



RESEARCH ARTICLE

Almost global well-posedness for quasilinear strongly coupled wave-Klein-Gordon systems in two space dimensions

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Abstract

We prove almost global well-posedness for quasilinear strongly coupled wave-Klein-Gordon systems with small and localized data in two space dimensions. We assume only mild decay on the data at infinity as well as minimal regularity. We systematically investigate all the possible quadratic null form type quasilinear strong coupling nonlinearities.

A key feature of the paper is our new, robust approach to the vector field method, which enables us to work at minimal regularity and decay in a quasilinear setting, and which, we believe, can be applied for a much wider class of problems.

1. Introduction

The problem we will address here is the Cauchy problem for the following quasilinear strongly coupled wave-Klein-Gordon system:

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t, x) = \mathbf{N}_1(v, \partial v) + \mathbf{N}_2(u, \partial v), \\ (\partial_t^2 - \Delta_x + 1)v(t, x) = \mathbf{N}_1(v, \partial u) + \mathbf{N}_2(u, \partial u), \end{cases} \quad (t, x) \in [0, +\infty) \times \mathbb{R}^2, \quad (1.1)$$

with initial conditions

$$\begin{cases} (u, v)(0, x) = (u_0(x), v_0(x)), \\ (\partial_t u, \partial_t v)(0, x) = (u_1(x), v_1(x)). \end{cases} \quad (1.2)$$

The nonlinearities $\mathbf{N}_1(\cdot, \cdot)$ and $\mathbf{N}_2(\cdot, \cdot)$ represent the wave-Klein-Gordon coupling via classical quadratic null structures. Precisely, $\mathbf{N}_1(\cdot, \cdot)$ and $\mathbf{N}_2(\cdot, \cdot)$ will be linear combinations of the classical quadratic null forms

$$\begin{cases} Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_i \psi \partial_j \phi, \\ Q_{0i}(\phi, \psi) = \partial_t \phi \partial_i \psi - \partial_t \psi \partial_i \phi, \\ Q_0(\phi, \psi) = \partial_t \phi \partial_t \psi - \nabla_x \psi \cdot \nabla_x \phi. \end{cases} \quad (1.3)$$

The main result we present in this paper asserts the almost global existence of solutions to the above system, when initial data are assumed to be small and localized. This is the first of a two-paper sequence,

where the aim of the second paper is to improve the almost-global well-posedness result to a global well-posedness result. The reason we structure this work in two papers is that they address very different aspects of the problem using essentially disjoint ideas and methods.

Compared with prior related works, our novel contributions here include the following:

- Our quasilinear structure provides a strong coupling between the wave and the Klein-Gordon equation, unlike any other prior works in two space dimensions (except for the second author's work [29], that only applies to the Q_0 type nonlinearities).
- We make no assumptions on the support of the initial data. Furthermore, we make very mild decay assumptions on the initial data at infinity. In particular, we use only two Klainerman vector fields in the analysis, which is optimal and below anything that has been done before.
- Rather than using arbitrarily high regularity, here we work with very limited regularity initial data (e.g., our two vector fields bound is simply in the energy space).
- In terms of methods, our work is based on a combination of energy estimates localized to dyadic space-time regions, and pointwise interpolation type estimates within the same regions. This is akin to ideas previously used by Metcalfe-Tataru-Tohaneanu [26] in a linear setting, and is also related to Alinhac's ghost weight method [1].

We remark that our methods allow for a larger array of weak quasilinear null form interactions in the equations, as well as non-null v - v interactions. We focus our exposition to the case of strong interactions above simply because this case is more difficult and has not been considered before except for [29].

1.1. Motivation and a brief history

The model we study here is physically motivated by problems arising in general relativity, where many similarly structured problems arise. Most of the results known so far concern the wave-Klein-Gordon systems in the 3-dimensional setting, but there is also a fair amount of work done in the 2-dimensional case, where systems akin to (1.1), but with different types of nonlinearities, have been considered. Since our result is set in the 2-dimensional setting, we will focus mostly in explaining what has been done in this direction and how it relates to our result.

Regardless of the spatial dimension considered, one always has to understand and deal with resonant interactions. In the wave-wave to wave bilinear interactions, resonance occurs for parallel waves. This is where the null condition plays a major role, as it cancels these interactions. In all other wave-Klein Gordon bilinear interactions, there is no true resonance; however, there is a near resonance for almost parallel waves in the high frequency limit, which becomes stronger in a quasilinear setting. For this reason, the null condition is still important in the wave-Klein Gordon quasilinear interactions, perhaps less so in the semilinear ones.

The main difference between the 2-dimensional setting and higher dimensions is due to the weaker dispersive decay in low dimension. In particular, this is the reason why our analysis and methods are much more involved than in the work done in the 3-dimensional case, and also why more is required in terms of the structure of the nonlinearity in two dimensions.

In what follows, we review some of the work which is relevant to our result, and which has been done for the wave-Klein-Gordon system. Some relatively recent work in the 3-dimensional setting that relates with this model started with the work of Georgiev [11], and Katayama [16], who proved the global existence of small amplitude solutions to coupled systems of wave and Klein-Gordon equations under certain suitable conditions on the nonlinearity. These include the *null condition* of Klainerman [17] on self-interactions between wave components. Katayama's conditions imposed on the nonlinearities are weaker than the *strong null condition* used by Georgiev. Relevant to our work is also Delort's work on Klein-Gordon systems [7, 4, 3, 5, 6]. More recently, a related problem was also studied by LeFloch, Ma [18] and Wang [31] as a model for the full Einstein-Klein-Gordon system. There the authors prove global existence of solutions to wave-Klein-Gordon systems with quasilinear quadratic nonlinearities satisfying suitable conditions, when initial data are small, smooth and compactly supported. An idea

used there is that of employing hyperbolic coordinates in the forward light cone; this was first introduced in the wave context in the work of Tataru [30], and later reintroduced by LeFloch-Ma in [18] under the name *hyperboloidal foliation method*. In [14], Ionescu and Pausader also prove global regularity and modified scattering in the case of small smooth initial data that decay at suitable rates at infinity, but not necessarily compactly supported. We also cite a work by Dong-Wyatt [8] in which global well-posedness is proved for a quadratic semilinear wave-Klein-Gordon interaction in which there are no derivatives on the wave component of the solution. Global stability for the full Einstein-Klein-Gordon system has been then proved by LeFloch-Ma [19] in the case of small smooth perturbations that agree with a Schwarzschild solution outside a compact set (see also Wang [32]), and by Ionescu-Pausader [15] in the case of unrestricted data.

Most of the results we know concerning global existence of small amplitude solutions in lower space dimension are due to Ma. His results apply to compactly supported Cauchy data (a restriction that our current result avoids) so that the hyperboloidal foliation method can be used; see [22]. In particular, in [21], Ma combines this method with a normal form argument to treat some quasilinear quadratic nonlinearities, while in [23], he treats wave-Klein-Gordon coupled system with more general quasilinear terms with null structure, but only in the case of a weakly coupled system. We also cite [20, 25, 10], in which Ma studies the case of some semilinear quadratic interactions. In [24], the restriction on the support of initial data is bypassed for the 1-dimensional problem, but there only a semilinear cubic model wave-Klein-Gordon system is discussed. An example of quadratic semilinear wave-Klein-Gordon system is also studied by Dong-Wyatt in [9]. The only global well-posedness result known at present for strongly-coupled quadratic and quasilinear wave-Klein-Gordon systems is an example studied by the second author in [29], where a Q_0 -type interaction is considered.

1.2. The linear system and energy functionals

The system (1.1) is a nonlinear version of the linear diagonal system

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t, x) = 0, \\ (\partial_t^2 - \Delta_x + 1)v(t, x) = 0, \end{cases} \quad (t, x) \in (0, +\infty) \times \mathbb{R}^2. \quad (1.4)$$

The linear system (1.4) has an associated conserved energy given by

$$E(t; u, v) = \int_{\mathbb{R}^2} u_t^2 + u_x^2 + v_t^2 + v_x^2 + v^2 \, dx. \quad (1.5)$$

This is no longer a conserved quantity for the nonlinear system (1.1), but we will still use it to define the associated energy space and also all our main function spaces. The system (1.4) is a well-posed linear evolution in the space $\mathcal{H}^0$ with norm

$$\|(u[t], v[t])\|_{\mathcal{H}^0}^2 := \|u\|_{H^1}^2 + \|u_t\|_{L^2}^2 + \|v\|_{H^1}^2 + \|v_t\|_{L^2}^2,$$

where we use the following notation for the Cauchy data in (1.1) at time t :

$$(u[t], v[t]) := (u(t), u_t(t), v(t), v_t(t)).$$

The higher-order energy spaces for the system (1.4) are the spaces $\mathcal{H}^n$ endowed with the norm

$$\|(u_0, u_1, v_0, v_1)\|_{\mathcal{H}^n}^2 := \sum_{|\alpha| \leq n} \|\partial_x^\alpha (u_0, u_1, v_0, v_1)\|_{\mathcal{H}^0}^2,$$

where $n \geq 1$. We will also use the energy spaces for the nonlinear system (1.1).

Above and in the sequel, we use notation conventions as follows: ∂ denotes time and spatial derivatives, ∂_x denotes only the spatial derivatives, ∇ represents the space-time gradient, and ∇_x represents

the spatial gradient only. Also, LHS (resp. RHS) will be an abbreviation for “left hand side” (resp. ‘right hand side’).

1.3. Scaling, criticality and local well-posedness

One important notion that will guide our efforts in proving optimal results in terms of regularity is given by the scaling of the problem. The nonlinear terms play a crucial role in the long time dynamics of the solution and also influence the critical regularity close to which we seek to prove our local and then global existence results. To properly explain the relation between the nonlinearity and the critical homogeneous Sobolev space, we connect the higher dimensions with the notion of criticality by means of the scaling symmetry which our system (1.1) possesses in the high frequency limit

$$\begin{cases} u(t, x) \rightarrow \lambda^{-1} u(\lambda t, \lambda x) \\ v(t, x) \rightarrow \lambda^{-1} v(\lambda t, \lambda x). \end{cases}$$

This, in particular, leads to the critical Sobolev space $\mathcal{H}^{s_c}$ with $s_c = d/2 + 1$. For our problem, the critical Sobolev exponent is $s_c = 2$. In particular, it is not too difficult to show that in two dimensions, (1.1) is locally well-posed in $\mathcal{H}^n$ for $n \geq 4$ (or $\mathcal{H}^{3+\epsilon}$ if we do not restrict ourselves to integers).

To describe the lifespan of the solutions, we define the time dependent control norms

$$A := \sum_{|\alpha|=1} \|\partial^\alpha u\|_{L^\infty} + \sum_{|\alpha|=1} \|\partial^\alpha v\|_{L^\infty}, \quad (1.6)$$

respectively,

$$B := \sum_{|\alpha|=2} \|\partial^\alpha u\|_{L^\infty} + \sum_{|\alpha|=2} \|\partial^\alpha v\|_{L^\infty}. \quad (1.7)$$

Here, A is a scale invariant quantity which will be required to remain small throughout in order to preserve the hyperbolicity of the problem. Then we have the following local result:

Theorem 1.

- The problem (1.1) is locally well-posed for initial data in $\mathcal{H}^n$, $n \geq 4$, with the additional property that A is small.*
- Uniform finite speed of propagation holds for as long as A remains small.*
- The solutions can be continued for as long as $\int B \, dt$ remains finite, and for each $k \geq 0$, we have the following energy estimate:*

$$\|(u, v)(t)\|_{\mathcal{H}^k} \lesssim e^{c \int_0^t B(s) \, ds} \|(u, v)(0)\|_{\mathcal{H}^k}. \quad (1.8)$$

We remark that this also shows the continuation of higher regularity of the solution for as long as $\int B \, dt$ remains finite.

1.4. The main result

To study the small data long-time well-posedness problem for the nonlinear evolution (1.1), one needs to add some decay assumptions for the initial data to the mix. At this point, we can already state a preliminary version of our main theorem, which clarifies the type of initial data we are considering.

Theorem 2. *Let $h \geq 8$. Assume that the initial data $(u[0], v[0])$ for (1.1) satisfies*

$$\|(u[0], v[0])\|_{\mathcal{H}^{2h}} + \|x \partial_x (u[0], v[0])\|_{\mathcal{H}^h} + \|x^2 \partial_x^2 (u[0], v[0])\|_{\mathcal{H}^0} \leq \epsilon \ll 1. \quad (1.9)$$

Then the equation (1.1) is almost globally well-posed in the same space (i.e., the solution exists up to time $T_\epsilon = e^{\frac{c}{\epsilon}}$, where c is a small positive universal constant).

Here, we made an effort to limit the decay assumptions (i.e. use only x^2 type decay), but we did not attempt to fully optimize the choice of h .

1.5. Vector fields and the main result revisited

To provide a better form of the above theorem, one should also describe the global bounds and decay properties of the solutions. This analysis is closely related to the family of Killing vector fields associated to our problem (i.e., of vector fields that commute with the linear evolution (1.4)). We will also add to the list below the scaling vector field S , which is not Killing but plays an important role in the proof of our main result in Theorem 3 below. The commuting vector fields together with the scaling vector field are as follows:

$$\partial_t, \partial_1, \partial_2, \quad (1.10)$$

$$\Omega_{ij} = x_j \partial_i - x_i \partial_j, \quad (1.11)$$

$$\Omega_{0i} = t \partial_i + x_i \partial_t \quad (1.12)$$

$$\mathcal{S} = t \partial_t + r \partial_r, \quad (1.13)$$

where $1 \leq i \neq j \leq 2$, $r = |x|$ and $\partial_r = \frac{x}{r} \cdot \nabla_x$. The expressions in (1.10) correspond to translations in the coordinate directions; (1.11) correspond to rotations in the space variable x ; (1.11) and (1.12) correspond to the Lorentz transformations; finally, (1.13) corresponds to dilations. To obtain symmetrical notations, we will sometimes write $t = x_0$ and $\partial_r = \partial_0$. Note that in (1.11), we can restrict to $1 \leq i < j \leq 2$ by skew-symmetry. Thus, we have a total of 8 different vector fields.

We refer to all the vector fields (1.11) and (1.12) as the *Klainerman vector fields*, and we will denote all of them by Z

$$Z := \{\Omega_{ij}, \Omega_{0i}\}. \quad (1.14)$$

We denote the full set of vector fields associated to the symmetries of the linear problem as

$$\mathcal{Z} := \{\partial_0, \partial_1, \partial_2, \Omega_{ij}, \Omega_{0i}\}. \quad (1.15)$$

For a multiindex $\gamma = (\alpha, \beta)$, we denote

$$\mathcal{Z}^\gamma = \partial^\alpha \mathcal{Z}^\beta,$$

and define the size of such a multi-index by

$$|\gamma| = |\alpha| + h|\beta|,$$

where h is a positive integer that will be specified later and describes the balance between Klainerman vector fields and regular derivatives in our analysis. We use these vector fields in order to define the higher-order counterparts of the energy functional (1.5):

a) the energy $E^n(t, u, v)$ measures the regularity in the function space $\mathcal{H}^n$ of the solutions,

$$E^n(t, u, v) := \sum_{|\alpha| \leq n} E(t; \partial^\alpha u, \partial^\alpha v). \quad (1.16)$$

b) the energy $E^{[n]}(t, u, v)$ keeps track of Z vector fields applied to the solution in addition to regular derivatives,

$$E^{[n]}(t, u, v) := \sum_{|\gamma| \leq n} E(t; \mathcal{Z}^\gamma u, \mathcal{Z}^\gamma v). \quad (1.17)$$

The energy functional (1.5) represents the natural energy of the Klein-Gordon equation together with the energy of the wave equation. The functional E^n is the energy associated to the differentiated variables and, as usual, helps us control the L^2 norm of these variables, equivalently saying it represents the higher-order energy that controls the H^{n+1} Sobolev norms of the solutions for $n \geq 3$. The last energy functional $E^{[n]}$ represents the energy associated to the system (1.1) to which we have also applied Klainerman vector fields. Using these energies, we are now able to state a more precise version of our main theorem:

Theorem 3. Assume that the initial data $(u[0], v[0])$ for (1.1) satisfies

$$\|(u[0], v[0])\|_{\mathcal{H}^{2h}} + \|x\partial_x(u[0], v[0])\|_{\mathcal{H}^h} + \|x^2\partial_x^2(u[0], v[0])\|_{\mathcal{H}^0} \leq \epsilon \ll 1. \quad (1.18)$$

Then the equation (1.1) is almost globally well-posed in $\mathcal{H}^{2h}$, with L^2 bounds as follows:

$$E^{[2h]}(t, u, v) \lesssim \epsilon^2, \quad (1.19)$$

and pointwise bounds

$$|\partial^j v| \lesssim \epsilon \langle t+r \rangle^{-1}, \quad j = \overline{0, 3}, \quad (1.20)$$

$$|\partial^j u| \lesssim \epsilon \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}}, \quad j = \overline{1, 3}, \quad (1.21)$$

$$|\partial^j Zu| \lesssim \epsilon, \quad j = \overline{0, 2}. \quad (1.22)$$

Remark 1.1. The pointwise bounds stated in the theorem represent baseline estimates. In fact, we obtain slightly better bounds in various regimes. These gains will be made specific later in the last section.

In the successor to this paper, we combine the bounds of this paper with asymptotic analysis for both the wave and the Klein-Gordon equation in order to convert the above result into a global result:

Theorem 4. Assume that the initial data $(u[0], v[0])$ for (1.1) satisfies

$$\|(u[0], v[0])\|_{\mathcal{H}^{2h}} + \|x\partial_x(u[0], v[0])\|_{\mathcal{H}^h} + \|x^2\partial_x^2(u[0], v[0])\|_{\mathcal{H}^0} \leq \epsilon. \quad (1.23)$$

Then the equation (1.1) is globally well-posed in $\mathcal{H}^{2h}$, with L^2 bounds as follows:

$$E^{[2h]}(t, u, v) \lesssim \epsilon^2 t^{C\epsilon}, \quad (1.24)$$

and pointwise bounds

$$|\partial^j v| \lesssim \epsilon \langle t+r \rangle^{-1}, \quad j = \overline{0, 3}, \quad (1.25)$$

$$|\partial^j u| \lesssim \epsilon \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}}, \quad j = \overline{1, 3}, \quad (1.26)$$

$$|\partial^j Zu| \lesssim \epsilon, \quad j = \overline{1, 2}. \quad (1.27)$$

1.6. The structure of the paper

We begin in the next section with an overview of the main steps of the proof. We follow a standard approach in which our proof has two main steps: (i) vector field energy estimates and (ii) pointwise bounds derived from energy estimates (sometimes called Klainerman-Sobolev inequalities). We depart from the standard setting in that our energy estimates are space-time L^2 local energy bounds, localized to dyadic regions $C_{TS}^\pm$, where T stands for dyadic time, S for the dyadic distance to the cone, and $\pm$ for the interior/exterior cone. Similarly, our pointwise bounds are akin to Sobolev embeddings or interpolation inequalities in the same type of regions.

The energy estimates are carried in the next three sections, in three steps: (i) for the linearized equation, (ii) for the solution and its higher derivatives, and finally (iii) for the vector fields applied to the solution.

Finally, the last section is devoted to the pointwise bounds, which are derived from the local energy bounds via interpolation inequalities in the same $C_{TS}^\pm$, with the extra step of also using the wave or Klein-Gordon equation in several interesting cases.

