits in V. The *normal quotient* Γ_N of Γ induced by N is defined as the digraph with vertex set V_N; and two vertices $B, C \in V_N$ are adjacent if there exist $u \in B$ and $v \in C$ that are adjacent in Γ. If Γ and Γ_N have the same valency, then Γ is called a *normal cover* of Γ_N. Obviously, if Γ is a cover of Γ_N, then Γ is undirected if and only if so is Γ_N.

A triple of distinct vertices of an undirected graph is called a 2-arc if one of them is adjacent to the other two. An undirected graph Γ is called $(G, 2)$-*arc transitive* if $G \le \mathrm{Aut}\,\Gamma$ is transitive on the set of 2-arcs of Γ. It easily follows that an undirected regular $(G, 2)$-arc transitive graph is G-vertex transitive and G-locally primitive.

In the literature, the classes of 2-arc transitive graphs and locally primitive graphs have been extensively studied; see [1, 11, 14] and references therein. In particular, undirected vertex primitive and vertex biprimitive 2-arc transitive Cayley graphs of elementary abelian p-groups are classified by Ivanov and Praeger [7]; a characterization of undirected 2-arc transitive graphs of prime-power order is given by the first author [8]. The main result of this paper is to extend the result of [8] to the class of undirected vertex transitive locally primitive graphs.

THEOREM 1.1. *Let Γ be a connected undirected graph of order p^n and valency at least three, with p prime. Assume that Γ is vertex transitive and locally primitive.*

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Then one of the following statements holds:

(i) Γ *is a normal Cayley graph of a* 2*-group;*
(ii) Γ *is a normal cover of* $\Sigma^{\times l}$*, where* $l \geq 1$ *and* $\Sigma = \mathbf{K}_{p^r}$ *or is the Schläfli graph; in particular,* Γ *is a Cayley graph; or*
(iii) Γ *is a normal cover of* $\mathbf{K}_{2^m,2^m}$*.*

This tells us that an undirected locally primitive graph of prime-power order is either a Cayley graph, or a normal cover of a complete bipartite graph. In particular, we have the following interesting corollary.

COROLLARY 1.2.

(i) *A connected undirected locally primitive graph of order a power of an odd prime is a Cayley graph.*
(ii) *A connected undirected locally primitive graph of order* p^n *with* $p \geq 5$ *prime is a normal cover of* $\mathbf{K}_{p^m}^{\times l}$*.*

Stimulated by Theorem 1.1, some further research problems naturally arise.

PROBLEM.

(1) Are all locally primitive normal covers of $\mathbf{K}_{2^m,2^m}$ Cayley graphs?
(2) Characterize normal Cayley graphs of 2-groups that are locally primitive.
(3) Study locally primitive normal covers of $\Sigma^{\times l}$, where Σ is a complete graph or the Schläfli graph.

2. Vertex quasiprimitive case

A permutation group $G \leq \mathrm{Sym}(\Omega)$ is called *quasiprimitive* if each nontrivial normal subgroup of G is transitive on Ω. In this section, we deal with the vertex quasiprimitive case. First, we give a characterization of quasiprimitive permutation groups of prime-power degree.

Let X be a quasiprimitive permutation group on Ω of degree p^n, where p is a prime. Let N be a minimal normal subgroup of X. Then $N \cong T^l$, where $l \geq 1$ and T is a simple group. Since X is quasiprimitive on Ω, N is transitive on Ω.

If T is abelian, then $T \cong \mathbb{Z}_p$, $l = n$, and $N \cong \mathbb{Z}_p^n$ is regular on Ω. Further, $\mathbb{Z}_p^n \lhd X \leq \mathrm{AGL}(n, p)$.

If $l = 1$ and T is nonabelian, then X is an almost simple group, and for $\alpha \in \Omega$, T_α has index p^n in T. The following theorem of Guralnick [5] presents the nonabelian simple groups with a subgroup of prime-power index.

THEOREM 2.1 [5]. *Let* T *be a nonabelian simple group that has a subgroup* H *of index* p^r *with* p *prime. Then one of the following holds:*

(i) $T \cong \mathrm{A}_{p^r}$*, and* $H \cong \mathrm{A}_{p^r-1}$*;*
(ii) $T \cong \mathrm{PSL}(d, q)$*,* H *is a maximal parabolic subgroup of* T*, and* $p^r = (q^d - 1)/(q - 1)$*;*
(iii) $T \cong \mathrm{PSL}(2, 11)$*,* $H \cong \mathrm{A}_5$*, and* $p^r = 11$*;*

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(iv) $T \cong \mathrm{M}_{11}$, $H \cong \mathrm{M}_{10}$, *and* $p^r = 11$;
(v) $T \cong \mathrm{M}_{23}$, $H \cong \mathrm{M}_{22}$, *and* $p^r = 23$; *or*
(vi) $T = \mathrm{PSU}(4, 2)$, $H \cong \mathbb{Z}_2^4{:}\mathrm{A}_5$ *and* $p^r = 27$.