2. An overview of the proof

We begin with a prerequisite for the proof, which has to do with the local in time theory for our evolution (1.1). The three main properties, also summarized in Theorem 1, are as follows:

- (1) Local well-posedness in $\mathcal{H}^4$ (also in $\mathcal{H}^n$ for $n \geq 4$).
- (2) Continuation of $\mathcal{H}^4$ solutions for as long as $\partial^2(u, v)$ remains bounded, also with propagation of higher regularity (i.e., bounds in $\mathcal{H}^n$ for all n).
- (3) Uniform finite speed of propagation as long as $|\nabla v|$ stays pointwise small.

Given these three facts, our proof is set up as a bootstrap argument, where the bootstrap assumption is on pointwise decay bounds for the solution. These bounds are as follows:

$$|Zu| \leq C\epsilon \langle t-r \rangle^\delta, \quad (2.1)$$

$$|\partial u| \leq C\epsilon \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}}, \quad (2.2)$$

$$|Z\partial^j u| \leq C\epsilon \langle t-r \rangle^{-\delta_1}, \quad j = \overline{1, 2}, \quad (2.3)$$

$$|\partial^j u| \leq C\epsilon \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}-\delta_1}, \quad j = \overline{2, 3}, \quad (2.4)$$

$$|\partial^j v| \leq C\epsilon \langle t+r \rangle^{-1-\delta_1} \langle t-r \rangle^{\delta_1}, \quad j = \overline{1, 3}. \quad (2.5)$$

Here, $0 < \delta \ll \delta_1$ are fixed small positive universal constant. However, C is a large universal constant, which will be improved as part of the conclusion of the proof. The proof is structured into two main steps. For expository purposes, we provide first a simplified outline of these two steps and refine this later.

1. Energy estimates. Here, one considers a solution to (1.1) in a time interval $[0, T_0]$, which is a-priori assumed to satisfy the bootstrap assumptions (2.1), (2.2), (2.3), (2.4) and (2.5). Then the conclusion is that the solution (u, v) satisfies the following energy estimates in $[0, T_0]$:

$$E^{[2h]}(u, v)(t) \lesssim \langle t \rangle^{\tilde{C}\epsilon} E^{[2h]}(u, v)(0), \quad t \in [0, T_0]. \quad (2.6)$$

Here, $\tilde{C}$ is a large constant which depends on C in our bootstrap assumption, $\tilde{C} \approx C$. However, the implicit constant in (2.6) cannot depend on C . No restriction is imposed on the lifespan bound T_0 .

2. Pointwise bounds. Here, we assume that we have a solution (u, v) to (1.1) in a time interval $[0, T_0]$, which satisfies the energy bounds

$$E^{[2h]}(u, v)(t) \lesssim \epsilon \langle t \rangle^{\tilde{C}\epsilon}, \quad t \in [0, T_0]. \quad (2.7)$$

Then we show that the solution (u, v) satisfies the pointwise bounds

$$\|Zu\|_{L^\infty} \leq \epsilon \langle t \rangle^{\tilde{C}\epsilon}, \quad (2.8)$$

$$|\partial u| \leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}}, \quad (2.9)$$

$$\|Z\partial^j u\|_{L^\infty} \leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t-r \rangle^{-2\delta_1}, \quad j = \overline{1, 2}, \quad (2.10)$$

$$|\partial^j u| \leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t+r \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}-2\delta_1}, \quad j = \overline{2, 3}, \quad (2.11)$$

$$|\partial^j v| \leq \epsilon \langle t \rangle^{\tilde{C}\epsilon} \langle t+r \rangle^{-1-2\delta_1} \langle t-r \rangle^{2\delta_1}, \quad j = \overline{0, 3}. \quad (2.12)$$

Here, the lifespan T_0 is again arbitrary.

In both steps, the time T_0 is arbitrary. However, in order to close the bootstrap argument, one needs to recover (2.1), (2.2), (2.3), (2.4) and (2.5) from (2.8), (2.9), (2.10), (2.11) and (2.12). This requires

$$T_0^{\tilde{C}\epsilon} \ll C,$$

which is satisfied provided that

$$T_0 \ll e^{\frac{C}{\epsilon}}$$

(i.e., our almost global result).

In the classical results on global or almost global well-posedness in $3+1$ dimensions, one uses a large number of vector fields both in the energy estimates and in the pointwise bounds, and the argument works exactly as outlined above. Notably, both steps require only fixed time bounds, and the pointwise bounds are akin to an improved form of the Sobolev embeddings, which are now referred to as Klainerman-Sobolev estimates.

By contrast, such a strategy would be too naive in our work, both because we work in $2+1$ dimensions and there is less dispersive decay, and because our problem is strongly quasilinear. Instead, a good portion of our analysis happens in space-time regions which are adapted to the light cone geometry. Thus, the next step is to describe our decomposition of the space-time.

We first consider a dyadic decomposition in time into sets

$$C_T := \{T \leq t \leq 2T\}. \quad (2.13)$$

Further, we dyadically decompose each of the C_T 's with respect to the size of $t-r$, which measures how far or close we are to the cone

$$\begin{aligned} C_{TS}^+ &:= \{(t, x) : S \leq t-r \leq 2S, T \leq t \leq 2T\}, \text{ where } 1 \leq S \lesssim T, \\ C_{TS}^- &:= \{(t, x) : S \leq r-t \leq 2S, T \leq t \leq 2T\}, \text{ where } 1 \leq S \lesssim T; \end{aligned} \quad (2.14)$$

see Figure 1.

That still leaves the exterior region

$$C_T^{out} := \{T \leq t \leq 2T, r \gg T\}. \quad (2.15)$$

Here, C_{TS}^+ represents a spherically symmetric dyadic region inside the cone with width S , distance S from the cone, and time length T . C_{TS}^- is the similar region outside the cone where, far from the cone, we would have $T \lesssim S$. To simplify the exposition, we will use the notation C_{TS} as a shorthand for either C_{TS}^+ or C_{TS}^- . Such a decomposition has been introduced before by Metcalfe-Tataru-Tohaneanu [26] in a linear setting; we largely follow their notations.

In the above definition of the C_{TS} sets, we limit S to $S \geq 1$ because our assumptions are invariant with respect to unit size translations. In particular, this leaves out a conical shell region along the side of the cone $t=r$, which intersects both the interior and the exterior of the cone. To also include this region in our analysis, we redefine

$$C_{T1} := \{(t, x) : |t-r| \leq 2, T \leq t \leq 2T\}, \text{ where } S \sim 1. \quad (2.16)$$

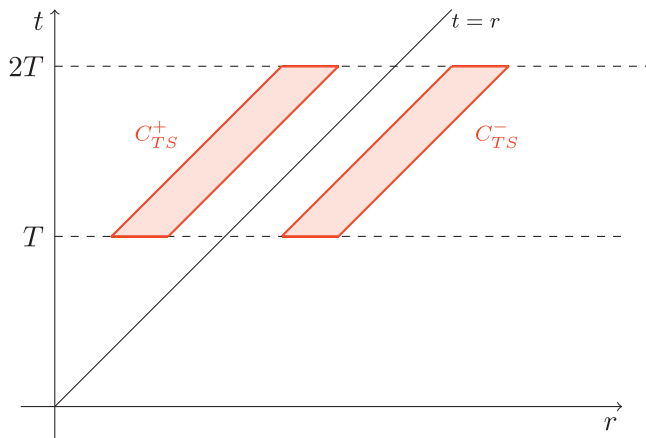


Figure 1. 1D vertical section of space-time regions $C_{TS}^\pm$.

This decomposition plays roles as follows in the two steps above:

- (1) While the energy estimate (2.6) holds as stated, a key part of its proof involves separately estimating the energy growth generated in each of the $C_{TS}^\pm$ set. This in turn requires improved local energy bounds for the solutions in such space-time regions. On the upside, at the conclusion of this step, we obtain not only fixed time energy estimates but also localized energy estimates in $C_{TS}^\pm$.
- (2) The pointwise bounds in the second step are proved locally in each of the $C_{TS}^\pm$ regions, based on the local energy bounds there rather than the global fixed time bounds.

One downside of the localizations required in our proof of the pointwise bounds is that it is rather delicate to close the arguments in the fixed time interval $[0, T_0]$, as it would require dealing with Sobolev type embeddings based on vector fields not necessarily compatible with the boundary. This is not a critical problem, and there are multiple ways to deal with it – for instance, by choosing the boundaries more carefully than simply time slices. Here, instead, we completely bypass the issue in a different way, by truncating the nonlinearity. Precisely, suppose we want to solve the equation (1.1) up to some time T_0 . Then we consider a smooth cutoff function χ_{T_0} , which is supported in $[0, 2T_0]$ and equals 1 in the time interval $[0, T_0]$. Thus, χ_{T_0} selects the region $t < T_0$ and replaces the equation (1.1) with the truncated version

$$\begin{cases} (\partial_t^2 - \Delta_x)u(t, x) = \chi_{T_0}(t)[\mathbf{N}_1(v, \partial v) + \mathbf{N}_2(u, \partial v)], \\ (\partial_t^2 - \Delta_x + 1)v(t, x) = \chi_{T_0}(t)[\mathbf{N}_1(v, \partial_1 u) + \mathbf{N}_2(u, \partial u)] \end{cases} \quad (t, x) \in [0, +\infty) \times \mathbb{R}^2. \quad (2.17)$$

Such a cutoff will make no difference in the proof but instead insures that beyond time $2T_0$ the solution (u, v) solves the corresponding linear constant coefficient problem.

This is similar to an idea introduced by Bourgain in the study of the semilinear dispersive equations [2] with a similar purpose (i.e., to avoid sharp time truncations in function spaces).

Another feature of our proof is that we use the finite speed of propagation to isolate and consider separately the exterior region $\{r \gg T\}$. Precisely, for large R , the problem localizes to the region

$$\{|x| \approx R, \quad |t| \ll R\}.$$

In this region the weights defining the initial data size are all constant, so it is enough to carry out standard energy estimates and obtain pointwise bounds via Sobolev embeddings at fixed time. This analysis is carried out in the next section, where we prove Theorem 1. As a consequence of Theorem 1 and Sobolev embeddings, we immediately obtain the following:

Proposition 2.1. Assume that the initial data $(u, v)[0]$ for (1.1) satisfy (1.9). Then the equations (1.1) and (2.17) are globally well-posed in the region $C^{out} := \{t \leq \frac{1+|x|}{4}\}$, with energy bounds

$$\|\partial^{\leq 2h}(u, v)[t]\|_{\mathcal{H}^0} + \|x\partial^{\leq h}\partial(u, v)[t]\|_{\mathcal{H}^0} + \|x^2\partial^2(u, v)[t]\|_{\mathcal{H}^0} \leq \epsilon, \quad (2.18)$$

and pointwise bounds

$$\begin{aligned} \|\langle x \rangle^{1+\delta} \partial^j u\|_{L^\infty} &\lesssim \epsilon, & j = \overline{2, 3}, \\ \|\langle x \rangle^{1+\delta} \partial^j v\|_{L^\infty} &\lesssim \epsilon, & j = \overline{0, 3}, \end{aligned} \quad (2.19)$$

and

$$\|\langle x \rangle \partial u\|_{L^\infty} \lesssim \epsilon. \quad (2.20)$$

For later use, we also state an alternative form of the above proposition, where the smallness assumption on the initial data is replaced by bootstrap assumptions akin to (2.2), (2.4), (2.5) but restricted to the exterior region. Denoting

$$\begin{aligned} E^{out, [2h]}(u, v)(t) &= \|\partial^{\leq 2h}(u, v)[t]\|_{\mathcal{H}^0(C^{out})} + \|x\partial^{\leq h}\partial(u, v)[t]\|_{\mathcal{H}^0(C^{out})} \\ &\quad + \|x^2\partial^2(u, v)[t]\|_{\mathcal{H}^0(C^{out})}, \end{aligned} \quad (2.21)$$

we have

Proposition 2.2. Let (u, v) be a solution for (1.1) in C^{out} which satisfies the bounds

$$|\partial u| + |\partial v| \ll 1, \quad |\partial^2 u| + |\partial^2 v| \ll \langle x \rangle^{-1}.$$

Then we have the uniform global bounds

$$E^{out, [2h]}(u, v)(t) \lesssim E^{out, [2h]}(u, v)(0).$$

As we will see in the next section, the proof of this proposition is a step in the proof of the previous proposition.

For the bulk part, where we track the evolution of the vector field energy $E^{[2h]}(u, v)$, we define a stronger norm X^T for (u, v) associated to dyadic time intervals, as well as a similar norm Y^T for the right-hand side of the equation. These norms will be introduced later in the paper; their definitions are given in (4.40) for X^T , respectively in (5.10) for Y^T . Then we replace the energy bound (2.6) with the stronger X^T and Y^T bounds:

Proposition 2.3. Let (u, v) be a solution to (1.1) or (2.17) in $[0, T_0]$ which satisfies the bootstrap bounds (2.1), (2.2), (2.3), (2.4) and (2.5). Then we have

$$\|\mathcal{Z}^\gamma(u, v)\|_{X^T} \lesssim \epsilon T^{\tilde{C}\epsilon}, \quad |\gamma| \leq 2h, \quad T \in [0, T_0]. \quad (2.22)$$

In addition,

$$\|\mathcal{Z}^\gamma(\square u, (\square + 1)v)\|_{Y^T} \lesssim \epsilon T^{\tilde{C}\epsilon}, \quad |\gamma| \leq h, \quad T \in [0, T_0]. \quad (2.23)$$

The bounds in this Proposition will be proved in Section 4 where we consider the linearized equations, in Section 5 which is devoted to the higher energy estimates, and in Section 6 where we establish the vector fields bounds.

In this context, our pointwise bounds will be linear and localized to dyadic time regions:

Proposition 2.4. *Let (u, v) be functions in C_T which satisfy the bounds*

$$\|\mathcal{Z}^\gamma(u, v)\|_{X^T} \leq 1, \quad |\gamma| \leq 2h, \quad (2.24)$$

as well as

$$\|\mathcal{Z}^\gamma(\square u, (\square + 1)v)\|_{Y^T} \leq 1, \quad |\gamma| \leq h. \quad (2.25)$$

Then we have the pointwise bounds

$$\|Zu\|_{L^\infty} \lesssim 1, \quad (2.26)$$

$$|\partial u| \lesssim T^{-\frac{1}{2}} S^{-\frac{1}{2}}, \quad (2.27)$$

$$\|Z\partial^j u\|_{L^\infty} \lesssim S^{-2\delta_1}, \quad j = \overline{1, 2}, \quad (2.28)$$

$$|\partial^j u| \lesssim T^{-\frac{1}{2}} S^{-\frac{1}{2}-2\delta_1}, \quad j = \overline{2, 3}, \quad (2.29)$$

$$|\partial^j v| \lesssim T^{-1-2\delta_1} S^{2\delta_1}, \quad j = \overline{0, 3}. \quad (2.30)$$

Taken together, the last three propositions imply the conclusion of our main result in Theorem 3. This Proposition is proved in the last section of the paper.

3. Local well-posedness, continuation and the exterior region C^{out}

In this section, we prove Theorem 1. As a consequence, we derive Proposition 2.1.

Proof of Theorem 1. The proof of this theorem is very similar to the proof of the local well-posedness for quasilinear wave equations (see, for instance, Hörmander [12], Sogge [28], Racke [27]), as well as the more modern treatment in Ifrim-Tataru [13]. We sketch here its main steps.

(i) *Energy estimates and the quasilinear energy.* A key part of the argument is played by energy estimates, which we discuss here in a simpler setting, for the inhomogeneous linear problem

$$\begin{cases} (\partial_t^2 - \Delta_x)U(t, x) = \mathbf{N}_1(v, \partial V) + \mathbf{N}_2(u, \partial V) + F \\ (\partial_t^2 - \Delta_x + 1)V(t, x) = \mathbf{N}_1(v, \partial U) + \mathbf{N}_2(u, \partial U) + G, \end{cases} \quad (3.1)$$

with initial conditions

$$\begin{cases} (U, V)(0, x) = (U_0(x), V_0(x)), \\ (\partial_t U, \partial_t V)(0, x) = (U_1(x), V_1(x)). \end{cases} \quad (3.2)$$

At leading order, this system agrees with the linearized equation discussed in the next section, Section 4.

Our starting point in the proof of the energy estimates is the energy functional associated to the corresponding linear equation

$$E(U, V) := \frac{1}{2} \int_{\mathbb{R}^2} U_t^2 + U_x^2 + V_t^2 + V_x^2 + V^2 \, dx = \int_{\mathbb{R}^2} e_0(t, x) \, dx,$$

where e_0 is the linear energy density

$$e_0(t, x) = \frac{1}{2} [U_t^2 + U_x^2 + V_t^2 + V_x^2 + V^2].$$

This would be the obvious candidate for the energy functional with respect to which we would like to prove the energy estimates required for the local well-posedness result. However, the right-hand side of

the equations (3.1) contains second-order derivatives of (U, V) , so if one tries to prove energy bounds via this functional, there would be a loss of derivatives. This is a common issue when working with quasilinear nondiagonalisable hyperbolic systems of PDEs. To avoid this loss of derivatives, we consider a quasilinear type modification of this energy, which has the form

$$E^{quasi}(U, V) := E(U, V) + \int_{\mathbb{R}^2} B_1(v; U, V) + B_2(u; U, V) dx = \int_{\mathbb{R}^2} e^{quasi}(t, x) dx, \quad (3.3)$$

where the quasilinear energy density is

$$e^{quasi}(t, x) := e_0(t, x) + B_1(v; U, V) + B_2(u; U, V).$$

Here, the trilinear forms B_1 and B_2 are associated to the null forms $\mathbf{N}_1, \mathbf{N}_2$ in (1.1) in a linear fashion. Precisely, corresponding to the three bilinear forms in (1.3), we have the associated corrections

$$\begin{cases} B_{0i}(w; U, V) := w_t U_i \partial V, \\ B_{ij}(w; U, V) := w_i U_j \partial V - w_j U_i \partial V, \\ B_0(w; U, V) := -w_x \cdot U_x \partial V, \end{cases} \quad (3.4)$$

for $w = u, v$.

We will use the above energy functional in order to study the well-posedness of (3.1) in $\mathcal{H}^0$. For this, we assume that (u, v) are known and that we control in a pointwise fashion their associated control parameters (A, B) introduced in (1.6), (1.7). Then we have

Lemma 3.1. *Assume that $A \leq \delta \ll 1$ and $B \in L^\infty$. Then the equation (3.1) is well-posed in $\mathcal{H}^0$ and the following properties hold:*

(i) *Energy equivalence:*

$$E^{quasi}(t, U, V) = (1 + O(\delta)) \|(U, V)[t]\|_{\mathcal{H}^0}^2. \quad (3.5)$$

(ii) *Energy estimate:*

$$\frac{d}{dt} E^{quasi}(t, U, V) \lesssim B(t) \|(U, V)[t]\|_{\mathcal{H}^0}^2 + \|(U, V)[t]\|_{\mathcal{H}^0} \|(F, G)[t]\|_{L^2}. \quad (3.6)$$

Proof. Part (i) is trivial. For clarity, we prove (ii) in the homogeneous case (i.e., when $F, G = 0$) and leave the minor inhomogeneous adaptation to the reader. To see what is needed for the energy estimate computation, we begin by deriving the density flux relation associated to our energy density e^{quasi} . We begin with the first component of e^{quasi} , which is e_0 :

$$\partial_t e_0(t, x) = \sum_{j=1}^2 \partial_{x_j} (U_t U_j + V_t V_j) + U_t \square U + V_t (\square + 1) V.$$

The last two terms can be expanded as follows:

$$\begin{cases} U_t \square U = U_t (\mathbf{N}_1(v, \partial V) + \mathbf{N}_2(u, \partial V)), \\ V_t (\square + 1) V = V_t (\mathbf{N}_1(v, \partial U) + \mathbf{N}_2(u, \partial U)). \end{cases} \quad (3.7)$$

Next, we turn our attention to the corrections

$$\partial_t B_i(w; U, V) = B_i(w_t; U, V) + B_i(w; U_t, V) + B_i(w; U, V_t), \quad w = u, v, \quad i = \overline{1, 2}. \quad (3.8)$$

Here, we will combine the first, respectively the second, terms on both RHS in the above equations with $\partial_t B_i(v; U, V)$, respectively with $\partial_t B_i(u; U, V)$. We obtain that

$$\begin{cases} U_t \mathbf{N}_1(v, \partial V) + V_t \mathbf{N}_1(v, \partial U) + \partial_t B_1(v; U, V) = \partial_x C_1(\partial v, \partial U, \partial V) + D_1(\partial^2 v, \partial U, \partial V), \\ U_t \mathbf{N}_2(u, \partial V) + V_t \mathbf{N}_2(u, \partial U) + \partial_t B_2(u; U, V) = \partial_x C_2(\partial u, \partial U, \partial V) + D_2(\partial^2 u, \partial U, \partial V), \end{cases}$$

where C_i and D_i are trilinear forms. Their structure is unimportant here, but will be investigated later in the next section.

Summing up all these terms, we obtain the following energy flux relation for solution to (3.1):

$$\partial_t e^{quasi}(t, x) = \sum_{j=1}^2 \partial_j f_j + g, \quad (3.9)$$

where the fluxes f_j have schematically the expressions

$$f_j = U_j U_t + V_j V_t + C_1(\partial v, \partial U, \partial V) + C_2(\partial u, \partial U, \partial V), \quad (3.10)$$

and the source g has the form

$$g = D_1(\partial^2 v, \partial U, \partial V) + D_2(\partial^2 u, \partial U, \partial V). \quad (3.11)$$

To complete the proof of the energy estimates, we use the relation (3.9) to obtain

$$\frac{d}{dt} \int_0^t E^{quasi}(t, U, V) = \int_0^t g \, dx,$$

where it remains to bound the RHS. We bound the $\partial^2 u$ and $\partial^2 v$ factors in g in L^∞ by $B(t)$. The ∂V and ∂U factors are bounded by the energy. Then the conclusion of the Lemma follows. $\square$

(ii) *Local existence.* We construct a local solution to (1.1)–(1.2) using an iteration scheme. We set

$$(u^{-1}, v^{-1}) \equiv 0,$$

and define (u^m, v^m) , $m = 0, 1, \dots$, inductively by

$$\begin{cases} (\partial_t^2 - \Delta_x) u^m(t, x) = \mathbf{N}_1(v^{m-1}, \partial v^m) + \mathbf{N}_2(u^{m-1}, \partial v^m), \\ (\partial_t^2 - \Delta_x + 1) v^m(t, x) = \mathbf{N}_1(v^{m-1}, \partial u^m) + \mathbf{N}_2(u^{m-1}, \partial u^m), \end{cases} \quad (t, x) \in [0, +\infty) \times \mathbb{R}^2, \quad (3.12)$$

with

$$\begin{cases} (u^m, v^m)(0, x) = (u_0(x), v_0(x)), \\ (\partial_t u^m, \partial_t v^m)(0, x) = (u_1(x), v_1(x)). \end{cases} \quad (3.13)$$

We assume the data to be in $\mathcal{S}$ so that, by the local existence theorem for linear equations, the above system admits a C^∞ solution for every m . We can later remove this assumption by an approximation argument.

To set the notations, we assume that at the initial time $t = 0$, we have the Lipschitz bound

$$A(0) := \sum_{|\alpha|=1} \|\partial^\alpha u(0)\|_{L^\infty} + \sum_{|\alpha|=1} \|\partial^\alpha v(0)\|_{L^\infty} \leq \delta \ll 1, \quad (3.14)$$

as well as the Sobolev bound

$$\|(u, v)[0]\|_{\mathcal{H}^n} \leq M. \quad (3.15)$$

Then we claim that there exists a time T_0 sufficiently small depending on δ and M such that the following bounds for the sequence (u^m, v^m) hold for all $t \in [0, T_0]$:

$$A^m(t) := \sum_{|\alpha|=1} \|\partial^\alpha u^m(t)\|_{L^\infty} + \sum_{|\alpha|=1} \|\partial^\alpha v^m(t)\|_{L^\infty} \leq 2\delta, \quad (3.16)$$

as well as the uniform energy bounds

$$\|(u^m(t), v^m(t))\|_{\mathcal{H}^n} \leq 2M \quad \forall 0 \leq t \leq T_0, \forall m \geq 0. \quad (3.17)$$

When $m = 0$, the functions (u^0, v^0) solve the linear system (1.4), for which the energy is conserved. Bound (3.17) is hence trivially satisfied. As for A^0 , we use Sobolev embeddings

$$A^0(t) \leq A^0(0) + \int_0^t \|(\partial u_t^0, \partial v_t^0)(s)\|_{L^\infty} ds \leq \delta + CMT_0,$$

where C is some positive universal constant. Then we can choose T_0 small enough (e.g., $T_0 < \delta/(CM)$) to obtain (3.16).

Let us now suppose that our claim holds true for $m - 1$, $m \geq 1$, and prove it for index m .

To measure $\|(u^m(t), v^m(t))\|_{\mathcal{H}^n}^2$, we will use the energies

$$E^{quasi,n}(t, u^m, v^m) = \sum_{|\alpha| \leq n} E_{m-1}^{quasi}(t, D_x^\alpha u^m, D_x^\alpha v^m), \quad (3.18)$$

where E_{m-1}^{quasi} is obtained from E^{quasi} by substituting (u, v) with (u^{m-1}, v^{m-1}) . Thanks to the smallness assumption on A^{m-1} , the bound (3.5) holds for E_{m-1}^{quasi} , so we will harmlessly replace $\|(u^m, v^m)[t]\|_{\mathcal{H}^n}^2$ by $E^{quasi,n}(t, u^m, v^m)$ in (3.15) and (3.17).

We start by differentiating the system (3.12). For any $0 \leq k \leq n$, the differentiated variables $(\partial_x^k u^m, \partial_x^k v^m)$ solve the following system:

$$\begin{cases} (\partial_t^2 - \Delta_x) \partial_x^k u^m(t, x) = \mathbf{N}_1(v^{m-1}, \partial \partial_x^k v^m) + \mathbf{N}_2(u^{m-1}, \partial \partial_x^k v^m) + \mathbf{F}_k \\ (\partial_t^2 - \Delta_x + 1) \partial_x^k v^m(t, x) = \mathbf{N}_1(v^{m-1}, \partial \partial_x^k u^m) + \mathbf{N}_2(u^{m-1}, \partial \partial_x^k u^m) + \mathbf{G}_k, \end{cases}$$

where $\mathbf{F}_k$ and $\mathbf{G}_k$ are given by

$$\begin{cases} \mathbf{F}_k := \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_1(\partial_x^i v^{m-1}, \partial \partial_x^j v^m) + \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_2(\partial_x^i u^{m-1}, \partial \partial_x^j v^m) \\ \mathbf{G}_k := \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_1(\partial_x^i v^{m-1}, \partial \partial_x^j u^m) + \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_2(\partial_x^i u^{m-1}, \partial \partial_x^j u^m). \end{cases}$$

We seek to apply Lemma 3.1 for this system. By Sobolev embeddings, we control

$$B^{m-1} := B(u^{m-1}, v^{m-1}) \lesssim M,$$

and therefore, by (3.6), we have

$$\frac{d}{dt} E_{m-1}^{quasi}(\partial_x^k u^m, \partial_x^k v^m) \lesssim M \|(\partial_x^k u^m, \partial_x^k v^m)\|_{\mathcal{H}^0}^2 + \|(\partial_x^k u^m, \partial_x^k v^m)\|_{\mathcal{H}^0} \|(\mathbf{F}_k, \mathbf{G}_k)\|_{L^2}. \quad (3.19)$$

We claim that $(\mathbf{F}_k, \mathbf{G}_k)$ can be estimated as follows:

$$\|(\mathbf{F}_k, \mathbf{G}_k)\|_{L^2} \lesssim M\|(u^m, v^m)\|_{\mathcal{H}^n}, \quad k \leq n. \quad (3.20)$$

Indeed, using Hölder inequality and the Gagliardo-Nirenberg interpolation inequality, we see that

$$\begin{aligned} \|\mathbf{N}(\partial_x^i v^{m-1}, \partial \partial_x^j v^m)\|_{L^2} &\leq \|\partial \partial_x^i v^{m-1}\|_{L^{\frac{2(k-1)}{i-1}}} \|\partial \partial_x^{j+1} v^m\|_{L^{\frac{2(k-1)}{j}}} \\ &\leq \|\partial \partial_x^k v^{m-1}\|_{L^2}^\alpha \|\partial^2 v^{m-1}\|_{L^\infty}^{1-\alpha} \|\partial \partial_x^k v^m\|_{L^2}^\beta \|\partial^2 v^m\|_{L^\infty}^{1-\beta} \end{aligned} \quad (3.21)$$

with $\alpha = \frac{i-1}{k-1}, \beta = \frac{j}{k-1}$, and by Sobolev embeddings,

$$\begin{aligned} \|\mathbf{N}(\partial_x^i v^{m-1}, \partial \partial_x^j v^m)\|_{L^2} &\leq \|(u^{m-1}, v^{m-1})(\tau)\|_{\mathcal{H}^k}^\alpha \|(u^{m-1}, v^{m-1})(\tau)\|_{\mathcal{H}^3}^{1-\alpha} \|(u^m, v^m)(\tau)\|_{\mathcal{H}^k}^\beta \|(u^m, v^m)(\tau)\|_{\mathcal{H}^3}^{1-\beta} \\ &\leq \|(u^{m-1}, v^{m-1})(\tau)\|_{\mathcal{H}^n} \|(u^m, v^m)(\tau)\|_{\mathcal{H}^n}. \end{aligned}$$

This estimate applies for $\mathbf{N} = \mathbf{N}_1$, and also for $\mathbf{N} = \mathbf{N}_2$ where v can be freely replaced by u . This proves (3.20). We substitute (3.20) in (3.22) and sum over $k \leq n$. Then we obtain the energy relation

$$\frac{d}{dt} E^{quasi,n}(u^m, v^m) \lesssim M\|(u^m, v^m)\|_{\mathcal{H}^n}^2 \approx M E^{quasi,n}(u^m, v^m),$$

and by Gronwall's lemma,

$$E^{quasi,n}(t, u^m, v^m) \leq e^{CMt} E^{quasi}(0, u^m, v^m) \quad \forall 0 \leq t \leq T_0$$

for some positive constant C . Then we can choose T_0 small enough (e.g., $T_0 < 1/(2CM)$) to obtain (3.17).

However, the uniform bound of $(\partial u^m, \partial v^m)$ is proved as for the case $m = 0$ using Sobolev embeddings,

$$A^m(t) \leq A^m(0) + \int_0^t \|(\partial u_t^m, \partial v_t^m)(s)\|_{L^\infty} ds \leq \delta + 2CMT_0,$$

which proves that $A^m(t) \leq 2\delta$ if T_0 is chosen small enough.

The remaining step is to prove the convergence of the sequence of approximate solutions (u^m, v^m) as $m \rightarrow \infty$. As the problem is quasilinear, the convergence can only be shown in a weaker topology. It is enough for our goal to show that $(u^m - u^{m-1}, v^m - v^{m-1})$ is a Cauchy sequence in $C^0([0, T_0]; \mathcal{H}^0)$. The limit (u, v) will hence automatically belong to $\mathcal{H}^n$ and satisfy (1.1) together with the uniform in time bound

$$\|(u, v)(t)\|_{\mathcal{H}^n} \leq 2M \quad \forall 0 \leq t \leq T_0.$$

From (3.12), we see that the differences $(\tilde{u}^m, \tilde{v}^m) = (u^m - u^{m-1}, v^m - v^{m-1})$ solve the following Cauchy problem:

$$\begin{cases} \square \tilde{u}^m(t, x) = \mathbf{N}_1(v^{m-1}, \partial \tilde{v}^m) + \mathbf{N}_2(v^{m-1}, \partial \tilde{v}^m) + \mathbf{N}_1(\tilde{v}^{m-1}, \partial v^m) + \mathbf{N}_2(\tilde{v}^{m-1}, \partial v^m), \\ (\square + 1) \tilde{v}^m(t, x) = \mathbf{N}_1(v^{m-1}, \partial \tilde{u}^m) + \mathbf{N}_2(v^{m-1}, \partial \tilde{u}^m) + \mathbf{N}_1(\tilde{u}^{m-1}, \partial v^m) + \mathbf{N}_2(\tilde{u}^{m-1}, \partial v^m) \end{cases}$$

with initial data $(\tilde{u}^m, \tilde{v}^m)[0] = 0$.

We now view the last two terms in each equation as source terms and apply Lemma 3.1 to obtain

$$\frac{d}{dt} E_{m-1}^{quasi}(\tilde{u}^m, \tilde{v}^m) \lesssim M(\|(\tilde{u}^m, \tilde{v}^m)\|_{\mathcal{H}^0}^2 + \|(\tilde{u}^m, \tilde{v}^m)\|_{\mathcal{H}^0} \|(\tilde{u}^{m-1}, \tilde{v}^{m-1})\|_{\mathcal{H}^0}).$$

Then by Gronwall's inequality, we obtain

$$\|(\tilde{u}^m, \tilde{v}^m)(t)\|_{\mathcal{H}^0} \leq CM e^{CMT_0} \int_0^t \|(\tilde{u}^{m-1}, \tilde{v}^{m-1})(\tau)\|_{\mathcal{H}^0} d\tau$$

for all $0 \leq t \leq T_0$. By iteration,

$$\begin{aligned} \|(\tilde{u}^m, \tilde{v}^m)(t)\|_{\mathcal{H}^0} &\lesssim (CM)^m e^{mCMT_0} \int_{0 \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_m \leq t} \|(u^0, v^0)(\tau_1)\|_{\mathcal{H}^0} d\tau_1 \dots d\tau_m \\ &\leq \frac{(CMt)^m}{m!} e^{mCMT_0} \sup_{t \in [0, T_0]} \|(u^0, v^0)(t)\|_{\mathcal{H}^0}, \end{aligned}$$

which implies that the series of general term (u^m, v^m) converges in $\mathcal{C}^0([0, T_0]; \mathcal{H}^0)$ and concludes the proof of the existence part (a) of the theorem.

(iii) *Uniqueness of solutions.* This follows by the same arguments as above. We assume we have two solutions of (1.1), (u^1, v^1) , (u^2, v^2) , we subtract them, and we obtain a similar system as above for the difference $(\tilde{u}, \tilde{v}) := (u^1, v^1) - (u^2, v^2)$ with zero Cauchy data. Then we apply the energy estimates in Lemma 3.1 followed by Gronwall's inequality to show $(\tilde{u}, \tilde{v}) = 0$. For more details, see the proof in (iv) below which yields a stronger result.

(iv) *Uniform finite speed of propagation.* Here, we consider two solutions of (1.1), (u^1, v^1) and (u^2, v^2) . We assume that their initial data coincide in a ball $B(x_0, R)$, and show that the two solutions have to agree in the cone

$$C = \{2t + |x - x_0| < R\}.$$

For the difference $(\tilde{u}, \tilde{v})$, we have the equation

$$\begin{cases} \square \tilde{u}(t, x) = \mathbf{N}_1(v^1, \partial \tilde{v}) + \mathbf{N}_2(u^1, \partial \tilde{v}) + \mathbf{N}_1(\tilde{v}, \partial v^2) + \mathbf{N}_2(\tilde{u}, \partial v^2), \\ (\square + 1)\tilde{v}(t, x) = \mathbf{N}_1(v^1, \partial \tilde{u}) + \mathbf{N}_2(u^1, \partial \tilde{u}) + \mathbf{N}_1(\tilde{v}, \partial u^2) + \mathbf{N}_2(\tilde{u}, \partial u^2). \end{cases}$$

We view the last two terms on the right as source terms and the rest as the equation (3.1) with $(u, v) = (u^1, v^1)$ and $(U, V) = (\tilde{u}, \tilde{v})$. Then the energy flux relation (3.9) remains valid, with the contribution of the source terms included in D_1 and D_2 in (3.11).

We integrate the energy flux relation (3.9) on the cone section $C_{[0, t_0]} = C \cap [0, t_0]$ to obtain

$$\int_{C_{t_0}} e^{quasi} dx = \int_{C_0} e^{quasi} dx + \int_{C_{[0, t_0]}} g dx dt + F,$$

where the flux F is an integral over the lateral surface of the cone section which we denote by $\partial C_{[0, t]}$,

$$F = \int_{\partial C_{[0, t_0]}} -e^{quasi} + \frac{1}{2} \frac{(x - x_0)_j}{|x - x_0|} f_j dx.$$

Since $A \ll 1$, it easily follows that the contribution of the cubic terms to F is negligible and then that $F \leq 0$. Then

$$\int_{C_{t_0}} e^{quasi} dx \leq \int_{C_0} e^{quasi} dx + B \int_{C_{[0, t_0]}} e^{quasi} dx dt.$$

At the initial time $t = 0$, we have $(\tilde{u}, \tilde{v}) = 0$, so by Gronwall's inequality, we obtain $e^{quasi} = 0$ inside C , which gives $(\tilde{u}, \tilde{v}) = 0$ in C .

(v) *Continuation of the solution.* We start by differentiating the system (3.12). For any $0 \leq k \leq n$, the differentiated variables $(\partial_x^k u, \partial_x^k v)$ solve the following system:

$$\begin{cases} (\partial_t^2 - \Delta_x) \partial_x^k u(t, x) = \mathbf{N}_1(v, \partial \partial_x^k v) + \mathbf{N}_2(u, \partial \partial_x^k v) + \mathbf{F}_k \\ (\partial_t^2 - \Delta_x + 1) \partial_x^k v(t, x) = \mathbf{N}_1(v, \partial \partial_x^k u) + \mathbf{N}_2(u, \partial \partial_x^k u) + \mathbf{G}_k, \end{cases}$$

where $\mathbf{F}_k$ and $\mathbf{G}_k$ are given by

$$\begin{cases} \mathbf{F}_k := \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_1(\partial_x^i v, \partial \partial_x^j v) + \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_2(\partial_x^i u, \partial \partial_x^j v) \\ \mathbf{G}_k := \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_1(\partial_x^i v, \partial \partial_x^j u) + \sum_{\substack{i+j=k \\ j < k}} \mathbf{N}_2(\partial_x^i u, \partial \partial_x^j u). \end{cases}$$

We seek to apply Lemma 3.1 for this system. By (3.6), we have

$$\frac{d}{dt} E^{quasi}(\partial_x^k u, \partial_x^k v) \lesssim B \|(\partial_x^k u, \partial_x^k v)\|_{\mathcal{H}^0}^2 + \|(\partial_x^k u, \partial_x^k v)\|_{\mathcal{H}^0} \|(\mathbf{F}_k, \mathbf{G}_k)\|_{L^2}. \quad (3.22)$$

It suffices to show that $(\mathbf{F}_k, \mathbf{G}_k)$ can be estimated as follows:

$$\|(\mathbf{F}_k, \mathbf{G}_k)\|_{L^2} \lesssim B \|(\partial_x^k u, \partial_x^k v)\|_{\mathcal{H}^0}, \quad k \leq n. \quad (3.23)$$

Indeed, using Hölder inequality and the Gagliardo-Nirenberg interpolation inequality, we see that

$$\begin{aligned} \|\mathbf{N}(\partial_x^i v, \partial \partial_x^j v)\|_{L^2} &\leq \|\partial \partial_x^i v\|_{L^{\frac{2(k-1)}{i-1}}} \|\partial \partial_x^{j+1} v\|_{L^{\frac{2(k-1)}{j}}} \\ &\leq \|\partial \partial_x^k v\|_{L^2}^\alpha \|\partial^2 v\|_{L^\infty}^{1-\alpha} \|\partial \partial_x^k v\|_{L^2}^\beta \|\partial^2 v\|_{L^\infty}^{1-\beta} \end{aligned} \quad (3.24)$$

with $\alpha = \frac{i-1}{k-1}, \beta = \frac{j}{k-1}$. Since $\alpha + \beta = 1$,

$$\|\mathbf{N}(\partial_x^i v, \partial \partial_x^j v)\|_{L^2} \leq \|(\partial^2 u, \partial^2 v)\|_{L^\infty} \|(\partial_x^k u, \partial_x^k v)\|_{\mathcal{H}^0}.$$

This estimate applies for $\mathbf{N} = \mathbf{N}_1$, and also for $\mathbf{N} = \mathbf{N}_2$, where v can be freely replaced by u . This proves (3.23).