Next we assume that N is nonabelian and $l \geq 2$. We will show that X is primitive of product action type. Let H be a group acting on Δ, and P a subgroup of the *symmetric group* S_l. Let $G = H \wr P$ be the *wreath product* of H by P. Then G acts naturally on $\Omega := \Delta^l$, called *product action*, as follows: for $(\delta_1, \ldots, \delta_l) \in \Omega$, $x = (h_1, \ldots, h_l) \in H^l$ and $\sigma \in P$,

$$(\delta_1, \ldots, \delta_l)^{(h_1,\ldots,h_l)\sigma} = (\varepsilon_1, \ldots, \varepsilon_l) \quad \text{where } \varepsilon_i = \delta_{i^{\sigma^{-1}}}^{h_{i^{\sigma^{-1}}}}.$$

It is known that G is primitive on Ω if and only if H acts primitively but not regularly on Δ, and P is a transitive subgroup of S_l; see [4, Lemma 2.7A].

A primitive permutation group is quasiprimitive, but the inverse is not necessarily true. In [9] and [10], it is shown that a quasiprimitive permutation group containing an abelian regular subgroup or a dihedral regular subgroup is primitive. The following theorem shows that a similar result holds for quasiprimitive permutation groups of prime-power degree.

THEOREM 2.2. *Let X be a quasiprimitive permutation group on Ω of degree p^n with p prime. Let N be a minimal normal subgroup of X. Then X is primitive, and one of the following holds:*

(i) *X is an affine group, $N = \mathbb{Z}_p^l$, and $X \leq \mathrm{AGL}(l, p)$, where $l \geq 1$;*
(ii) *X is an almost simple group, and $N \cong T$ is as in Theorem 2.1; in particular, either X is 2-transitive, or $X = \mathrm{PSU}(4, 2)$ or $\mathrm{PSU}(4, 2).\mathbb{Z}_2$; or*
(iii) *X is of product action type, $N = T^l$ with $l \geq 2$, and T lies in the list of Theorem 2.1.*

Moreover, if $|\Omega|$ is a power of 2 and N is nonabelian, then $N = T^l$ with $l \geq 1$, and $T = \mathrm{A}_{2^s}$ or $\mathrm{PSL}(2, p)$ with $p + 1 = 2^s$ for $s \geq 3$ and $p \equiv 3 \pmod 4$, and N has a subgroup that is regular on Ω.

PROOF. Since N is a minimal normal subgroup of X, $N \cong T^l$ for some simple group T and $l \geq 1$. Since X is quasiprimitive, N is transitive on Ω. If N is abelian, it is known and easily shown that X is primitive and part (i) holds.

Thus we assume that N is nonabelian. If $N \cong T$ is simple, then the stabilizer N_α, where $\alpha \in \Omega$, has index p^m in N. Hence by Theorem 2.1, $N \cong T$ is listed in Theorem 2.1, and N_α is maximal in N. So N and X are primitive on Ω.

Now, we further assume that N is not simple. Then $N = T_1 \times \cdots \times T_l \cong T^l$, where $l \geq 2$ and T is a nonabelian simple group. Since $|N : N_\alpha| = |\Omega| = p^m$ and $|T_1 : (T_1)_\alpha| = |N : ((T_1)_\alpha \times T_2 \times \cdots \times T_l)|$ divides $|N : N_\alpha|$, we conclude that $(T_1)_\alpha$ has index p-power in T. Hence by Theorem 2.1, $(T_1)_\alpha$ is a maximal subgroup of T_1. Similarly, for all i with $1 \leq i \leq l$, $(T_i)_\alpha$ is maximal and has index p-power in T_i. By the O'Nan–Scott theorem (see [4]), X is primitive of product action type.