We substitute to obtain the energy relation

$$\frac{d}{dt} E^{quasi}(\partial_x^k u, \partial_x^k v) \lesssim B \|(\partial_x^k u, \partial_x^k v)\|_{\mathcal{H}^0}^2 \approx B E^{quasi}(\partial_x^k u, \partial_x^k v),$$

and by Gronwall's lemma,

$$E^{quasi}(t, \partial_x^k u, \partial_x^k v) \leq e^{CBt} E^{quasi}(0, \partial_x^k u, \partial_x^k v) \quad \forall 0 \leq t \leq T$$

for some positive constant C . □

Proof of Proposition 2.1. The exterior region corresponds to $t \leq \frac{1+|x|}{4}$. Because of the finite speed of propagation property, this region can be treated separately as long as ∂u and ∂v remain small. Precisely, fix a dyadic $R > 1$ and consider the solution (u, v) to (1.1) in the region

$$C_R^{out} = \left\{ R < |x| < 2R, 0 \leq t < \frac{1+|x|}{4} \right\}.$$

By the finite speed of propagation previously proved, the solution in this region is uniquely determined by the data in

$$A_R = \{R/2 \leq |x| \leq 4R\}.$$

In the region A_R , our hypothesis guarantees that we have the bounds

$$\|(u, v)[0]\|_{\mathcal{H}^{2h}} \lesssim \epsilon, \quad \|(\partial^2 u, \partial^2 v)[0]\|_{\mathcal{H}^0} \lesssim \epsilon R^{-2}. \quad (3.25)$$

We first restrict the data to A_R and then extend them to all $\mathbb{R}^2$ so that (3.25) still holds. To obtain a bound in C_R^{out} , it suffices to solve the equation up to time $T_R = R/2$. A local in time solution exists. For this solution, we make the bootstrap assumption

$$\|\partial(u, v)\|_{L^\infty} \leq \sqrt{\epsilon}, \quad \|\partial^2(u, v)\|_{L^\infty} \leq R^{-1}\sqrt{\epsilon} \quad (3.26)$$

in a time interval $[0, T]$ with $T \leq T_R$. Applying the energy estimates in Theorem 1, we can propagate the energy bounds in (3.25) up to time T . Then we can use Sobolev embeddings to get pointwise bounds from the energy bounds.

For the first derivatives, this yields

$$\|\partial u\|_{L^\infty} \lesssim \|\partial u\|_{L^2}^{\frac{1}{2}} \|\partial^3 u\|_{L^2}^{\frac{1}{2}} \lesssim \epsilon R^{-1},$$

and similarly for v . This suffices in order to improve the bootstrap assumption.

For the second derivatives, this yields

$$\|\partial^2 u\|_{L^\infty} \lesssim \log R \|\partial^3 u\|_{L^2} + R^{-2} \|\partial^2 u\|_{H^2} \lesssim \epsilon R^{-2} \log R,$$

and similarly for v . This again suffices in order to improve the bootstrap assumption.

As the bootstrap assumption can be improved for all $T < T_R$, it follows that the solution (u, v) extends to time T_R and satisfies the above pointwise bounds. The proof of the proposition is concluded. $\square$

4. Linearized equation: Alinhac's approach

In this section, we derive the linearized wave-Klein-Gordon system and prove the energy estimates for them. More precisely, we prove quadratic energy estimates in $\mathcal{H}^0$, which apply to large data problem.

The solutions for the linearized wave-Klein-Gordon system around a solution (u, v) are denoted by (U, V) . With this notation in place, the linearized system takes the form

$$\begin{cases} (\partial_t^2 - \Delta_x)U(t, x) = \mathbf{N}_1(v, \partial V) + \mathbf{N}_1(V, \partial v) + \mathbf{N}_2(u, \partial V) + \mathbf{N}_2(U, \partial v) \\ (\partial_t^2 - \Delta_x + 1)V(t, x) = \mathbf{N}_1(v, \partial U) + \mathbf{N}_1(V, \partial u) + \mathbf{N}_2(u, \partial U) + \mathbf{N}_2(U, \partial u). \end{cases} \quad (4.1)$$

We recall that for $\mathbf{N}_1(\cdot, \cdot)$ and $\mathbf{N}_2(\cdot, \cdot)$, we will take linear combinations of the classical admissible quadratic null forms

$$\begin{cases} Q_{ij}(\phi, \psi) = \partial_i \phi \partial_j \psi - \partial_i \psi \partial_j \phi, \\ Q_{0i}(\phi, \psi) = \partial_t \phi \partial_i \psi - \partial_t \psi \partial_i \phi, \\ Q_0(\phi, \psi) = \partial_t \phi \partial_t \psi - \nabla_x \psi \cdot \nabla_x \phi. \end{cases} \quad (4.2)$$

To obtain energy estimates for the linearized equation, we consider the same energy and energy density as in the proof of the local well-posedness result,

$$E^{quasi}(U, V) := E(U, V) + \int_{\mathbb{R}^2} B_1(v; U, V) + B_2(u; U, V) dx = \int_{\mathbb{R}^2} e^{quasi}(t, x) dx, \quad (4.3)$$

where the quasilinear energy density is

$$e^{quasi}(t, x) := e_0(t, x) + B_1(v; U, V) + B_2(u; U, V),$$

and e_0 is the linear energy density

$$e_0(t, x) = \frac{1}{2} [U_t^2 + U_x^2 + V_t^2 + V_x^2 + V^2].$$

Our energy estimates will be proven under the following uniform bound assumptions (which are a part of our bootstrap argument):

$$|Zu| \leq C\epsilon \langle t - r \rangle^\delta \quad (4.4)$$

$$|Z\partial u| \leq C\epsilon \langle t - r \rangle^{-\delta_1} \quad (4.5)$$

$$|Z\partial^2 u| \leq C\epsilon \langle t - r \rangle^{-\delta_1} \quad (4.6)$$

$$|\partial u| \leq C\epsilon \langle t + r \rangle^{-\frac{1}{2}} \langle t - r \rangle^{-\frac{1}{2}} \quad (4.7)$$

$$|\partial^j u| \leq C\epsilon \langle t + r \rangle^{-\frac{1}{2}} \langle t - r \rangle^{-\frac{1}{2} - \delta_1}, \quad j = \overline{2, 3}, \quad (4.8)$$

$$|\partial^j v| \leq C\epsilon \langle t + r \rangle^{-1 - \delta_1} \langle t - r \rangle^{\delta_1}, \quad j = \overline{1, 3}, \quad (4.9)$$

where C is a large positive constant and $0 < \delta \ll \delta_1$.

We observe that under the above assumptions, and for ϵ sufficiently small, we have

$$E^{quasi}(U, V) \approx E(U, V)$$

in the sense that

$$\frac{1}{2} E^{quasi}(U, V) \leq E(U, V) \leq 2 E^{quasi}(U, V).$$

Our main result for the linearized equation is as follows:

Proposition 4.1. *Assume the solutions to the main equations (1.1) or (2.17) satisfy the bounds (4.4)-(4.9) in some time interval $[0, T]$. Then the linearized equation (4.1), subject to the constraints in (4.2), is well-posed in $[0, T]$, and the solution satisfies*

$$E^{quasi}(U, V)(t) \lesssim t^{\tilde{C}\epsilon} E^{quasi}(U, V)(0), \quad t \in [0, T], \quad (4.10)$$

where $\tilde{C} \approx C$ is a positive constant.

Along the way, we will establish a larger family of bounds for U, V . These are collected together at the end of the section in Corollaries 4.6, 4.7, which can be viewed as a stronger form of the above proposition.

Proof. The difficulty we encounter here is that we do not have a good estimate at a fixed time for the time derivative of the energy in order to directly apply a Gronwall's type inequality. To address this issue, the key idea is to obtain a 'good' energy inequality. We are led to consider the energy growth on dyadic time scales $[T, 2T]$. Within such a dyadic time interval, it suffices to prove

$$\sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) \leq (1 + \epsilon C) E^{quasi}(U, V)(T). \quad (4.11)$$

As a preliminary step, we determine the growth of the $E^{quasi}(U, V)(t)$ on such a time interval:

$$E^{quasi}(U, V)(\tilde{T}) - E^{quasi}(U, V)(T) = \int_T^{\tilde{T}} \frac{d}{dt} E^{quasi}(U, V)(t) dt \quad \tilde{T} \in [T, 2T], \quad (4.12)$$

where, after expanding, the RHS is a trilinear form integrated in space time, rather than at fixed time. To estimate this integral, that is, to get

$$\left| \int_T^{\tilde{T}} \frac{d}{dt} E^{quasi}(U, V)(t) dt \right| \lesssim \epsilon E^{quasi}(U, V)(T), \quad (4.13)$$

we will first need to obtain the L^2 space-time bounds for U and V and their derivatives over various space-time regions relative to the distance to the cone.

To understand what is needed for the energy estimate computation, we begin by derivating the density flux relation associated to our energy density e^{quasi} . We begin with the first component of e^{quasi} , which is e_0 :

$$\partial_t e_0(t, x) = \sum_{j=1}^2 \partial_{x_j} (U_t U_j + V_t V_j) + U_t \square U + V_t (\square + 1)V.$$

The last two terms can be expanded as follows:

$$\begin{cases} U_t \square U = U_t (\mathbf{N}_1(v, \partial V) + \mathbf{N}_1(V, \partial v) + \mathbf{N}_2(u, \partial V) + \mathbf{N}_2(U, \partial v)), \\ V_t (\square + 1)V = V_t (\mathbf{N}_1(v, \partial U) + \mathbf{N}_1(V, \partial u) + \mathbf{N}_2(u, \partial U) + \mathbf{N}_2(U, \partial u)). \end{cases} \quad (4.14)$$

Next, we turn our attention to the quasilinear correction

$$\begin{aligned} \partial_t B_1(v; U, V) &= B_1(v_t; U, V) + B_1(v; U_t, V) + B_1(v; U, V_t) \\ \partial_t B_2(u; U, V) &= B_2(u_t; U, V) + B_2(u; U_t, V) + B_2(u; U, V_t). \end{aligned} \quad (4.15)$$

Here, we will combine the first, respectively third, terms in both RHS in equations (4.14) with $\partial_t B_1(v; U, V)$, respectively with $\partial_t B_2(u; U, V)$. Schematically, we obtain that

$$\begin{cases} U_t \mathbf{N}_1(v, \partial V) + V_t \mathbf{N}_1(v, \partial U) + \partial_t B_1(v; U, V) = \partial_x C_1(\partial v, \partial U, \partial V) + D_1(\partial^2 v, \partial U, \partial V) \\ U_t \mathbf{N}_2(u, \partial V) + V_t \mathbf{N}_2(u, \partial U) + \partial_t B_2(u; U, V) = \partial_x C_2(\partial u, \partial U, \partial V) + D_2(\partial^2 u, \partial U, \partial V), \end{cases}$$

where C_1, C_2 and D_1, D_2 are algebraic trilinear forms. Here, we need to take a closer look at the structure of D_1 and D_2 . Indeed, a simple direct computation shows that both of them have a null structure,

$$\begin{cases} D_1(\partial^2 v, \partial U, \partial V) = \mathbf{N}(\partial v, U) \partial V + \mathbf{N}(\partial v, V) \partial U \\ D_2(\partial^2 u, \partial U, \partial V) = \mathbf{N}(\partial u, U) \partial V + \mathbf{N}(\partial u, V) \partial U \end{cases} \quad (4.16)$$

where $\mathbf{N}(\cdot, \cdot)$ denote null forms (i.e., linear combinations of the null forms in (4.2)). The relation for D_2 is important for us, while the one for D_1 is less critical because of the better t^{-1} decay enjoyed by the Klein-Gordon component v .

We also note that B_j and C_j do not have a null structure in general, as it can be seen by examining (3.4). However, they are matched, and we will take advantage of this later on.

Summing up all these terms, we obtain the following energy flux relation for solution to inhomogeneous linearized problem:

$$\partial_t e^{quasi}(t, x) = \sum_{j=1}^2 \partial_j f_j + g, \quad (4.17)$$

where the fluxes f_j have the expressions

$$f_j = U_j U_t + V_j V_t + C_1(\partial v, \partial U, \partial V) + C_2(\partial u, \partial U, \partial V), \quad (4.18)$$

and the source g is a trilinear form with null structure and has components as follows:

$$g = D_1(\partial^2 v, \partial U, \partial V) + D_2(\partial^2 u, \partial U, \partial V). \quad (4.19)$$

To complete the proof of the energy estimates, we will have to bound the source terms using the energy. The v -terms $D_1(\partial^2 v, \partial U, \partial V)$ will be well-behaved because $\partial^2 v$ has t^{-1} decay, but the u -terms $D_2(\partial^2 u, \partial U, \partial V)$ do not share this property. Instead, for these terms, we need a different idea which takes advantage of their null structure.

This leads us to Alinhac's approach which establishes an improved version of the 'standard' energy inequality (by this we mean the inequality corresponding to the multiplier ∂_t case); such an inequality yields, besides the usual fixed time energy bound, a bound of the (weighted) L^2 norm in both variables x and t of some *good derivatives* of (U, V) in special regions.

The special regions mentioned above are exactly the sets $C_{TS}^\pm$ introduced earlier in (2.13), (2.14), (2.15) and (2.16), which provide a double dyadic decomposition of the space-time relative to the size of t and the size of $t - r$ which measures how far or close we are to the cone. To simplify the exposition, we will use the notation C_{TS} as a shorthand for either C_{TS}^+ or C_{TS}^- .

The good derivatives alluded to above are exactly the tangential derivatives relative to the cones $\{t - r = \text{const}\}$, or equivalently,¹ relative to the hyperboloids $\{t^2 - x^2 = \text{const}\}$. These surfaces can be viewed as providing nearly equivalent foliations of the sets $C_{TS}^\pm$.

Lemma 4.2. *Assume the solutions (u, v) to the main equations (1.1) or (2.17) satisfy the bounds (4.4)-(4.9) over the space-time regions C_T . Then the solution (U, V) of the linearized equation (4.1) satisfies*

$$\sup_{1 \leq S \leq T} \int_{C_{TS}} \frac{1}{S} \left\{ \left(V_j + \frac{x_j}{r} V_t \right)^2 + \left(U_j + \frac{x_j}{r} U_t \right)^2 + V^2 \right\} dx dt \lesssim \sup_{t \in [T, 2T]} \text{Quasi}(U, V)(t). \quad (4.20)$$

Here, we only consider the C_{TS} regions with $S \lesssim T$, as the outer region C_T^{out} is uninteresting from this perspective. Concerning the directional derivatives in the lemma, we note that they have a special structure:

Remark 4.3. The quantities appearing in (4.20) (i.e., $V_j + \frac{x_j}{r} V_t$ and $U_j + \frac{x_j}{r} U_t$) represent the derivatives of V , respectively U , in the tangential directions to the cones $C = \{t - r = \text{const}\}$. We denote them by

$$\mathcal{T}_j = \partial_j + \frac{x_j}{r} \partial_t.$$

We further remark that we have the trivial bound

$$\sup_{1 \leq S \leq T} \int_{C_{TS}} \frac{1}{T} \left(|\nabla U|^2 + |\nabla V|^2 \right) dx dt \lesssim \sup_{t \in [T, 2T]} \text{Quasi}(U, V)(t), \quad (4.21)$$

which can be viewed as the natural complement of (4.20) for nontangential derivatives. In other words, (4.20) and (4.21) should be viewed as a pair. This last bound also shows that (4.20) becomes trivial if $S \approx T$. Thus, in the proof, we will be concerned with the case $1 \leq S \ll T$.

Closely related to the last comment, an important observation that applies in the C_{TS} regions is that we can connect the vector fields $\mathcal{T}$, tangent to the cones, to the corresponding vector fields Z , tangent to the hyperboloids that foliate both the interior or the exterior of the cone:

$$H_\rho = \{t^2 - x^2 = \pm \rho^2\}.$$

¹from the perspective of the estimates

Here, we consider a hyperboloid which intersects C_{TS}^+ provided that $\rho^2 \approx TS$. Since $S \geq 1$, this in particular requires that

$$T \lesssim \rho^2 \lesssim T^2.$$

In this setting, we note that vector fields Z and $\mathcal{T}$ are related in general via

$$Z = t\mathcal{T} - \frac{x}{r}(t-r)\partial_t \quad (4.22)$$

and, in particular in the C_{TS} regions, by

$$Z \approx T\mathcal{T} - S\partial_t. \quad (4.23)$$

As the tangent planes to both the hyperboloids and cones are close to each other, via the estimate given in (4.21), we give an alternative statement of Lemma 4.2 in terms of the Z vector fields.

Lemma 4.4. *Under the same assumptions as in Lemma 4.2, we have*

$$\sup_{1 \leq S \leq T} \int_{C_{TS}} \frac{1}{S} \{T^{-2}(|ZU|^2 + |ZV|^2) + V^2\} + \frac{1}{T}(|\nabla U|^2 + |\nabla V|^2) dx dt \lesssim \sup_{t \in [T, 2T]} E^{quasi}(U, V)(t). \quad (4.24)$$

Proof of Lemma 4.2. We consider the following weighted version of the energy $E(U, V)$,

$$E_a(U, V) := \int_{\mathbb{R}^2} e^a e_0(t, x) dx, \quad (4.25)$$

and similarly the weighted version of the quasilinear energy $E^{quasi}(U, V)$,

$$E_a^{quasi}(U, V) := \int_{\mathbb{R}^2} e^a e^{quasi}(t, x) dx. \quad (4.26)$$

Here, e^a is a *ghost weight*, which will be chosen such that a is bounded and the weight e^a ultimately disappears from the inequalities. Precisely, we will choose a of the form

$$a(t, r) := -A(t - r), \quad (4.27)$$

where A is a bounded nondecreasing function. Then the gain in the estimates will come from the contribution of $A'(t - r)$, which will be chosen to be positive.

We can further specialize the choice of the function $A(t - r)$ and separately adapt it to each dyadic space-time regions C_{TS} for $1 \leq S \ll T$. Precisely, we can chose it so that

$$A'(t - r) \approx \frac{1}{S}, \quad \text{for } |t - r| \approx S, \quad \text{and } A'(t - r) = 0 \text{ elsewhere.} \quad (4.28)$$

For such functions A , we need to understand the time derivative of $E_a^{quasi}(U, V)$. Thus, using the equation (4.17), we have

$$\begin{aligned} \frac{d}{dt} E_a^{quasi}(U, V) &= \int_{\mathbb{R}^2} \frac{d}{dt} [e^a e^{quasi}(t, x)] dx \\ &= \int_{\mathbb{R}^2} e^a a_t e^{quasi} + e^a (\partial_j f_j + g) dx. \\ &= \int_{\mathbb{R}^2} e^a (a_t e^{quasi} - a_j f_j) dx + \int_{\mathbb{R}^2} e^a g dx. \end{aligned}$$

The second integrand involves the function g , which is a trilinear form with a null structure. The terms in the first integrand do not separately have a null structure, so we will take a closer look at them together for our choice of a as above. Separating the quadratic and the cubic contributions, we write

$$\begin{aligned} a_t e^{quasi} - a_j f_j &= -A'(t-r) \left(e^{quasi} + \frac{x_j}{r} f_j \right) \\ &= -A'(t-r) (Q_2(\partial U, \partial V) + Q_{3,1}(\partial v, \partial U, \partial V) + Q_{3,2}(\partial u, \partial U, \partial V)), \end{aligned}$$

where Q_2 represents the quadratic term,

$$Q_2(U, V) = e_0(\partial U, \partial V) + \frac{x_j}{r} (U_t U_j + V_j V_t),$$

and $Q_{3,1}, Q_{3,2}$ represent the cubic terms,

$$Q_{3,i}(\partial v, \partial U, \partial V) = B_i(\partial v, \partial U, \partial V) + \frac{x}{r} C_i(\partial w, \partial U, \partial V).$$

Recombining the terms in Q_2 one obtains

$$Q_2(U, V) = \left(V_j + \frac{x_j}{r} V_t \right)^2 + \left(U_j + \frac{x_j}{r} U_t \right)^2 + V^2,$$

which is exactly as in (4.20). However, a short algebraic computation reveals the following structure for $Q_{3,i}$:

$$Q_{3,i}(\partial w, \partial U, \partial V) = D_{1,i}(\mathcal{T}w, \partial U, \partial V) + D_{2,i}(\partial w, \partial U, \partial V),$$

where

- $D_{1,i}$ has no null structure but only uses a tangential derivative of w ,
- $D_{2,j}$ has a null structure – that is, can be represented as

$$D_{2,j}(\partial w, \partial U, \partial V) = \mathbf{N}(w, U) \partial V + \mathbf{N}(w, V) \partial U.$$

Thus, we obtain

$$\begin{aligned} \frac{d}{dt} E_a^{quasi}(U, V) &+ \int_{\mathbb{R}^2} e^a A'(t-r) Q_2(U, V) dx \\ &= - \int_{\mathbb{R}^2} e^a A'(t-r) (D_{1,1}(\mathcal{T}v, U, V) + D_{1,2}(\mathcal{T}u, U, V)) dx \\ &\quad - \int_{\mathbb{R}^2} e^a A'(t-r) (D_{2,1}(\partial v, \partial U, \partial V) + D_{2,2}(\partial u, \partial U, \partial V)) dx \\ &\quad + \int_{\mathbb{R}^2} e^a (D_1(\partial^2 v, \partial U, \partial V) + D_2(\partial^2 u, \partial U, \partial V)) dx. \end{aligned} \tag{4.29}$$

Now we integrate this relation between T and $2T$. With our choice for A , the first integral in the left-hand side controls the expression on the left in Lemma 4.2. It remains to estimate the remaining terms on the right perturbatively.

1. *The contributions of $D_{1,j}$.* Here, we use our bootstrap assumption to estimate

$$|\mathcal{T}u| + |\mathcal{T}v| \lesssim \epsilon T^{-1} S^\delta,$$

which implies that

$$|D_{1,1}(\mathcal{T}v, \partial U, \partial V)| + |D_{1,2}(\mathcal{T}u, \partial U, \partial V)| \lesssim \epsilon T^{-1} S^\delta |\partial U| |\partial V|.$$

Since $|A'| \lesssim S^{-1}$, this suffices in order to bound their contribution by the energy.

2. The v -terms in $D_{2,1}$ and D_1 . Their contribution is

$$\int_T^{2T} \int_{\mathbb{R}^2} e^a [A'(t-r) D_{2,1}(\partial v, \partial U, \partial V) + D_1(\partial^2 v, \partial U, \partial V)] dx dt \quad (4.30)$$

which are all bounded using (4.9) for the v -factors and the energy E_a^{quasi} for the U and V terms by

$$\lesssim \epsilon \sup_{t \in [T, 2T]} E^{quasi}(U, V)(t). \quad (4.31)$$

3. The u -terms in $D_{2,2}$ and D_2 . These have the form

$$\int_T^{2T} \int_{\mathbb{R}^2} e^a [A'(t-r) D_{2,2}(\partial u, \partial U, \partial V) + D_2(\partial^2 u, \partial U, \partial V)] dx dt, \quad (4.32)$$

which we need to process further. In the region C_T^{out} , the pointwise bounds (4.7) and (4.8) give a t^{-1} decay for both ∂u and $\partial^2 u$, so this is identical to the case of the v terms above. It remains to consider the contribution over $C_T^{in} := C_T \setminus C_T^{out}$, where we will exploit the null structure of $D_{2,2}$ and D_2 ; see (4.16).

The key property is that all null forms can be expressed in the form

$$\mathbf{N}(\phi, \psi) = \partial \phi \cdot \mathcal{T} \psi + \mathcal{T} \phi \cdot \partial \psi, \quad (4.33)$$

or equivalently,

$$\mathbf{N}(\phi, \psi) = \frac{1}{t} (\partial \phi \cdot Z \psi + Z \phi \cdot \partial \psi + (t-r) \partial \phi \cdot \partial \psi). \quad (4.34)$$

We begin with D_2 , which contains terms of the form $\mathbf{N}(\partial u, V) \partial U$ and $\mathbf{N}(\partial u, U) \partial V$. We consider the first term, as the second will be similar. By (4.33), we have

$$\mathbf{N}(\partial u, V) = \partial^2 u \cdot \mathcal{T} V + \mathcal{T} \partial u \cdot \partial V. \quad (4.35)$$

For the last term, we can directly use our bootstrap assumptions in (4.5) and (4.8) to obtain the pointwise bounds

$$|\mathcal{T} \partial u| + \left| \frac{t-r}{t} \partial^2 u \right| \lesssim \epsilon T^{-1} S^{-\delta_1}.$$

Hence, the contributions of those terms are estimated as in Case 1 by (4.31).

The contribution of the first term in (4.35) to the integral in (4.32) is more delicate because now the $\mathcal{T}$ vector field applies to V . So instead, we split the integral over C_T^{in} into the sum of integrals over the C_{TS} space-time regions, apply the Cauchy-Schwartz inequality in space-time, and apply Hölder's inequality in time in each such region, as well as (4.8), to bound each of these integrals by

$$\begin{aligned} \left| \int_{C_{TS}} \partial U \partial^2 u \mathcal{T} V dx dt \right| &\lesssim \|\partial^2 u\|_{L_{CTS}^\infty} \|\partial U\|_{L_{CTS}^2} \|\mathcal{T} V\|_{L_{CTS}^2} \\ &\lesssim \epsilon T^{-\frac{1}{2}} S^{-\frac{1}{2}-\delta_1} T^{\frac{1}{2}} \left(\sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) \right)^{\frac{1}{2}} S^{\frac{1}{2}} \left\| S^{-\frac{1}{2}} \mathcal{T} V \right\|_{L_{CTS}^2} \\ &\lesssim \epsilon S^{-\delta_1} \left(\sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) \right)^{\frac{1}{2}} \left\| S^{-\frac{1}{2}} \mathcal{T} V \right\|_{L_{CTS}^2}, \end{aligned}$$

and after a straightforward S summation over $1 \leq S \leq T$, we get

$$\left| \int_{C_T^{in}} e^a D_2(\partial^2 u, \partial U, \partial V) dx dt \right| \lesssim \epsilon \left(\sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) \right)^{\frac{1}{2}} \sup_{1 \leq S \leq T} \|S^{-\frac{1}{2}} \mathcal{T}(U, V)\|_{L_{C_{TS}}^2} + \epsilon \sup_{t \in [T, 2T]} E^{quasi}(U, V)(t).$$

The bound for the contribution of $D_{2,2}$ is similar, with the difference that the integrand is now localized to a fixed dyadic region C_{TS} and that we are using the bootstrap assumption (4.7) for ∂u instead of (4.8) for $\partial^2 u$. Using also (4.28), we obtain

$$\left| \int_{C_T^{in}} e^a A'(t-r) D_{2,2}(\partial u, \partial U, \partial V) dx dt \right| \lesssim \epsilon S^{-1+\delta} \left(\sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) \right)^{\frac{1}{2}} \sup_{1 \leq S \leq T} \|S^{-\frac{1}{2}} \mathcal{T}(U, V)\|_{L_{C_{TS}}^2} + \epsilon S^{-1+\delta} \sup_{t \in [T, 2T]} E^{quasi}(U, V)(t),$$

where the $S^{-1+\delta}$ gain insures the summation in S , but it is not otherwise needed in the sequel.

Overall, we have proved that

$$(4.32) \lesssim \epsilon \sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) + \epsilon \sup_{1 \leq S \leq T} \|S^{-\frac{1}{2}} \mathcal{T}(U, V)\|_{L_{C_{TS}}^2}^2.$$

Summing all up all the contributions to the integrated form of (4.29), we obtain

$$\int_{C_{TS}} S^{-1} Q_2(U, V) dx dt \lesssim \epsilon \sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) + \epsilon \sup_{1 \leq S \leq T} \|S^{-\frac{1}{2}} \mathcal{T}(U, V)\|_{L_{C_{TS}}^2}^2.$$

Finally we take the supremum over $1 \leq S \leq T$. Then the last term on the right can be absorbed on the left, which concludes the proof of the lemma. $\square$

Now we conclude the proof of the Proposition 4.1. For this, we repeat the computation above with $a(t, r) = 0$. We integrate the relation (4.29) from T up to an arbitrary $t \in [T, 2T]$, and estimate the RHS exactly as in the proof of the Lemma 4.4. We obtain

$$E_a^{quasi}(t) - E_a^{quasi}(T) \lesssim \epsilon \sup_{t_0 \in [T, 2T]} E_a^{quasi}(t_0), \quad (4.36)$$

and taking the supremum over $t \in [T, 2T]$ gives

$$\sup_{t \in [T, 2T]} E_a^{quasi}(t) \lesssim E_a^{quasi}(T),$$

which concludes the proof of the proposition. $\square$

A consequence of the proof of Proposition 4.1 is that, in addition to the uniform energy bound in $[T, 2T]$, we also gain uniform control of the localized energies in the left-hand side of (4.20). It will be useful in effect to obtain a slight improvement over (4.20), where we think of C_{TS}^+ as foliated by hyperboloids and obtain uniform L^2 bounds over each such hyperboloid.

To set the notations, consider a hyperboloid

$$H_\rho = \{t^2 - x^2 = \rho^2\},$$

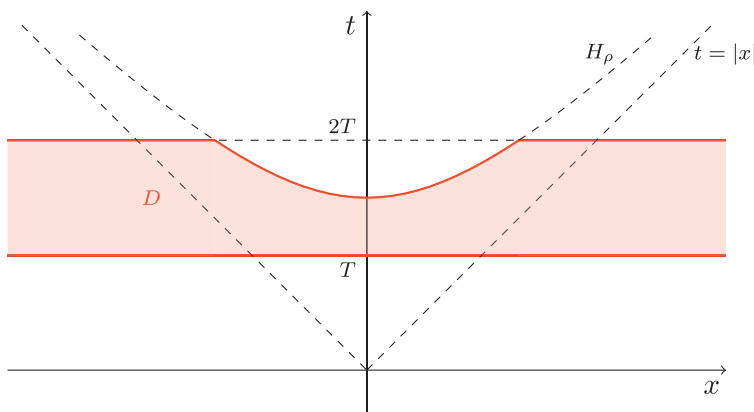


Figure 2. Region D in $1+1$ space-time dimension.

which intersects $C_{T,S}^+$ provided that $\rho^2 \approx TS$. Since $S \geq 1$, this in particular requires that

$$T \lesssim \rho^2 \lesssim T^2. \quad (4.37)$$

Then we have the following:

Lemma 4.5. *Under the same assumptions as Proposition 4.1, the solution (U, V) to (4.1) satisfies*

$$\sup_{T \lesssim \rho^2 \lesssim T^2} \int_{H_\rho \cap C_T} T^{-2} (|ZU|^2 + |ZV|^2 + \rho^2 (|\nabla U|^2 + |\nabla V|^2)) + V^2 \, dx \lesssim E(U, V)(T). \quad (4.38)$$

Proof. With small differences, the proof mimics the proof of Proposition 4.1. We consider the domain

$$D = \{(x, t) \in C_T; t^2 - x^2 \leq \rho^2\},$$

which represents the portion of C_T below the hyperboloid H_ρ (see Figure 2, which depicts the case when the hyperboloid intersects the surface $t = 2T$, but not the $t = T$). Then we integrate the relation (4.17) over D , estimating the contribution of g exactly as in the proof of Proposition 4.1. The contributions on the bottom $t = T$ and the top $t = 2T$ are simply the energies. This yields the bound

$$\int_{H_\rho \cap C_T} e^{quasi}(t, x) - \frac{x_j}{t} f_j \, dx \lesssim E(U, V)(T). \quad (4.39)$$

Here, we have used the normal vector $n = (1, -\frac{x}{t})$ on H_ρ . It remains to verify that the integrand is positive definite and controls the integrand in the left-hand side of (4.38).

The leading contribution comes from e_0 and the quadratic part f_{j0} of the f_j and gives exactly the correct expressions. It remains to perturbatively estimate the cubic contributions to the integrand in (4.39), which under the assumption (4.37), has size

$$\lesssim (|\nabla v| + |\nabla u|) |\nabla U| |\nabla V| \lesssim \frac{\epsilon}{\sqrt{ST}} (|\nabla U|^2 + |\nabla V|^2) \ll \epsilon \frac{\rho^2}{T^2} (|\nabla U|^2 + |\nabla V|^2),$$

as needed. □

To streamline the notations, it will help to introduce the following norm for functions (U, V) in C_T :

$$\|(U, V)\|_{X_T}^2 := \sup_{t \in [T, 2T]} E(U, V)(t) + LHS(4.20) + LHS(4.38). \quad (4.40)$$

Then we have the following.

Corollary 4.6. *Under the same assumptions as Proposition 4.1, the solution (U, V) to (4.1) satisfies*

$$\|(U, V)\|_{X^T}^2 \lesssim E(U, V)(T). \quad (4.41)$$

Another direct consequence of Proposition (4.1) and of above corollary is the following corollary which derives similar energy estimates but for the non-homogeneous analogue of (4.1):

$$\begin{cases} (\partial_t^2 - \Delta_x)U(t, x) = \mathbf{N}_1(v, \partial V) + \mathbf{N}_1(V, \partial v) + \mathbf{N}_2(u, \partial V) + \mathbf{N}_2(U, \partial v) + \mathbf{F}(t, x) \\ (\partial_t^2 - \Delta_x + 1)V(t, x) = \mathbf{N}_1(v, \partial U) + \mathbf{N}_1(V, \partial u) + \mathbf{N}_2(u, \partial U) + \mathbf{N}_2(U, \partial u) + \mathbf{G}(t, x), \end{cases} \quad (4.42)$$

where $\mathbf{F}$ and $\mathbf{G}$ are arbitrary functions of t and x .

Corollary 4.7. *Assume the solutions to the main equations (1.1) satisfy the bounds (4.4)–(4.9) in some time interval $[0, T]$. Then the non-homogeneous linearized equation (4.42) is well-posed in $[0, T]$, and the solution satisfies*

$$\sup_{t \in [T, 2T]} E^{quasi}(U, V)(t) \leq (1 + \epsilon \tilde{C}) E^{quasi}(U, V)(T) + \|(\mathbf{F}, \mathbf{G})\|_{L_t^1 L_x^2}^2, \quad (4.43)$$

where $\tilde{C} \approx C$ with C as in (4.4)–(4.9). In addition, we have

$$\|(U, V)\|_{X^T}^2 \lesssim E(U, V)(T) + \|(\mathbf{F}, \mathbf{G})\|_{L_t^1 L_x^2}^2. \quad (4.44)$$

Proof. The proof of the corollary is a direct consequence of Proposition 4.1 and of the variation of parameters principle (i.e., Duhamel's principle). $\square$

5. Higher-order energy estimates

The main goal of this section is to establish energy bounds for (u, v) and their higher derivatives. We will compare this system with the linearized system which was studied in Section 4 and use a large portion of the estimates already obtained for the non-homogeneous linearized system (4.42), as in Corollary 4.7.

We start with the equations (1.1) and differentiate them n times. Here, the variables that play the role of the linearized variables (U, V) are the n times differentiated variables $(\partial^n u, \partial^n v)$, which we will denote by (u^n, v^n) . We differentiate (1.1) n times and separate the terms into leading-order and lower-order contributions, interpreting the differentiated equation as a linearized equation with a source term, as in (4.42):

$$\begin{cases} (\partial_t^2 - \Delta_x)u^n(t, x) = \mathbf{N}_1(v, \partial v^n) + \mathbf{N}_1(v^n, \partial v) + \mathbf{N}_2(u, \partial v^n) + \mathbf{N}_2(u^n, \partial v) + \mathbf{F}^n \\ (\partial_t^2 - \Delta_x + 1)v^n(t, x) = \mathbf{N}_1(v, \partial u^n) + \mathbf{N}_1(v^n, \partial u) + \mathbf{N}_2(u, \partial u^n) + \mathbf{N}_2(u^n, \partial u) + \mathbf{G}^n, \end{cases} \quad (5.1)$$

where the source terms have the form

$$\mathbf{F}^n(t, x) = \sum_{k=1}^{n-1} \mathbf{N}_1(v^k, \partial v^{n-k}) + \mathbf{N}_2(u^k, \partial v^{n-k}), \quad \mathbf{G}^n(t, x) = \sum_{k=1}^{n-1} \mathbf{N}_1(v^k, \partial u^{n-k}) + \mathbf{N}_2(u^k, \partial u^{n-k}).$$

Our energy estimates for the differentiated system will be proved under the same bootstrap assumptions we previously imposed (4.4)–(4.9). The main result of this section is as follows:

Proposition 5.1. *Let $n \geq 4$. Assume the solutions (u, v) to the original main equations (1.1) or (2.17) are defined in $\mathcal{H}^n$ in some time interval $[0, T]$, and satisfy the bootstrap bounds (4.4)–(4.9). Then the following bound holds:*

$$E^n(u, v)(t) \lesssim t^{\tilde{C}\epsilon} E^n(u, v)(0), \quad t \in [0, T], \quad (5.2)$$

for some positive constant $\tilde{C}$.

Remark 5.2. As defined, the energies $E^n(u, v)(t)$ measure not only higher-order spatial derivatives for (u, v) but also higher-order time derivatives of (u, v) . However, at the initial time, we want to measure only the size of the Cauchy data, which means at most one time derivative of (u, v) . However, our a priori bounds (bootstrap assumptions (4.4)–(4.9)) suffice in order to estimate the Cauchy data to higher-order time derivatives of (u, v) in terms of the corresponding bounds for the Cauchy data of (u, v) . This is a relatively straightforward exercise which is left for the reader.

Proof. The proof heavily uses the energy estimates for the linearized system (4.1) in the previous section. We will use the differentiated equations (5.1) to inductively prove bounds on the differentiated functions (u^n, v^n) . For this, we will rely on the energy E^{quasi} and on the bounds in Corollary 4.7.

We begin by discussing the case when $n = 0$. The easy way to handle this case is to relate it to the linearized equation, by simply thinking at (1.1) as being written as in Corollary 4.7 where $\mathbf{F}$ and $\mathbf{G}$ are given by

$$\mathbf{F} := -\mathbf{N}_1(V, \partial v) - \mathbf{N}_2(U, \partial v), \quad \mathbf{G} := -\mathbf{N}_1(V, \partial u) - \mathbf{N}_2(U, \partial u).$$

The non-homogeneous term $\mathbf{F}$ can be easily bound in $L_t^1 L_x^2(C_{TS})$ using a priori estimate (4.9), while $\mathbf{G}$ is bounded in $L_t^1 L_x^2(C_{TS})$ using the null structure highlighted in (4.33), (4.34) as in the proof of Lemma 4.4.