Next suppose that $|\Omega|$ is a power of 2. Since T is a normal subgroup of N, we conclude that T is half-transitive on Ω, so $|T : T_\alpha|$ divides 2^d. By Theorem 2.1, $T \cong \mathrm{A}_{2^s}$ or $\mathrm{PSL}(d, q)$ and $(q^d - 1)/(q - 1) = 2^s$ for some s. Suppose that $T = \mathrm{PSL}(d, q)$ with $d \geq 3$. Then $(q, d) \neq (2, 6)$, and hence $q^d - 1$ has a primitive prime divisor r, that is, r divides $q^d - 1$ but not $q^i - 1$ for each $i < d$; see [6, p. 508]. It follows that $(q^d - 1)/(q - 1)$ is not a power of 2, which is not possible. Hence $d = 2$. Now, $q + 1 = (q^2 - 1)/(q - 1) = 2^s$, and it then follows that $q = 2^s - 1$ is a prime. □

The following result was proved by Praeger [13].

LEMMA 2.3 [13, Theorem 2.1(a)]. *Let $X \leq H \wr \mathrm{S}_l$ be a primitive permutation group of product action type on $\Omega := \Delta^l$, where H is almost simple and primitive on Δ. Let $\alpha = (\gamma, \ldots, \gamma) \in \Delta^l$. Suppose that Λ is an X_α-orbit on $\Omega \setminus \{\alpha\}$, and X_α is quasiprimitive on Λ. Then $\Lambda = \Lambda(\gamma)^l$, where $\Lambda(\gamma)$ is an orbit of H_γ on Δ.*

The next lemma shows that the direct product of locally primitive graphs is locally primitive.

LEMMA 2.4. *Let Σ be a Y-locally primitive digraph with vertex set Δ, where $Y \leq \mathrm{Aut}\,\Sigma$ is almost simple and primitive on Δ. Let $\Gamma = \Sigma^{\times l}$, with vertex set Δ^l. Let $X = Y^l.P \leq Y \wr \mathrm{S}_l$ act on Δ^l in product action, where P is a transitive subgroup of the symmetric group S_l. Then $X \leq \mathrm{Aut}\,\Gamma$ and Γ is an X-locally primitive digraph.*

Further, if Σ is a Cayley graph of a group H, then Γ is a Cayley graph of the group H^l.

PROOF. Let $V = \Delta^l$. It is easily shown that $X \leq \mathrm{Aut}\,\Gamma$, and X is transitive on V. Further, for $v = (\delta, \ldots, \delta) \in V$, we have $X_v = (Y_\delta)^l.P$. Since Σ is a Y-locally primitive graph, Y_δ is primitive on $\Sigma(\delta)$. By [4, Lemma 2.7A], X_v is primitive on $\Gamma(v)$ as P is a transitive subgroup of S_l. So Γ is an X-locally primitive digraph.

Further, suppose that Σ is a Cayley graph of a group H. Then $H \leq \mathrm{Aut}\,\Sigma$ is regular on Δ, so $H^l \leq (\mathrm{Aut}\,\Sigma)^l.P \leq \mathrm{Aut}\,\Gamma$ and is regular on V. Therefore, Γ is a Cayley graph of the group H^l. □

The *socle* of a group X is the normal subgroup generated by all minimal normal subgroups of X, denoted by $\mathrm{soc}(X)$.

LEMMA 2.5. *Let Γ be an X-locally primitive digraph with vertex set V. Suppose that X is a primitive permutation group on V of product action type. Suppose further that $\mathrm{soc}(X) = \mathrm{PSL}(d, q)^l$ with $l \geq 1$, and $|V| = ((q^d - 1)/(q - 1))^l$. Then $d = 2$.*

PROOF. It is easily shown that X is almost simple or of product action type. Let $N = \mathrm{soc}(X)$, $T = \mathrm{PSL}(d, q)$, and $O = X/N$.

Suppose that X is almost simple, and $d \geq 3$. For $u, v \in V$, the stabilizers

$$T_u \cong [q^{d-1}] : (\mathbb{Z}_{(q-1)/(d,q-1)}.\mathrm{PGL}(d-1, q)),$$
$$T_{uv} \cong [q^{2(d-2)}] : (\mathbb{Z}_{(q-1)/(d,q-1)}.\mathbb{Z}_{q-1}.\mathrm{PGL}(d-2, q)),$$

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and $X_u \cong T_u.O$, $X_{uv} \cong T_{uv}.O$. Then there exists a group

$$H = \mathbf{O}_p(T_u)T_{uv} \cong [q^{2d-3}] : (\mathbb{Z}_{(q-1)/(d,q-1)}.\mathbb{Z}_{q-1}.\mathrm{PGL}(d-2, q))$$

such that $X_{uv} < H.O < X_u$. Thus, X_{uv} is not a maximal subgroup of X_u, which is impossible as X_u is primitive on $\Gamma(u)$. Thus, if X is almost simple, then $d = 2$.