The case $n = 1$ is a trivial consequence as $(\nabla u, \nabla v)$ exactly solve the linearized equation. So from here on, we will assume that $n \geq 2$.

Since we do not have a way to prove a good energy estimate working only at fixed time, we will focus on proving a good energy estimate on dyadic time scales $[T, 2T]$. Precisely, arguing by induction on n , it suffices to show that

$$\sup_{t \in [T, 2T]} E^{quasi}(u^n, v^n)(t) \leq E^{quasi}(u^n, v^n)(T) + \epsilon C E^{quasi}(u^{\leq n}, v^{\leq n})(T). \quad (5.3)$$

Just as in Proposition 4.1, to prove this, we will use the stronger auxiliary norm X^T in (4.40), which captures more of the structure of linearized waves. So instead of (5.3), we will prove the pair of bounds

$$\sup_{t \in [T, 2T]} E^{quasi}(u^n, v^n)(t) \leq E^{quasi}(u^n, v^n)(T) + \epsilon C \|(u^{\leq n}, v^{\leq n})\|_{X^T}^2. \quad (5.4)$$

$$\|(u^n, v^n)\|_{X^T}^2 \lesssim E^{quasi}(u^n, v^n)(T) + \epsilon C \|(u^{\leq n}, v^{\leq n})\|_{X^T}^2. \quad (5.5)$$

To prove both of these bounds, we rely on the results of the Corollary 4.7, which shows that it suffices to obtain $L_t^1 L_x^2$ bounds for the non-homogeneous contributions $(\mathbf{F}^n, \mathbf{G}^n)$. Precisely, we will show the bound

$$\|(\mathbf{F}^n, \mathbf{G}^n)\|_{L^1 L_x^2}^2 \lesssim C \epsilon \|(u^{\leq n}, v^{\leq n})\|_{X^T}^2, \quad (5.6)$$

which by Corollary 4.7 allows us to recover the estimates (5.4), (5.5).

Analogous to the discussion of the terms in (4.30), (4.32), we need to deal with two types of terms:

1. The v -terms have the form

$$\mathbf{N}_1(v^k, \partial v^{n-k}), \quad k = \overline{1, n-1}, \quad (5.7)$$

which we want to bound in $L_t^1 L_x^2$. Here, we do not need to use the null structure. We discuss two possible terms:

a) The case when $k = 1$. In this case, we use the bound (4.9) for the $\partial^2 v$ term. We also control $\|\partial v^n\|_{L_x^2}$ from the v^n energy. Thus, we get fixed time bound

$$\|\mathbf{N}_1(\partial v, \partial v^{n-1})\|_{L_x^2} \lesssim \epsilon T^{-1} \|\nabla v^n\|_{L_x^2}.$$

We obtain the $L_t^1 L_x^2$ bound by integrating over the time interval $[T, 2T]$.

b) The case when $k \geq 2$ and $n - k \geq 1$. It suffices to estimate at fixed time

$$\|\mathbf{N}_1(v^k, \partial v^{n-k})\|_{L_x^2},$$

which is done by placing ∂v^k in $L^{\frac{2(n-1)}{k-1}}$ and $\partial^2 v^{n-k}$ in $L^{\frac{2(n-1)}{n-k}}$. If all derivatives were spatial derivatives, then we can bound these terms at fixed time using interpolation by the energy of v^n and the bounds in (4.9)

$$\begin{aligned} \|\mathbf{N}_1(v^k, \partial v^{n-k})\|_{L_x^2} &\lesssim \|\partial v^k\|_{L_x^{\frac{2(n-1)}{k-1}}} \|\partial^2 v^{n-k}\|_{L_x^{\frac{2(n-1)}{n-k}}} \\ &\lesssim \|\partial v^n\|_{L_x^2} \|\partial^2 v\|_{L_x^\infty} \\ &\lesssim C\epsilon T^{-1} \|\partial v^n\|_{L_x^2}. \end{aligned}$$

If some of these derivatives are time derivatives, then the same argument applies with the one difference that on the right we use uniform norms over the interval $[T, 2T]$. Integrating in time over the $[T, 2T]$ time interval leads to the $L_t^1 L_x^2$ bound.

2. The u -terms are as follows:

$$\mathbf{N}_2(u^k, \partial v^{n-k}), \quad \mathbf{N}_1(v^k, \partial u^{n-k}), \quad \mathbf{N}_2(u^k, \partial u^{n-k}), \quad k = \overline{1, n-1}, \quad (5.8)$$

and also need to be bounded $L_t^1 L_x^2(C_T)$. The second term is treated as in Case 1 for $k = 1$, so we will only consider it for $k = 2, n-1$. The analysis of the first two terms is almost identical, so we just discuss the first and third terms. In fact, we will estimate them in $L_t^2 L_x^2$ and then use Hölder's inequality:

$$\|\mathbf{N}(u^k, \partial w^{n-k})\|_{L_t^1 L_x^2} \lesssim T^{\frac{1}{2}} \|\mathbf{N}(u^k, \partial w^{n-k})\|_{L_t^2 L_x^2},$$

where $w = u, v$, and $\mathbf{N} = \mathbf{N}_1$ or $\mathbf{N} = \mathbf{N}_2$ accordingly to (5.8). We need to estimate the nonlinearity $\mathbf{N}(u^k, \partial w^{n-k})$ in $L_t^2 L_x^2$. The difficulty here is that the second derivatives of u do not have t^{-1} uniform decay; instead, they decay like $t^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}-\delta_1}$.

We begin by noting that in the region C_T^{out} , the second derivatives of u do have t^{-1} uniform decay, so again, the argument in Case 1 applies. From here on, we will consider the remaining region C_T^{in} which is near the cone and corresponds to dyadic scales $1 \leq S \ll T$. Here is where we make use of the null structure (4.33) as done in Lemma 4.4. We successively consider the two terms in (4.33),

$$\mathbf{N}(u^k, \partial w^{n-k}) = \partial u^k \cdot \mathcal{T} \partial w^{n-k} + \mathcal{T} u^k \cdot \partial^2 w^{n-k}. \quad (5.9)$$

We consider the C_{TS} partition of the C_T^{in} and will estimate the $L_{t,x}^2(C_{ST})$ norms separately. As discussed above, we can assume that $S \ll T$; also, we will not distinguish between the $\pm$ (i.e., the interior vs. the exterior of the cone).

We first consider the case where $w = v$. We estimate the first term in C_{TS} using interpolation restricted to C_{TS} :

$$\begin{aligned} \|\partial u^k \cdot \mathcal{T} \partial v^{n-k}\|_{L^2} &\lesssim \|\partial^2 u\|_{L^\infty}^{\frac{n-k}{n-1}} \|\partial u\|_{L^2}^{\leq n} \|\mathcal{T} \partial v\|_{L^\infty}^{\frac{k-1}{n-1}} \|\mathcal{T} v\|_{L^2}^{\leq n} \|\mathcal{T} v\|_{L^2}^{\frac{n-k}{n-1}} \\ &\lesssim C\epsilon \left(T^{-\frac{1}{2}} S^{-\frac{1}{2}-\delta_1}\right)^{\frac{n-k}{n-1}} T^{-\frac{1}{2}} \frac{k-1}{n-1} S^{\frac{1}{2}} \frac{n-k}{n-1} \|S^{-\frac{1}{2}} \mathcal{T} v\|_{L^2}^{\leq n} \|\mathcal{T} v\|_{L^2}^{\frac{n-k}{n-1}} \sup_{t \in [T, 2T]} \|\nabla u^{\leq n}(t)\|_{L^2}^{\frac{k-1}{n-1}} \\ &\lesssim C\epsilon T^{-\frac{1}{2}} S^{-\delta_1} \frac{n-k}{n-1} \|(u^{\leq n}, v^{\leq n})\|_{X^T}, \end{aligned}$$

and similarly for the second, where we also use that $S \ll T$ in the region C_{TS} :

$$\begin{aligned} \|\mathcal{T}u^k \cdot \partial^2 v^{n-k}\|_{L^2} &\lesssim \|\mathcal{T}\partial u\|_{L^\infty}^{\frac{n-k}{n-1}} \|\mathcal{T}u^{\leq n}\|_{L^2}^{\frac{k-1}{n-1}} \|\partial^2 v\|_{L^\infty}^{\frac{k-1}{n-1}} \|\partial^2 v^{\leq n-1}\|_{L^2}^{\frac{n-k}{n-1}} \\ &\lesssim C\epsilon \left(T^{-1}S^{-\delta_1}\right)^{\frac{n-k}{n-1}} S^{\frac{1}{2}\frac{k-1}{n-1}} T^{-\frac{k-1}{n-1}+\frac{1}{2}\frac{n-k}{n-1}} \|S^{-\frac{1}{2}}\mathcal{T}u^{\leq n}\|_{L^2}^{\frac{k-1}{n-1}} \sup_{t \in [T, 2T]} \|\nabla v^{\leq n}(t)\|_{L^2}^{\frac{n-k}{n-1}} \\ &\lesssim C\epsilon T^{-\frac{1}{2}} S^{-\delta_1 \frac{n-k}{n-1}} \|(u^{\leq n}, v^{\leq n})\|_{X^T}. \end{aligned}$$

Note that commutators between $\mathcal{T}$ and derivatives yield extra T^{-1} factors and hence give negligible contributions. The dyadic summation over S is trivial, and hence,

$$\|\mathbf{N}(u^k, \partial v^{n-k})\|_{L_t^2 L_x^2} \lesssim C\epsilon T^{-\frac{1}{2}} \|(u^{\leq n}, v^{\leq n})\|_{X^T}.$$

A similar analysis is done on the $L_t^2 L_x^2$ norm of (5.9) when $w = u$. Using interpolation restricted to C_{TS} , we find for the second term that

$$\begin{aligned} \|\partial^2 u^{n-k} \cdot \mathcal{T}u^k\|_{L^2} &\lesssim \|\mathcal{T}u^k\|_{L^{\frac{2(n-1)}{k-1}}} \|\partial^2 u^{n-k}\|_{L^{\frac{2(n-1)}{n-k}}} \\ &\lesssim \|\mathcal{T}\partial u\|_{L^\infty}^{\frac{n-k}{n-1}} \|\mathcal{T}u^{\leq n}\|_{L^2}^{\frac{k-1}{n-1}} \|\partial^2 u\|_{L^\infty}^{\frac{k-1}{n-1}} \|\partial^2 u^{\leq n-1}\|_{L^2}^{\frac{n-k}{n-1}} \\ &\lesssim C\epsilon (T^{-1}S^{-\delta_1})^{\frac{n-k}{n-1}} \left(T^{-\frac{1}{2}}S^{-\frac{1}{2}-\delta_1}\right)^{\frac{k-1}{n-1}} T^{\frac{1}{2}\frac{n-k}{n-1}} S^{\frac{1}{2}\frac{k-1}{n-1}} \|S^{-\frac{1}{2}}\mathcal{T}u^{\leq n}\|_{L^2}^{\frac{k-1}{n-1}} \sup_{t \in [T, 2T]} \|\nabla u^{\leq n}(t)\|_{L^2}^{\frac{n-k}{n-1}} \\ &\lesssim C\epsilon T^{-\frac{1}{2}} S^{-\delta_1} \|u^{\leq n}\|_{X^T}. \end{aligned}$$

The first estimate is treated very similarly and satisfies the same estimate.

The dyadic summation over S is once again trivial, and we find

$$\|\mathbf{N}(u^k, \partial u^{n-k})\|_{L_t^2 L_x^2} \lesssim C\epsilon T^{-\frac{1}{2}} \|(u^{\leq n}, v^{\leq n})\|_{X^T}.$$

This completes the proof of (5.6) and thus of our proposition. $\square$

We remark that exactly the same argument also applies to the truncated equation (2.17). Also, implicit in the proof is the fact that we obtain also control over the X^T norm of the solutions. In addition to that, we will also obtain good control of the localized L^2 norms for the right-hand side of the equation (2.17). To best summarize those, we introduce the norms Y^T for functions in $[T, 2T] \times \mathbb{R}^2$ by

$$\|(\mathbf{F}, \mathbf{G})\|_{Y^T} = \sup_{1 \leq S \leq T} T^{\frac{1}{2}} \|(\mathbf{F}, \mathbf{G})\|_{L^2(C_{TS})}. \quad (5.10)$$

We introduce this norm as a way to measure the RHS of (2.17), interpreted as a source term. Such an estimate will be needed later in the proof of the pointwise estimates derived in Section 7. To formulate the next result, we will turn to the set up of the Proposition 2.3, and we set $n = 2h$. Then we have

Proposition 5.3. *Let $n \geq 4$. Assume the solutions (u, v) to either the equations (2.17) or (1.1) are defined in $\mathcal{H}^n$ in the time interval $[0, 2T_0]$, and satisfy the bootstrap bounds (4.4)–(4.9). Then the following bounds hold:*

$$E^{2h}(u, v)(t) \lesssim t^{\tilde{C}\epsilon} E^{2h}(u, v)(0), \quad t \in [0, T]. \quad (5.11)$$

In addition, for all $1 \leq T \leq T_0$, we have the following estimates in C_T :

$$\|\partial^{\leq 2h}(u, v)\|_{X^T}^2 \lesssim T^{\tilde{C}\epsilon} E^{2h}(u, v)(0), \quad (5.12)$$

and

$$\|\partial^{\leq 2h-1}(\square u, (\square + 1)v)\|_{Y^T}^2 \lesssim T^{\tilde{C}\epsilon} E^{2h}(u, v)(0). \quad (5.13)$$

Remark 5.4. The loss of derivative in the bound (5.13) is a consequence of the RHS of (1.1) or (2.17) being one derivative higher than what we can control with the X^T norms.

Proof. The bounds (5.11) and (5.13) are direct consequences of Proposition 5.1.

The bound (5.13) is only partially implicit in the proof, as one also needs to estimate the first four terms in RHS (5.1). $\square$

We also remark that the result in Proposition 5.3 proves the bounds (2.22) and (2.23) in Proposition 2.3, but only when $\mathcal{Z}^\gamma$ are regular derivatives.

6. Vector field energy estimates

The main goal of this section is to establish energy bounds for the solution (u, v) to which we have applied a certain number of vector fields from the family of vector field (1.15). Precisely, we want to prove an energy bound for $(\mathcal{Z}^\gamma u, \mathcal{Z}^\gamma v)$, where γ counts the number of Klainerman vector fields and spatial and time derivatives applied to the solution (u, v) of the equation (1.1).

All the vector fields Z in (1.14) are related to the geometry of the problem, and they are the generators of the Lorentz transformations of the Minkowski space $\mathbb{R}^{1+2}$ which preserve the equation (1.4). Under these circumstances, our aim is to prove an energy inequality for the energy functional $E^{[2h]}$, which we introduced in (1.17).

Recall the notations already introduced in the Introduction, where

$$\mathcal{Z} = \{Z, \partial\}$$

is the collection of vector fields we work with. We will use multiindices α to count the number of spatial derivatives and β for Z derivatives. We put these together in

$$\gamma = (\alpha, \beta),$$

and set

$$\mathcal{Z}^\gamma = \partial^\alpha Z^\beta.$$

We use weights to measure the total number of derivatives

$$|\gamma| = |\alpha| + h|\beta|.$$

Here, we use the parameter h to choose a balance between the relative strength of vector fields versus regular derivatives. This will allow us to work with only two vector fields, provided that we have a larger number of regular derivatives, thus enabling us to use very weak decay assumptions on the initial data. For this, we will use the range $|\gamma| \leq 2h$ which corresponds exactly to two vector fields. Ideally, we will want h to be as small as possible; its size will be dictated by the Klainerman-Sobolev inequalities in the next section.

One might wish to compare the system satisfied by $(\mathcal{Z}^\gamma u, \mathcal{Z}^\gamma v)$ with the linearized system which was studied before in Section 4. We start by applying γ vector fields $\mathcal{Z}$ to the equation (1.1). Here, the variables that play the role of the linearized variables (U, V) are $(\mathcal{Z}^\gamma u, \mathcal{Z}^\gamma v)$. One difference when working with the spatial rotations or with the Lorentz generators are the commutative properties of these vector fields with respect to the null structure nonlinearity.

Thus, applying $\mathcal{Z}^\gamma$ vector fields to both hand sides of (1.1) and denoting $\mathcal{Z}^\gamma u =: u^\gamma$, we obtain the inhomogeneous equations

$$\begin{cases} \square u^\gamma(t, x) = \mathbf{N}_1(v, \partial v^\gamma) + \mathbf{N}_1(v^\gamma, \partial v) + \mathbf{N}_2(u, \partial v^\gamma) + \mathbf{N}_2(u^\gamma, \partial v) + \mathbf{F}^\gamma \\ (\square + 1)v^\gamma(t, x) = \mathbf{N}_1(v, \partial u^\gamma) + \mathbf{N}_1(v^\gamma, \partial u) + \mathbf{N}_2(u, \partial u^\gamma) + \mathbf{N}_2(u^\gamma, \partial u) + \mathbf{G}^\gamma, \end{cases} \quad (6.1)$$

where the source terms $(\mathbf{F}^\gamma, \mathbf{G}^\gamma)$ are of the form

$$\mathbf{F}^\gamma := \sum_{\substack{|\gamma_1|+|\gamma_2| \leq |\gamma| \\ |\gamma_1|, |\gamma_2| < |\gamma|}} \mathbf{N}(v^{\gamma_1}, \partial v^{\gamma_2}), \quad \mathbf{G}^\gamma := \sum_{\substack{|\gamma_1|+|\gamma_2| \leq |\gamma| \\ |\gamma_1|, |\gamma_2| < |\gamma|}} \mathbf{N}(v^{\gamma_1}, \partial u^{\gamma_2}). \quad (6.2)$$

Here, $\mathbf{N}(\cdot, \cdot)$ is a new linear combination of quadratic null forms (1.3) arising from the commutator terms.

To better understand the structure of the system (6.1), we need a full description of the quadratic nonlinearities in $(\mathbf{F}^\gamma, \mathbf{G}^\gamma)$.