Assume now that X is of product action type. Then $X_u \cong T_u^l.O$ and $X_{uv} \cong T_{uv}^l.O$. Therefore, if $d \geq 3$, we have $X_{uv} < H^l.O < X_u$, which is impossible as X_u is primitive on $\Gamma(u)$. So $d = 2$. □

For a digraph Γ and $X \leq \mathrm{Aut}\,\Gamma$, the action of the vertex stabilizer X_v on $\Gamma(v)$ may be unfaithful. As usual, the kernel of X_v on $\Gamma(v)$ is denoted by $X_v^{[1]}$. Then $X_v^{\Gamma(v)} \cong X_v/X_v^{[1]}$.

Lemma 2.6. *Let Γ be a Y-locally primitive digraph with vertex set V. Assume that Y is primitive on V, $|V| = 27$, and $\mathrm{soc}(Y) = \mathrm{PSU}(4, 2)$. Then Γ is the Schläfli graph, which is a locally primitive Cayley graph of $\mathbb{Z}_9{:}\mathbb{Z}_3$ of valency 16.*

Proof. It is known that $Y = \mathrm{PSU}(4, 2).O$ with $O \leq \mathbb{Z}_2$, Y has rank 3, and $Y_v = \mathbb{Z}_2^4{:}\mathrm{A}_5$ or $\mathbb{Z}_2^4{:}\mathrm{S}_5$; see the Atlas [3]. Further, the two orbital graphs are the Schläfli graph Γ and its complement, Σ say; refer to [2]. Then Σ has valency 10. We claim that Σ is not locally primitive. Suppose that $Y_v^{\Sigma(v)}$ is primitive. Then Y_v is unfaithful on $\Sigma(v)$ and the kernel $Y_v^{[1]} \cong \mathbb{Z}_2^4$. So $Y_v^{\Sigma(v)} \cong \mathrm{A}_5$. Since $|\Sigma(v)| = 10$, we conclude that $Y_{vw}^{\Sigma(v)} \cong \mathrm{S}_3$, where $w \in \Sigma(v)$. Hence

$$1 \neq (Y_v^{[1]})^{\Sigma(w)} \lhd Y_{vw}^{\Sigma(w)} \cong \mathrm{S}_3,$$

and thus $(Y_v^{[1]})^{\Sigma(w)}$ is a normal 2-subgroup of S_3. However, S_3 has no normal 2-subgroup, which is a contradiction.

For the Schläfli graph Γ, $Y_v = \mathbb{Z}_2^4{:}\mathrm{A}_5.O$ is faithful on $\Gamma(v)$. Since $(Y_v)_w \cong \mathrm{A}_5.O$ is a maximal subgroup of Y_v, where $w \in \Gamma(v)$, $Y_v^{\Gamma(v)}$ is primitive. Hence Γ is a Y-locally primitive graph. Further, it follows from [12] that Y contains a 3-group $\mathbb{Z}_9{:}\mathbb{Z}_3$, which is regular on V, so Γ is a Y-locally primitive Cayley graph of $\mathbb{Z}_9{:}\mathbb{Z}_3$. □

The final lemma of this section shows that locally primitive digraphs of prime-power order in the vertex quasiprimitive case are all Cayley graphs.

Lemma 2.7. *Let $\Gamma = (V, E)$ be a connected X-locally primitive digraph of order p^n, where p is a prime. Assume further that X is quasiprimitive on V. Then X is primitive on V and has a subgroup that is regular on V, and Γ is a Cayley graph. Moreover, one of the following statements holds:*

(i) *Γ is a normal Cayley graph of an elementary abelian p-group, and further Γ is undirected if and only if $p = 2$;*

(ii) *$\Gamma = \mathbf{K}_{p^n}$, $\mathrm{Aut}\,\Gamma = \mathrm{S}_{p^n}$, and either $p = 2$ and X is a 2-primitive affine group, or* $\mathrm{soc}(X) = \mathrm{PSL}(2, 11)$, M_{11}, M_{23}, A_{p^n} *or* $\mathrm{PSL}(2, q)$*;*

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(iii) $\Gamma = \mathbf{K}_{p^r}^{\times l}$ *with* $l \geq 2$ *and* $n = rl$, *and* $\mathrm{Aut}\,\Gamma = \mathrm{S}_{p^r} \wr \mathrm{S}_l$, *and* X *is a blow-up of a* 2*-primitive group as in part* (ii)*; or*

(iv) $\Gamma = \Sigma^{\times l}$, *where* $l \geq 1$ *and* Σ *is the Schläfli graph, and* $\mathrm{PSU}(4, 2)^l \lhd X \leq \mathrm{Aut}\,\Gamma = (\mathrm{PSU}(4, 2).2) \wr \mathrm{S}_l$.