Lemma 6.1. *All the nonlinear terms in (6.1) are linear combinations of the quadratic null forms (1.3).*

Proof. Simple computations show that this is indeed the case. Iterations of the following formulas for the Klainerman vector field Ω_{0i}

$$\begin{cases} \Omega_{0i} Q_{12}(\phi, \psi) = Q_{12}(\Omega_{0i}\phi, \psi) + Q_{12}(\phi, \Omega_{0i}\psi) - (-1)^i Q_{0j}(\phi, \psi), & i, j \in \{1, 2\}, i \neq j, \\ \Omega_{0i} Q_{0j}(\phi, \psi) = Q_{0j}(\Omega_{0i}\phi, \psi) + Q_{0j}(\phi, \Omega_{0i}\psi) + Q_{ij}(\phi, \psi), & i, j \in \{1, 2\}, \\ \Omega_{0i} Q_0(\phi, \psi) = Q_0(\Omega_{0i}\phi, \psi) + Q_0(\phi, \Omega_{0i}\psi), & i = 1, 2, \end{cases}$$

as well for the Ω_{12} vector field

$$\begin{cases} \Omega_{12} Q_{12}(\phi, \psi) = Q_{12}(\Omega_{12}\phi, \psi) + Q_{12}(\phi, \Omega_{12}\psi), \\ \Omega_{12} Q_{0i}(\phi, \psi) = Q_{0i}(\Omega_{12}\phi, \psi) + Q_{0i}(\phi, \Omega_{12}\psi) + (-1)^i Q_{0j}(\phi, \psi), & i, j \in \{1, 2\}, i \neq j, \\ \Omega_{12} Q_0(\phi, \psi) = Q_0(\Omega_{12}\phi, \psi) + Q_0(\phi, \Omega_{12}\psi), \end{cases}$$

show that indeed the quadratic nonlinearities have a null structure. Useful in our computations are also the commutator properties of individual vector fields acting on the null structures (1.3), which we list below

$$[\Omega_{0i}, \partial_t] = -\partial_i, \quad [\Omega_{0i}, \partial_j] = -\delta_{ij}\partial_t, \quad [\Omega_{12}, \partial_t] = 0, \quad [\Omega_{12}, \partial_1] = \partial_2, \quad [\Omega_{12}, \partial_2] = -\partial_1. \quad (6.3)$$

□

Our main vector field energy estimate is as follows:

Proposition 6.2. *Let (u, v) be solutions to the equations (2.17) in the time interval $[0, 2T_0]$, which in addition satisfy the corresponding bootstrap bounds (4.4)-(4.9) Then, for $T' < T_0$, they must also satisfy the bound*

$$E^{[2h]}(u, v)(t) \lesssim t^{\tilde{C}\epsilon} E^{[2h]}(u, v)(0), \quad t \in [0, T'], \quad (6.4)$$

where $\tilde{C}$ is a positive constant.

The content of Remark 5.2 remains valid in the context of the above proposition (i.e., we do not distinguish between space and time derivatives in the choice of our vector fields). Observe also that the product $\mathcal{Z}^\gamma$ with $|\gamma| \leq 2h$ can at most contain two vector fields Z of the family (1.14).

Proof. As a preliminary step, we remark that our initial data energy $E^{[2h]}(u, v)(0)$ is equivalent to the square of the norm in (1.18),

$$E^{[2h]}(u, v)(0) \approx \|(u, v)[0]\|_{\mathcal{H}^{2h}}^2 + \|x\partial_x(u, v)[0]\|_{\mathcal{H}^h}^2 + \|x^2\partial_x^2(u, v)[0]\|_{\mathcal{H}^0}^2. \quad (6.5)$$

This is a straightforward elliptic computation which is left for the reader. We will take advantage of this observation in order to simplify the analysis in the exterior region C^{out} . Indeed, in this region, we can directly apply the result of Proposition 2.2 to obtain the desired bounds on the solution, and we can dispense with the vector field analysis. Precisely, we conclude that we have the exterior fixed time uniform estimate

$$E_{ext}^{[2h]}(u, v)(t) \lesssim E^{[2h]}(u, v)(0), \quad (6.6)$$

where

$$E_{ext}^{[2h]}(u, v)(t) = \|(u, v)[t]\|_{\mathcal{H}^{2h}(C^{out})}^2 + \|x\partial_x(u, v)[t]\|_{\mathcal{H}^h(C^{out})}^2 + \|x^2\partial_x^2(u, v)[t]\|_{\mathcal{H}^0(C^{out})}^2.$$

This bound is stronger than the needed one in (6.4) in the exterior region in two ways: (i) it does not have the $t^{\tilde{C}\epsilon}$ loss, and (ii) it applies to all vector fields of size $|x|$ rather than only the Z vector fields.

After these preliminaries, we turn our attention to the evolution of the full vector field energies of (u, v) . We begin with several reductions which follow the pattern of previous sections. First we recall that our problem is quasilinear and the energy that can be propagated for the derivatives of (u, v) is the quasilinear energy. So denoting

$$E_{quasi}^{[2h]}(u, v) := \sum_{|\gamma| \leq 2h} E_{quasi}^{[2h]}(t; \mathcal{Z}^\gamma u, \mathcal{Z}^\gamma v), \quad (6.7)$$

we will replace the bound (6.4) with the equivalent bound

$$E_{quasi}^{[2h]}(v, u)(t) \lesssim t^{\tilde{C}\epsilon} E_{quasi}^{[2h]}(v, u)(0), \quad t \in [0, T']. \quad (6.8)$$

As before, this reduces to a bound on a dyadic time interval

$$E_{quasi}^{[2h]}(u, v)(t) \leq (1 + \tilde{C}\epsilon) E_{quasi}^{[2h]}(u, v)(T), \quad t \in [T, 2T] \subset [0, T']. \quad (6.9)$$

Applying Corollary 4.7, bound (6.9) in turn would follow from a bound for the source terms in the equation (6.1) in the time interval $[T, 2T]$, which we separate into an interior and an exterior part:

$$\|(\mathbf{F}^\gamma, \mathbf{G}^\gamma)\|_{L_t^1 L_x^2(C_T^{int})} \lesssim C\epsilon \|\mathcal{Z}^{\leq 2h}(u, v)\|_{X^T}, \quad |\gamma| \leq 2h, \quad (6.10)$$

respectively

$$\|(\mathbf{F}^\gamma, \mathbf{G}^\gamma)\|_{L_t^1 L_x^2(C_T^{out})} \lesssim C\epsilon \|\mathcal{Z}^{\leq 2h}(u, v)[0]\|_{\mathcal{H}^0}, \quad |\gamma| \leq 2h. \quad (6.11)$$

This separation is convenient here because in the exterior region, we have access to the stronger bounds in (6.6) which simplifies matters somewhat. A similar separation could have been implemented in the previous two sections, but there it would have made less of a difference.

Since (u, v) play symmetric roles in this analysis, we will use the notations w and ω for either u or v . Then we need to estimate

$$\|\mathbf{N}(w^{\gamma_1}, \partial\omega^{\gamma_2})\|_{L_t^1 L_x^2} \lesssim C\epsilon \|\mathcal{Z}^{\leq 2h}(u, v)\|_{X^T},$$

where γ_1 and γ_2 are restricted to the range

$$|\gamma_1| + |\gamma_2| \leq 2h, \quad |\gamma_1|, |\gamma_2| < 2h.$$

Setting $\gamma_i = (\alpha_i, \beta_i)$, we distinguish three cases:

- I) $\beta_1 = \beta_2 = 0$.
- II) $|\beta_1| = 0, |\beta_2| = 1$ or viceversa.
- III) $|\beta_1| = |\beta_2| = 1$.

The case **I**) was already considered in Section 5. For the case **II**), we enumerate the possibilities:

$$\mathbf{N}(\partial^{n_1} w, \partial^{n_2} Z\omega) \quad 1 \leq n_1 + n_2 \leq h + 1, \quad n_2 \leq h.$$

The case $n_1 = 0$ may occur here only if we started with exactly Z^2 and one Z was commuted. In this case, we get the terms

$$\mathbf{N}(w, \partial Z\omega),$$

which have not been covered yet. We will still discard this case because it is similar but simpler than case (6.14) below. The case $n_1 = 1$ is also identical to the estimates in Section 5, using directly the bootstrap assumptions for the second-order derivatives for u and v , so we can also discard it. Thus, we are left with

$$\mathbf{N}(\partial^{n_1} w, \partial^{n_2} Z\omega) \quad 2 \leq n_1 \leq h, \quad 1 \leq n_1 + n_2 \leq h + 1, \quad (6.12)$$

$$\mathbf{N}(\partial^{h+1} w, Z\omega). \quad (6.13)$$

Finally, in case **III**), the only terms to consider are

$$\mathbf{N}(Zw, \partial Z\omega). \quad (6.14)$$

□

To bound each of these source terms, we follow the same outline as the proofs of the corresponding results discussed in Sections 7.6, 4 and 5. One minor difference is that we will separate the exterior region C_T^{out} from C_T^{in} . For the main region C_T^{int} , we refine the analysis even further and prove estimates on the space-time regions $C_{TS}^\pm$. In all cases, the dyadic summation in S will be trivial. We will repeatedly use the following interpolation Lemma:

Lemma 6.3. Assume that $n \geq 0$ and

$$\frac{2}{p} = \frac{1}{2} + \frac{1}{q}, \quad 2 \leq q \leq \infty.$$

Then we have

$$\|\partial^{n+1} Z\phi\|_{L^p(C_{TS})} \lesssim \|Z^{\leq 2} \phi\|_{L^2(C_{TS})}^{\frac{1}{2}} \left(\|\partial^{\leq 2n} \partial^2 \phi\|_{L^q(C_{TS})}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial \phi\|_{L^q(C_{TS})}^{\frac{1}{2}} \right). \quad (6.15)$$

The same holds in C_T^{int} .

Proof. Using hyperbolic polar coordinates adapted to C_{TS} (see the next section), this lemma reduces to variants of the classical Gagliardo-Nirenberg inequality in a unit sized domain. The details of the proof are included in the appendix. □

A. The null form (6.14).

Here, we consider the terms in (6.14) and separate the exterior region C_T^{out} and the dyadic interior regions C_{TS} . In the latter case, we will not differentiate between C_{TS}^+ and C_{TS}^- .

a) The exterior region C_T^{out} . Here, we prove the bound (6.11) in C_T^{out} . We neglect the null structure as well as the vector field structure so that we are left with the fixed time bound

$$\|x^2 \partial^2 w \partial^3 \omega\|_{L_x^2} \lesssim \epsilon T^{-1} (E_{ext}^{[2h]}(\omega))^{\frac{1}{2}}.$$

But this is straightforward, as for the first factor, we can use the ϵr^{-1} bound in our bootstrap assumption, and for the second, we use the $x^{-2} L^2$ bound in the above exterior norm.

b) The interior region C_T^{int} , v -terms. Here, we consider the terms (6.14) for $w = \omega = v$. This is the simpler case, and it is instructive to consider it first. We are considering expressions of the form

$$\mathbf{N}(Zv, \partial Zv), \quad (6.16)$$

and we want to bound them in $L_t^1 L_x^2(C_T^{int})$. We actually estimate them in $L_t^2 L_x^2(C_T^{int})$ using that

$$\|\mathbf{N}(Zv, \partial Zv)\|_{L_t^1 L_x^2(C_T^{int})} \lesssim T^{\frac{1}{2}} \|\mathbf{N}(Zv, \partial Zv)\|_{L_t^2 L_x^2(C_T^{int})}.$$

We neglect the null condition and will place both factors in $L^4(C_T^{int})$

$$\|\mathbf{N}(Zv, \partial Zv)\|_{L^2(C_T^{int})} \lesssim \|\partial Zv\|_{L^4(C_T^{int})} \|\partial^2 Zv\|_{L^4(C_T^{int})},$$

where the two terms are estimated using interpolation inequalities as follows:

$$\|\partial^2 Zv\|_{L^4(C_T^{int})} \lesssim \|Z^{\leq 2} \partial v\|_{L^2(C_T^{int})}^{\frac{1}{2}} \|\partial^{\leq 2} \partial v\|_{L^\infty(C_T^{int})}^{\frac{1}{2}}, \quad (6.17)$$

respectively

$$\|\partial Zv\|_{L^4(C_T^{int})} \lesssim \|Z^{\leq 2} \partial v\|_{L^2(C_T^{int})}^{\frac{1}{2}} \|\partial v\|_{L^\infty(C_T^{int})}^{\frac{1}{2}}. \quad (6.18)$$

Here, we can use slightly larger sets for the interpolation on the right, which allows us to localize the estimates. Combining the two and using our bootstrap assumption, we obtain

$$\|\mathbf{N}(Zv, \partial Zv)\|_{L_t^2 L_x^2(C_T^{int})} \lesssim C \epsilon T^{-\frac{1}{2}} \|Z^{\leq 2} v\|_{X^T},$$

as needed.

c) The C_T^{int} region, the u terms. Here, we consider the nonlinear term (6.14) in the case where $w = \omega = u$

$$\mathbf{N}(Zu, \partial Zu), \quad (6.19)$$

which we want to bound them in $L_t^1 L_x^2(C_T^{int})$. We consider separately each of the C_{TS} regions. This is simpler if $S \approx T$, as the null condition is not needed there and the argument in case (b) applies.

It remains to separately consider the regions C_{TS} with $1 \leq S \ll T$, in which case we use the null form structure via the representation (4.34)

$$\mathbf{N}(Zu, \partial Zu) = \partial Zu \cdot \mathcal{T} \partial Zu + \mathcal{T} Zu \cdot \partial^2 Zu. \quad (6.20)$$

Again, we interpolate between L^2 and L^∞ to get L^4 , but this time in C_{TS} . This is acceptable because the Z vector fields are well adapted to C_{TS} . We harmlessly commute $\mathcal{T}$ and ∂ with Z at the expense of much better error terms. The bootstrap assumptions (4.4) and (4.7) yield that

$$|\mathcal{T}u| \lesssim \frac{1}{t} |Zu| + \frac{t-r}{t} |\partial u| \lesssim C \epsilon t^{-1} \langle t-r \rangle^\delta,$$

hence, for the second term in (6.20), we have

$$\begin{aligned} \|Z\mathcal{T}u \cdot \partial^2 Zu\|_{L^2(C_{TS})} &\lesssim \|Z\mathcal{T}u\|_{L^4(C_{TS})} \|\partial^2 Zu\|_{L^4(C_{TS})} \\ &\lesssim \|Z^{\leq 2}\mathcal{T}u\|_{L^2(C_{TS})}^{\frac{1}{2}} \|\mathcal{T}u\|_{L^\infty}^{\frac{1}{2}} \|Z^{\leq 2}\partial u\|_{L^2(C_{TS})}^{\frac{1}{2}} (\|\partial^3 u\|_{L^\infty}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial u\|_{L^\infty}^{\frac{1}{2}}) \\ &\lesssim \epsilon C S^{\frac{1}{4}} T^{-\frac{1}{4}} S^{-\frac{1}{4}-\frac{1}{2}} (\delta_1 - \delta) T^{-\frac{1}{4}} \|Z^{\leq 2}(u, v)\|_{X^T}, \end{aligned}$$

and the S summation is straightforward since $\delta \ll \delta_1$. For the first term in (6.20), we have

$$\begin{aligned} \|Z\partial u \cdot \partial Z\mathcal{T}u\|_{L^2(C_{TS})} &\lesssim \|Z\partial u\|_{L^4(C_{TS})} \|\partial Z\mathcal{T}u\|_{L^4(C_{TS})} \\ &\lesssim \|Z^{\leq 2}\partial u\|_{L^2(C_{TS})}^{\frac{1}{2}} \|\partial u\|_{L^\infty(C_{TS})}^{\frac{1}{2}} \|Z^{\leq 2}\mathcal{T}u\|_{L^2(C_{TS})}^{\frac{1}{2}} \\ &\quad \cdot \left(\|\partial^2 \mathcal{T}u\|_{L^\infty(C_{TS})}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial \mathcal{T}u\|_{L^\infty(C_{TS})}^{\frac{1}{2}} \right) \\ &\lesssim C \epsilon T^{\frac{1}{4}} S^{\frac{1}{4}} T^{-\frac{1}{4}} S^{-\frac{1}{4}} T^{-\frac{1}{2}} S^{-\frac{\delta_1}{2}} \|Z^{\leq 2}(u, v)\|_{X^T} \\ &\lesssim C \epsilon T^{-\frac{1}{2}} S^{-\frac{\delta_1}{2}} \|Z^{\leq 2}(u, v)\|_{X^T}, \end{aligned}$$

where for the last factors, we used the bootstrap assumptions (4.6) and (4.8), which imply

$$|\partial^j \mathcal{T}u| \lesssim \frac{1}{t} |\partial^j Zu| + \frac{t-r}{t} |\partial^{j+1} u| \lesssim C \epsilon t^{-1} \langle t-r \rangle^{-\delta}, \quad j = \overline{1, 2}.$$

(d) The C_T^{int} region, the mixed u, v terms. Consider first the quadratic term

$$\mathbf{N}(Zu, \partial Zv)$$

which we need to estimate separately in each C_{TS} region. Again, we only consider the case $1 \leq S \ll T$, the one where $S \approx T$ are simpler, and use the null structure via the representation (4.33)

$$\mathbf{N}(Zu, \partial Zv) = \partial Zu \cdot \mathcal{T} \partial Zv + \mathcal{T} Zu \cdot \partial^2 Zv.$$

For the first term, we use interpolation inequalities and our bootstrap assumptions (4.7) and (4.9) to obtain

$$\begin{aligned} \|Z\partial u \cdot \partial Z\mathcal{T}v\|_{L^2(C_{TS})} &\lesssim \|Z\partial u\|_{L^4(C_{TS})} \|\partial Z\mathcal{T}v\|_{L^4(C_{TS})} \\ &\lesssim \|Z^{\leq 2}\partial u\|_{L^2(C_{TS})}^{\frac{1}{2}} \|\partial u\|_{L^\infty(C_{TS})}^{\frac{1}{2}} \|Z^{\leq 2}\mathcal{T}v\|_{L^2(C_{TS})}^{\frac{1}{2}} \\ &\quad \cdot \left(\|\partial^2 \mathcal{T}v\|_{L^\infty(C_{TS})}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial \mathcal{T}v\|_{L^\infty(C_{TS})}^{\frac{1}{2}} \right) \\ &\lesssim C \epsilon T^{\frac{1}{4}} T^{-\frac{1}{4}} S^{-\frac{1}{4}} S^{\frac{1}{4}} T^{-\frac{1}{2}} S^{-\frac{\delta_1}{2}} S^{\frac{\delta_1}{2}} \|Z^{\leq 2}(u, v)\|_{X^T} \\ &\lesssim C \epsilon T^{-\frac{1}{2}-\frac{\delta_1}{2}} S^{\frac{\delta_1}{2}} \|Z^{\leq 2}(u, v)\|_{X^T}, \end{aligned}$$

where the S summation is straightforward. The second quadratic term is similar.

The other mixed quadratic term to consider in this scenario is

$$\mathbf{N}(Zv, \partial Zu),$$

which is easy to treat as shown below:

$$\begin{aligned} \|\mathbf{N}(Zv, \partial Zu)\|_{L^2(C_{TS})} &\lesssim \|\partial Zv \cdot \partial^2 Zu\|_{L^2(C_{TS})} \lesssim \|\partial Zv\|_{L^4(C_{TS})} \|\partial^2 Zu\|_{L^4(C_{TS})} \\ &\lesssim \|Z^{\leq 2} v\|_{L^2(C_{TS})}^{\frac{1}{2}} (\|\partial^2 v\|_{L^\infty(C_{TS})}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial v\|_{L^\infty(C_{TS})}^{\frac{1}{2}}) \|Z^{\leq 2} \partial u\|_{L^2(C_{TS})}^{\frac{1}{2}} \\ &\quad \cdot \left(\|\partial^3 u\|_{L^2(C_{TS})}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial u\|_{L^2(C_{TS})}^{\frac{1}{2}} \right) \\ &\lesssim C\epsilon S^{\frac{1}{4}} T^{-\frac{1}{2}} T^{\frac{1}{4}} T^{-\frac{1}{4}} S^{-\frac{1}{4} - \frac{\delta_1}{2}} \|Z^{\leq 2}(u, v)\|_{X^T} \\ &\lesssim C\epsilon T^{-\frac{1}{2}} S^{-\frac{\delta_1}{2}} \|Z^{\leq 2}(u, v)\|_{X^T}. \end{aligned}$$

B. The null form (6.13)

The arguments here are similar to the ones above. In the exterior region C_T^{ext} , we have

$$|\mathbf{N}(\partial^{h+1} w, Z\omega)| \lesssim |x| |\partial^{h+2} w| |\partial^2 \omega|,$$

and we can bound the first factor in L^2 ,

$$\| |x| \partial^{h+2} w \|_{L^2(C_T^{out})} \lesssim (E_{ext}^{[2h]}(w))^{\frac{1}{2}},$$

and the second factor pointwise by $C\epsilon |x|^{-1}$.