In particular, all graphs Γ *that appear in parts* (ii)–(iv) *are undirected.*

PROOF. Let N be a minimal normal subgroup of X, and let $Y = \mathrm{Aut}\,\Gamma$. By Theorem 2.2, X is primitive on V, and thus Y is primitive on V.

Suppose that N is nonabelian simple. Then by Theorem 2.1 and Lemma 2.5, $N = \mathrm{PSL}(2, 11)$, M_{11}, M_{23}, $\mathrm{PSU}(4, 2)$, A_{p^r} or $\mathrm{PSL}(2, q)$. In the first five cases, N has a regular subgroup that is isomorphic to $\mathbb{Z}_{11}$, $\mathbb{Z}_{11}$, $\mathbb{Z}_{23}$, $\mathbb{Z}_9{:}\mathbb{Z}_3$, or $\mathbb{Z}_p^r$, respectively. Suppose that $N = \mathrm{PSL}(2, q)$. If q is even, then $N = \mathrm{PSL}(2, q) = \mathrm{PGL}(2, q)$ contains a regular subgroup $\mathbb{Z}_{q+1}$. If q is odd, as $q + 1 = p^r$, it follows that $p = 2$ and $q \equiv 3 \pmod 4$, so N contains a regular subgroup D_{q+1}. Further, by Theorem 2.2(ii), either Γ is a complete graph, or Γ is the Schläfli graph, as in part (ii) or part (iv) with $l = 1$, respectively. In particular, Γ is undirected.

Suppose next that X is nonabelian and nonsimple. Then by Theorem 2.2, X^V is of product action type. Thus, $V = \Delta^l$ and $N = T^l$ with $l \geq 2$, such that $T = \mathrm{PSL}(2, 11)$, M_{11}, M_{23}, $\mathrm{PSU}(4, 2)$, A_{p^r} or $\mathrm{PSL}(2, q)$, and $|\Delta| = 11$, 11, 23, 27, p^r or $q + 1$, respectively. The previous paragraph shows that T has a subgroup G that is regular on Δ. Thus G^l is a subgroup of N and regular on V, and Γ is a Cayley graph.

For a vertex $\alpha = (\delta, \ldots, \delta) \in V$, since $X_\alpha^{\Gamma(\alpha)}$ is primitive, we have that $\Gamma(\alpha)$ is an orbit of X_α on $V \setminus \{\alpha\}$. By Lemma 2.3, $\Gamma(\alpha) = \Delta(\delta)^l$, where $\Delta(\delta)$ is an orbit of H_δ in $\Delta \setminus \{\delta\}$. It follows that $\Gamma = \Sigma^{\times l}$. Moreover, since either T is 2-transitive on Δ, or $T = \mathrm{PSU}(4, 2)$, we conclude that either Σ is a complete graph, or Σ is the Schläfli graph, as in part (iii) or part (iv) with $l \geq 2$, respectively. In particular, Γ is undirected.

Finally, assume that N is abelian. Then N is regular on V, and Γ can be expressed as a Cayley graph of N. It follows since Γ is X-locally primitive that Γ is undirected if and only if N is a 2-group. Further, by Theorem 2.2, the primitive permutation group $Y = \mathrm{Aut}\,\Gamma$ is affine, almost simple, or of product action type. If Y is affine, then Γ is a normal Cayley graph, as in part (i). If Y is almost simple, then Y is 2-transitive on V, as in part (ii). If Y is of product action type, then Y is a blow-up of the almost simple group case, as in part (iii). □

3. Bi-quasiprimitive case

A transitive permutation group X on Ω is called *bi-quasiprimitive* if each nontrivial normal subgroup of X has at most two orbits, and there exists a normal subgroup of X that has two orbits on Ω. Further, X is called *biprimitive* if Ω has a nontrivial X-invariant partition $\Omega = U \cup W$ such that $X_U = X_W$ is primitive on U and W. Let $X^+ = X_U = X_W$. Then X^+ is a normal subgroup of Y of index 2.

The next result, proved in [11, Theorems 1.4 and 1.5], gives some properties of bi-quasiprimitive permutation groups.

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THEOREM 3.1. *Let X be a bi-quasiprimitive permutation group on Ω. Then either:*

(i) *X^+ acts unfaithfully on U and W; or*

(ii) *X^+ acts faithfully on U and W, and one of the following holds:*

  (a) *X^+ is quasiprimitive on U and W, or*

  (b) *X^+ has two minimal normal subgroups M_1 and M_2 that are conjugate in X and semiregular on Ω; moreover, $\langle M_1, M_2\rangle = M_1 \times M_2$ is a minimal normal subgroup of X and transitive on both U and W.*

We need the following special case.

COROLLARY 3.2. *Let X be a bi-quasiprimitive permutation group on Ω with bipartition $\Omega = U \cup W$, where $|\Omega| = 2^m$. Suppose further that X^+ acts faithfully on U and W. Then either X^+ is primitive and has a subgroup that is regular on U and W, or X^+ has a normal elementary abelian 2-group that is regular on both U and W.*

PROOF. If X^+ is quasiprimitive on both U and W, by Theorem 2.2, X^+ is primitive on both U and W and has a regular subgroup. If X^+ is not quasiprimitive, by Theorem 3.1(ii)(b), X^+ has two minimal normal subgroups M_1, M_2 that are semiregular on Ω. Thus M_1, M_2 are both 2-groups, and so M_1 and M_2 are elementary abelian 2-groups. It then follows that $\langle M_1, M_2\rangle$ is a normal elementary abelian 2-group and regular on both U and W. □

A permutation group $G \le \mathrm{Sym}(\Omega)$ is called *biregular* if it is semiregular and has exactly two orbits on Ω.

LEMMA 3.3. *Let $\Gamma = (V, E)$ be a connected undirected X-locally primitive graph of order 2^n. Assume that X is transitive and bi-quasiprimitive on V, associated with the bipartition $V = U \cup W$. Then X^+ has a subgroup G that is biregular on V, and one of the following statements holds:*

(i) *$\Gamma \cong \mathbf{K}_{2^{n-1},2^{n-1}}$;*

(ii) *X^+ is faithful on both U and W, and G is an elementary normal 2-subgroup; or*

(iii) *X^+ is faithful and primitive on both U and W.*

PROOF. Since X is bi-quasiprimitive on V, the graph Γ is bipartite with biparts U and W, say.

Suppose that X^+ is unfaithful on U. Let K_1 be the kernel of X^+ acting on U. Then $K_1 \ne 1$ and K_1 acts faithfully on W. For an edge $\{\alpha, \beta\}$ of Γ, where $\alpha \in U$ and $\beta \in W$, let B be the K_1-orbit of β in W. Since K_1 fixes α, we conclude that $B \subseteq \Gamma(\alpha)$. Further, as

$$1 \ne K_1^{\Gamma(\alpha)} \lhd (X_\alpha^+)^{\Gamma(\alpha)} = X_\alpha^{\Gamma(\alpha)}$$

and $X_\alpha^{\Gamma(\alpha)}$ is primitive, we obtain $B = \Gamma(\alpha)$. Since this holds for every vertex α adjacent to a vertex of B, by the connectivity of Γ, it is easily shown that $B = W$. It then follows that $\Gamma \cong \mathbf{K}_{2^{n-1},2^{n-1}}$, as in part (i). Noting that $X_\alpha^{\Gamma(\alpha)}$ is now a primitive permutation group of degree 2^{n-1}, by Lemma 2.7, we have that $X_\alpha^{\Gamma(\alpha)}$ has a subgroup

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that is regular on $\Gamma(\alpha) = W$. It follows that K_1 has a regular subgroup G_1 on W. Similarly, K_2 has a regular subgroup G_2 on U. Since $K_1 \cong K_2$, we may assume that $G_1 \cong G_2$. Let ϕ be an isomorphism between G_1 and G_2. Then X has a biregular subgroup $G = \{(x, x^{\phi}) \mid x \in G_1\}$.

Assume now that X^+ is faithful on U and W. Then by Corollary 3.2, either X^+ is primitive and has a subgroup that is regular on both U and W, as in part (iii), or X^+ has a normal elementary abelian 2-group that is regular on both U and W. For the latter case, by Lemma 4.2, either $\Gamma \cong \mathbf{K}_{2^{n-1},2^{n-1}}$, as in part (i), or X^+ is faithful on both U and W, as in part (ii). □

4. Proof of Theorem 1.1

To prove Theorem 1.1, we need a lemma regarding the normal quotient, which is a generalization of [14, Theorem 4.1] and whose proof is easy and omitted.

LEMMA 4.1. *Let Γ be an undirected X-locally primitive graph, and let $N \lhd X$ have at least three orbits on V. Then Γ_N is X/N-locally primitive and Γ is a normal cover of Γ_N.*

A graph Γ is called the *bi-Cayley graph* of a group G, denoted by $\mathrm{BiCay}(G, S)$, if there is a nonempty set S of G such that the vertex set of Γ is $\{(g, i) \mid g \in G, i = 1, 2\}$; and two vertices (g, i), (h, j) are adjacent if and only if $hg^{-1} \in S$ and $i \neq j$. It easily follows that $\mathrm{BiCay}(G, S)$ is the standard double cover of the Cayley graph $\mathrm{Cay}(G, S)$, and so $\mathrm{BiCay}(G, S) = \mathrm{Cay}(G, S) \times \mathbf{K}_2$.

LEMMA 4.2. *Let $\Gamma = (V, E)$ be a connected undirected bipartite graph with biparts $U \cup W$ that is not a complete bipartite graph. Let $X = \mathrm{Aut}\,\Gamma$, and $X^+ = X_U = X_W$. Suppose that X^+ has a subgroup G that is regular on both U and W. Then the following statements hold:*

(i) *$\Gamma = \mathrm{BiCay}(G, S) = \mathrm{Cay}(G, S) \times \mathbf{K}_2$ for some subset S of G;*
(ii) *letting $\Sigma = \mathrm{Cay}(G, S)$, we have $\mathrm{Aut}\,\Sigma = X^+$;*
(iii) *if Γ is locally primitive, then so is $\mathrm{Cay}(G, S)$; and*
(iv) *if $\mathrm{Cay}(G, S)$ is undirected, then $X = X^+ \times \mathbb{Z}_2$, and Γ is a Cayley graph of $G \times \mathbb{Z}_2$.*

PROOF. Since Γ is not a complete bipartite graph, there exist vertices $u \in U$ and $w \in W$ that are not adjacent in Γ. Label the elements of G as $g_1, g_2, \ldots, g_n$ with $g_1 = 1$. Then label the vertices in U as $u_j = u^{g_j}$, and the vertices in W as $w_j = w^{g_j}$, for $j = 1, 2, \ldots, n$. Let $S = \{g_j \in G \mid (u, w^{g_j}) \in E\}$. Then

$$
\begin{aligned}
\{u_i, w_j\} \in E \quad &\Longleftrightarrow \quad \{u^{g_i}, w^{g_j}\} \in E \\
&\Longleftrightarrow \quad \{u, w^{g_j g_i^{-1}}\} \in E \\
&\Longleftrightarrow \quad g_j g_i^{-1} \in S \\
&\Longleftrightarrow \quad (g_i, 1) \sim (g_j, 2) \text{ in } \mathrm{BiCay}(G, S).
\end{aligned}
$$

Thus, $\Gamma \cong \mathrm{BiCay}(G, S) = \mathrm{Cay}(G, S) \times \mathbf{K}_2$, as in part (i).

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Let $\Sigma = \mathrm{Cay}(G, S)$. By definition, for any elements g_i, g_j of G, the vertices g_i, g_j of $\mathrm{Cay}(G, S)$ are adjacent if and only if the vertices $(g_i, 1)$ and $(g_j, 2)$ of $\mathrm{BiCay}(G, S)$ are adjacent. For any permutation x of U and any edge $\{g_i, g_j\}$ of Σ, we have that $(g_i, 1)$ and $(g_j, 2)$ are adjacent in $\mathrm{BiCay}(G, S)$, and

$$\begin{aligned}
x \in X^+ \quad &\Longleftrightarrow \quad (g_i, 1)^x \sim (g_j, 2)^x \text{ in } \mathrm{BiCay}(G, S)\\
&\Longleftrightarrow \quad (g_i^x, 1) \sim (g_j^x, 2) \text{ in } \mathrm{BiCay}(G, S)\\
&\Longleftrightarrow \quad g_j^x (g_i^x)^{-1} \in S\\
&\Longleftrightarrow \quad g_i^x \sim g_j^x \text{ in } \mathrm{Cay}(G, S)\\
&\Longleftrightarrow \quad x \in \mathrm{Aut}\,\Sigma.
\end{aligned}$$

So $X^+ = \mathrm{Aut}\,\Sigma$, as in part (ii).

Identify elements $g_i \in G$ with points $(g_i, 1)$ of U, and identify u with the identity of G. We have $\Sigma(u) = S = \{g_j \in G \mid \{u, w^{g_j}\} \in E\}$, and $\Gamma(w) = \{(g_j, 1) \mid g_j \in S\}$. If Γ is locally primitive, then $X_w = X_w^+$ acts primitively on $\Gamma(w)$. It follows that X_u^+ acts primitively on $\Sigma(u)$, and Σ is X^+-locally primitive.

Finally, suppose that $\mathrm{Cay}(G, S)$ is undirected. It is easily shown that the map

$$\tau : (g, j) \mapsto (g, 3 - j), \quad \text{for } g \in G \text{ and } j = 1 \text{ or } 2,$$

is an automorphism of Γ. Further, τ is an involution and centralizes X^+, and it then follows that $X = X^+ \times \langle \tau \rangle \cong X^+ \times \mathbb{Z}_2$. □

Now, we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let Γ be a connected undirected vertex transitive and locally primitive graph with vertex set V, such that $|V| = p^n$ with p prime.

Let $X = \mathrm{Aut}\,\Gamma$, and let $N \lhd X$ be maximal subject to the condition that N has at least three orbits on V. Let $\overline{X} = X/N$, and V_N the set of N-orbits on V. Then $\overline{X}$ is quasiprimitive or bi-quasiprimitive on V_N. By Lemma 4.1, the normal quotient Γ_N is $\overline{X}$-locally primitive, and Γ is a normal cover of Γ_N.

Assume that $\overline{X}$ is quasiprimitive on V_N. Then, by Lemma 2.7, $\overline{X}$ has a subgroup $\overline{G}$ that is regular on V_N. Thus the extension $N.\overline{G}$ is regular on V, and Γ is a Cayley graph. Again, by Lemma 2.7, either $\overline{G}$ is normal in $\overline{X}$, or $\Gamma_N = \mathbf{K}_{p^n}^{\times l}$ with $l \geq 2$ or $\Sigma^{\times l}$ where $l \geq 1$ and Σ is the Schläfli graph. For the former, $N.\overline{G}$ is regular on V and normal in $X = \mathrm{Aut}\,\Gamma$, and so Γ is a normal Cayley graph of the 2-group $N.\overline{G}$. For the latter, Γ is a normal cover of $\mathbf{K}_{p^n}^{\times l}$ or $\Sigma^{\times l}$.

Assume that $\overline{X}$ is bi-quasiprimitive on V_N. Then Γ is bipartite with biparts U and W. By Lemma 3.3, $\overline{X}$ has a subgroup $\overline{G}$ that is biregular on V_N. Let $G = N.\overline{G} < N.\overline{X} = X$. It follows that the subgroup G is biregular on V. Suppose that Γ is not a complete bipartite graph. By Lemma 4.2, Γ is a bi-Cayley graph of G, say $\Gamma = \mathrm{BiCay}(G, S) = \mathrm{Cay}(G, S) \times \mathbf{K}_2$ for some subset S of G. Let $\Sigma = \mathrm{Cay}(G, S)$.

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Then Σ is X^+-locally primitive, and Σ_N is $\overline{X}^+$-locally primitive. Further, Γ_N and $\overline{X}$ satisfy Lemma 3.3.

If $\Gamma_N = \mathbf{K}_{2^m,2^m}$, as in Lemma 3.3(i), then Γ is a normal cover of a complete bipartite graph, as in Theorem 1.1(i). Thus assume next that Γ_N is not a complete bipartite graph.

Suppose that $\overline{X}^+$ has an elementary abelian normal 2-subgroup that is regular on U_N. Then the normal quotient Σ_N is undirected, and so is $\mathrm{Cay}(G, S)$. By Lemma 4.2, we have that $X = X^+ \times \mathbb{Z}_2$, and $G \times \mathbb{Z}_2$ is a normal subgroup of X and regular on V. So Γ is a normal Cayley graph of $G \times \mathbb{Z}_2$, as in Theorem 1.1(ii).

Suppose that $\overline{X}^+$ is a primitive permutation group on U_N that is almost simple or of product action type. By Lemma 2.7, the quotient Σ_N is $\mathbf{K}_{p^r}^{\times l}$, and so they are undirected. Thus Σ is undirected, and by Lemma 4.2, $X = X^+ \times \mathbb{Z}_2$. So $G \times \mathbb{Z}_2 < X$ is regular on V, and Γ is a Cayley graph of $G \times \mathbb{Z}_2$. □

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