The argument in the region C_T^{int} is also similar. The bounds for $\mathbf{N}(\partial^{h+1} w, Z\omega)$ in each C_{TS} are obtained in the same way as the bounds for $\mathbf{N}(Zw, \partial Z\omega)$, replacing the inequalities for ∂Zw and $\mathcal{T}Zw$ with the following interpolation inequality

$$\|\partial^{h+1} \psi\|_{L^4} \lesssim \|\partial^{\leq 2h-2} \partial^2 \psi\|_{L^2}^{\frac{1}{2}} \|\partial^2 \psi\|_{L^\infty}^{\frac{1}{2}}, \quad \psi = \partial w, \mathcal{T}w \quad (6.21)$$

which holds in each region C_{TS} .

C. The null form (6.12)

In this case, we follow the same strategy as above, but with the difference that we can no longer rely on L^4 interpolation, and instead we must use other L^p norms. To fix the notations, we simply consider the worst case when $n_1 + n_2 = h + 1$.

We start with the exterior region, where we estimate the terms in (6.12) as follows:

$$|\mathbf{N}(\partial^{n_1} w, \partial^{n_2} Z\omega)| \lesssim |x| |\partial^{n_1+1} w| |\partial^{n_2+2} \omega|.$$

We chose exponents p_1, p_2 so that

$$p_1 = \frac{2(h-1)}{n_1-2}, \quad p_2 = \frac{2(h-1)}{n_2}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$$

and will place the two factors in L^{p_1} respectively in L^{p_2} . At a fixed time $t \in [T, 2T]$, we use Holder's inequality and interpolation to get the exterior bound

$$\begin{aligned} \| |x| \mathbf{N}(\partial^{n_1} w, \partial^{n_2} Z\omega) \|_{L^2(C_T^{out})} &\lesssim \| |x| \partial^{n_1+1} w \|_{L^{p_1}(C_T^{out})} \|\partial^{n_2+2} \omega\|_{L^{p_2}(C_T^{out})} \\ &\lesssim \| |x| \partial^2 w \|_{L^\infty(C_T^{out})}^{\frac{n_2}{h-1}} \| |x| \partial^{\leq h} \partial^2 w \|_{L^2(C_T^{out})}^{\frac{n_1-2}{h-1}} \|\partial^2 \omega\|_{L^\infty(C_T^{out})}^{\frac{n_1-2}{h-1}} \|\partial^{\leq h} \partial^2 \omega\|_{L^2(C_T^{out})}^{\frac{n_2}{h-1}}, \end{aligned}$$

where the L^∞ norms are estimated using the bootstrap assumption and the L^2 norms are estimated using the outer energy bound, – in particular, that

$$\|\partial^{\leq h} \partial^2 \omega\|_{L^2(C_T^{out})} \lesssim T^{-1} (E_{ext}^{[2h]}(\omega))^{\frac{1}{2}}.$$

For the interior region C_T^{int} , we consider separately the sets C_{TS} as before and use the null form representation (4.34) in the case where $(w, \omega) = \{(u, u), (u, v), (v, u)\}$. Then we need to estimate the expressions

$$\mathbf{N}(\partial^{n_1} w, \partial^{n_2} Z\omega) = \partial^{n_1+1} w \cdot \mathcal{T} \partial^{n_2} Z\omega + \mathcal{T} \partial^{n_1} \omega \cdot \partial^{n_2+1} Z\omega \quad (6.22)$$

in $L^2(C_{TS})$ for $1 \leq S \leq T$. When $n_2 = 0$, we use again the $L^4 - L^4$ interpolation argument, so we focus here on the case $n_2 \geq 1$. Both terms are treated in a similar manner. For simplicity, we consider the latter one, in order to facilitate comparison with case (a). We begin with two exponents p_1 and p_2 given by

$$p_1 = \frac{4(h-1)}{n_1-1}, \quad p_2 = \frac{4(h-1)}{h+n_2-2}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2},$$

and we estimate

$$\|\mathcal{T} \partial^{n_1} w \cdot \partial^{n_2+1} Z\omega\|_{L^2(C_{TS})} \lesssim \|\mathcal{T} \partial^{n_1} w\|_{L^{p_1}(C_{TS})} \|\partial^{n_2+1} Z\omega\|_{L^{p_2}(C_{TS})},$$

where the two terms are estimated using interpolation inequalities as follows:

$$\|\mathcal{T} \partial^{n_1} w\|_{L^{p_1}(C_{TS})} \lesssim \|\partial^{\leq 2h-2} \mathcal{T} \partial w\|_{L^2(C_{TS})}^{\frac{2}{p_1}} \|\mathcal{T} \partial w\|_{L^\infty(C_{TS})}^{1-\frac{2}{p_1}}, \quad (6.23)$$

respectively

$$\|\partial^{n_2+1} Z\omega\|_{L^{p_2}(C_{TS})} \lesssim \|Z^{\leq 2} \partial \omega\|_{L^2(C_{TS})}^{\frac{1}{2}} (\|\partial^{\leq 2(n_2-1)} \partial^3 \omega\|_{L^{p_3}(C_{TS})}^{\frac{1}{2}} + S^{-\frac{1}{2}} \|\partial^2 \omega\|_{L^{p_3}(C_{TS})}^{\frac{1}{2}}), \quad (6.24)$$

where p_3 is given by

$$\frac{1}{p_3} + \frac{1}{2} = \frac{2}{p_2}.$$

Finally, the last terms in (6.24) are interpolated again as

$$\begin{aligned} \|\partial^{\leq 2(n_2-1)} \partial^2 \omega\|_{L^{p_3}(C_{TS})} &\lesssim \|\partial^{\leq 2(h-1)} \partial^3 \omega\|_{L^2(C_{TS})}^{\frac{4}{p_2}-1} \|\partial^3 \omega\|_{L^\infty(C_{TS})}^{2-\frac{4}{p_2}}, \\ \|\partial^2 \omega\|_{L^{p_3}(C_{TS})} &\lesssim \|\partial^2 w\|_{L^2(C_{TS})}^{\frac{4}{p_2}-1} \|\partial^2 \omega\|_{L^\infty(C_{TS})}^{2-\frac{4}{p_2}}. \end{aligned}$$

Combining the last five relations and using our bootstrap assumptions, we obtain

$$\|\mathcal{T} \partial^{n_1} w \cdot \partial^{n_2+1} Z\omega\|_{L^2(C_{TS})} \lesssim C \epsilon T^{-\frac{1}{2}} S^{-c \delta_1} \|Z^{\leq 2} \omega\|_{X_T}^{\frac{1}{2}} \|\partial^{\leq 2h}(w, \omega)\|_{X_T}^{\frac{1}{2}},$$

as needed, where

$$c = \begin{cases} 1, & (w, \omega) = (u, u) \\ \frac{2}{p_2}, & (w, \omega) = (u, v) \\ \frac{2}{p_1}, & (w, \omega) = (v, u). \end{cases}$$

The term $\mathbf{N}(\partial^{n_1} v, \partial^{n_2} Zv)$, however, does not use the null structure and can be bounded using the same interpolation argument as the one employed above for the product $\mathcal{T}\partial^{n_1} w \cdot \partial^{n_2+1} Z$ but carried in the whole interior region C_T^{int} rather than in each C_{TS} . From the pointwise bound (4.9), we get

$$\begin{aligned} \|\mathbf{N}(\partial^{n_1} v, \partial^{n_2} Zv)\|_{L^2(C_T^{int})} &\lesssim \|\partial^{n_1+1} v \cdot \partial^{n_2+1} Zv\|_{L^2(C_T^{int})} \\ &\lesssim \left\| \partial^{\leq 2(h-1)} \partial^2 v \right\|_{L^2(C_T^{int})}^{\frac{2}{p_1}} \left\| \partial^2 v \right\|_{L^\infty(C_T^{int})}^{1-\frac{2}{p_1}} \left\| Z^{\leq 2} \partial v \right\|_{L^2(C_T^{int})}^{\frac{1}{2}} \left\| \partial^{\leq 2(h-1)} \partial^3 v \right\|_{L^2(C_T^{int})}^{\frac{2}{p_2}-\frac{1}{2}} \left\| \partial^3 v \right\|_{L^\infty(C_T^{int})}^{1-\frac{2}{p_2}} \\ &\lesssim \epsilon T^{-1} \left\| \partial^{\leq 2h} v \right\|_{X^T}^{\frac{1}{2}} \left\| Z^{\leq 2} v \right\|_{X^T}^{\frac{1}{2}}. \end{aligned}$$

Now we are able to finish the proof of Proposition 2.3. The proof of Proposition 6.2 already gives us the X^T bound of (u, v) . It remains to consider the Y^T bound. The Y^T bounds without any Z vector fields were already discussed in Section 5, so we are left with the single Y^T bound that involves the Z vector field

$$\|Z(\square u, (\square + 1)v)\|_{Y^T} \lesssim \epsilon C T^{\epsilon \tilde{C}}. \quad (6.25)$$

This is the same as

$$\|\mathbf{N}(Zw, \partial\omega)\|_{Y^T} + \|\mathbf{N}(w, Z\partial\omega)\|_{Y^T} \lesssim \epsilon C T^{\epsilon \tilde{C}}. \quad (6.26)$$

These bounds have already been proved in the proof of Proposition 6.2 above in the worst case scenario, which is that of the u -terms; see (4.33).

7. Klainerman-Sobolev inequalities

To recover the bootstrap bounds (4.4) to (4.9) on the almost global time scale, we need appropriate Klainerman-Sobolev inequalities, where the aim is to obtain pointwise bounds from the integral bounds. Our main result here is a linear result. Unfortunately, we cannot work at fixed time, so instead we work in a dyadic time region C_T with $T \geq 1$. The pointwise bounds in the exterior region C_t^{out} were already established in Proposition 2.1, so here it remains to concentrate on the region C_T^{in} .

Theorem 5. *Let $h \geq 8$. Assume that the functions (u, v) in C_T^{in} satisfy the bounds*

$$\|\mathcal{Z}^\gamma(u, v)\|_{X^T} \leq 1, \quad |\gamma| \leq 2h, \quad (7.1)$$

as well as

$$\|\mathcal{Z}^\gamma(\square u, (\square + 1)v)\|_{Y^T} \leq 1, \quad |\gamma| \leq h. \quad (7.2)$$

Then they also satisfy the pointwise bounds

$$|Zu| \lesssim 1, \quad (7.3)$$

$$|Z\partial^j u| \lesssim \langle t-r \rangle^{-\delta} \quad j = \overline{1, 2}, \quad (7.4)$$

$$|\partial u| \lesssim \langle t \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}}, \quad (7.5)$$

$$|\partial^j u| \lesssim \langle t \rangle^{-\frac{1}{2}} \langle t-r \rangle^{-\frac{1}{2}-\delta}, \quad j = \overline{2, 3} \quad (7.6)$$

$$|\partial^j v| \lesssim \langle t \rangle^{-1-\delta} \langle t-r \rangle^\delta, \quad j = \overline{0, 3}. \quad (7.7)$$

Here, $\delta > 0$ is a fixed small constant. This suffices to prove the almost global well-posedness result. The remainder of this Section is devoted to the proof of Theorem 5.

To prove the theorem, we divide the forward half-space $\mathbb{R}^+ \times \mathbb{R}^2$ in two regions: $\mathcal{C}^+$ inside the cone, and $\mathcal{C}^-$ outside the cone, and solve the problem separately in each of the regions. Here, we will allow for a small ambiguity in that the region at distance at most one from the cone can be treated both ways; this corresponds to the fact that the bulk of the X^T norm is invariant with respect to unit (space and time) translations. This is related to the fact that the main result of this paper is also invariant with respect to unit translations. To a large extent, we will think of the bounds of this Theorem as consequences of Sobolev type embeddings or Bernstein type inequalities. Since our vector fields include the Z vector fields, in order to be able to interpret the bounds in Theorem 5, we need to work in coordinates which are adapted to the vector fields Z . In practice, this means working in hyperbolic coordinates, both in $\mathcal{C}^+$ and in $\mathcal{C}^-$. Because of this, in what follows, we first introduce the hyperbolic coordinates inside the cone and prove our bounds there, and then introduce the hyperbolic coordinates for the $\mathcal{C}^-$ region and again prove the corresponding estimates. Fortunately, there will be many similarities between the two regions, and some proofs will actually be completely identical.

7.1. Normalized coordinates inside the cone

For the analysis inside the cone, it is very convenient to work in the so-called *spherical hyperbolic coordinates* in $\mathbb{H}^2 \times \mathbb{R}$:

$$\begin{cases} t = e^\sigma \cosh(\phi), \\ x_1 = e^\sigma \sinh(\phi) \sin(\theta), \\ x_2 = e^\sigma \sinh(\phi) \cos(\theta), \end{cases} \quad (7.8)$$

where θ and ϕ are the polar coordinates in the hyperbolic space; ϕ measures the distance from a point on the hyperboloid to the origin $(1, 0, 0)$ (the origin is also called the *pole*), and θ is the angle from a reference direction. Finally, σ represents the time in the hyperbolic coordinates. The Jacobian of the change of variable is

$$J(\sigma, \phi, \theta) = e^{3\sigma} \sinh(\phi).$$

We will also need to express the wave operator into this new variables, as it will later become important in our analysis:

$$-\square = e^{-2\sigma} \left(-\partial_\sigma^2 + \partial_\phi^2 + \frac{1}{\sinh^2(\phi)} \partial_\theta^2 - \partial_\sigma + \frac{\cosh(\phi)}{\sinh(\phi)} \partial_\phi \right). \quad (7.9)$$

One can verify very easily the formula in (7.9) as well as the following correspondence in between the derivatives relative to the Euclidean or to the hyperbolic coordinates:

$$\begin{cases} \partial_\sigma = t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2} = t\partial_t + r\partial_r, \\ \partial_\phi = r\partial_t + \frac{t}{r} [x_1\partial_{x_1} + x_2\partial_{x_2}] = r\partial_t + t\partial_r, \\ \partial_\theta = -x_2\partial_{x_1} + x_1\partial_{x_2}, \end{cases} \quad (7.10)$$

and respectively

$$\begin{cases} \partial_t = e^{-\sigma} \cosh(\phi) \partial_\sigma - e^{-\sigma} \sinh(\phi) \partial_\phi, \\ \partial_{x_1} = -e^{-\sigma} \sinh(\phi) \sin(\theta) \partial_\sigma + e^{-\sigma} \cosh(\phi) \sin(\theta) \partial_\phi + \frac{e^{-\sigma} \cos(\theta)}{\sinh(\phi)} \partial_\theta, \\ \partial_{x_2} = -e^{-\sigma} \sinh(\phi) \cos(\theta) \partial_\sigma + e^{-\sigma} \cosh(\phi) \cos(\theta) \partial_\phi - \frac{e^{-\sigma} \sin(\theta)}{\sinh(\phi)} \partial_\theta. \end{cases} \quad (7.11)$$

Once in these coordinates, the regions C_{TS}^+ become essentially rectangular regions of size 1. Precisely, in spherical hyperbolic coordinates, the regions C_{TS}^+ are represented as follows:

$$C_{TS}^+ \longrightarrow D := \{(\sigma, \phi, \theta) : (\sigma, \phi, \theta) \in I_\sigma \times I_\phi \times [0, 2\pi]\},$$

where I_σ and I_ϕ are intervals of size 1. As discussed earlier, we are assuming we are at distance at least one from the cone, which corresponds to $S \geq 1$. Furthermore, in the region C_{TS}^+ , we have

$$e^{2\sigma} = t^2 - |x|^2 \approx ST, \quad e^\sigma \cosh(\phi) \approx T, \quad J \approx ST^2. \quad (7.12)$$

Observation: Please note² the connection between the new coordinates and the vector fields associated to our problem:

- the scaling vector field (1.13) and the derivative with respect to the time-like variable σ :

$$\partial_\sigma = \mathcal{S}.$$

- the rotations Ω_{12} and Ω_{21} are nothing more than the derivative in the θ direction:

$$\partial_\theta = \Omega_{12} = -\Omega_{21}.$$

- the Lorentz boosts Ω_{01} and Ω_{02} are closely connected to the derivative in the ϕ direction, which we will denote by Ω_{0r} :

$$\Omega_{0r} := \partial_\phi = r\partial_t + t\partial_r = \frac{x_1}{r}\Omega_{01} + \frac{x_2}{r}\Omega_{02}.$$

A more useful relation is given by

$$\begin{cases} \Omega_{01} = \cos \theta \partial_\phi - \sin \theta \frac{\cosh \phi}{\sinh \phi} \partial_\theta, \\ \Omega_{02} = \cos \theta \frac{\cosh \phi}{\sinh \phi} \partial_\theta + \sin \theta \partial_\phi. \end{cases}$$

Many of our estimates involve the regular gradient, which is very simple in the standard Minkowski coordinates where it can be expressed in the basis $\{\partial_t, \partial_{x_1}, \partial_{x_2}\}$, but not as simple when expressed in spherical hyperbolic coordinates. Because of this, it is very useful to have an alternative basis to measure the size of the gradient which is well-adapted to the geometry of the hyperboloids. It is natural to choose two vector fields which are tangent to the hyperboloids, but then it is not obvious how to complete this to a basis with a third vector field. For simplicity, let us restrict ourselves to the region where $\phi \geq 1$. Then two natural vectors which are tangent to the hyperboloids are ∂_ϕ and ∂_θ , both of which have Euclidean length T , and we can choose the first two elements of our new basis to be $T^{-1}\partial_\phi$ and $T^{-1}\partial_\theta$. For the third vector field, we cannot choose ∂_σ because it is too close in direction to ∂_ϕ . Instead, we could choose ∂_r , which we can rewrite in the form

$$2\partial_r = -\frac{1}{t-r}(\partial_\sigma - \partial_\phi) + \frac{1}{t+r}(\partial_\sigma + \partial_\phi). \quad (7.13)$$

Thus, in the C_{TS}^+ regions, we can use the following three vector fields as a substitute for the gradient:

$$\nabla_{t,x} \approx \left\{ \frac{1}{T}\partial_\theta, \frac{1}{T}\partial_\phi, \frac{1}{S}(\partial_\sigma - \partial_\phi) \right\}. \quad (7.14)$$

²At least when $\phi > 1$ which corresponds to $S \ll T$. A small variation of this computation is needed in the interior region C_{TT}^+ .

In the region $\phi < 1$, which corresponds to $S \approx T$, the matters are simpler because we can simply use $T^{-1}\partial_\sigma$ for the third vector field.

From here, we split the analysis in two components: one that deals with the Klein-Gordon pointwise estimates and one that establishes the wave pointwise bounds.

7.2. Pointwise bounds for the Klein-Gordon component inside the cone

We work in a dyadic region C_{TS}^+ , which is foliated by hyperboloids H_σ with

$$e^{2\sigma} \approx ST.$$

We begin by recalling the components of the X^T norms in C_{TS}^+ . From the localized energies, we have

$$\|\mathcal{Z}^\gamma v\|_{L_t^2 L_x^2} + \|\mathcal{Z}^\gamma \mathcal{T}v\|_{L_t^2 L_x^2} \lesssim S^{\frac{1}{2}}, \quad |\gamma| \leq 2h, \quad (7.15)$$

and

$$\|\mathcal{Z}^\gamma \nabla v\|_{L_t^2 L_x^2} \lesssim T^{\frac{1}{2}}, \quad |\gamma| \leq 2h. \quad (7.16)$$

Correspondingly, we have the stronger L^2 bounds on the hyperboloids

$$\|\mathcal{Z}^\gamma v\|_{L^2(H)} + \|\mathcal{Z}^\gamma \mathcal{T}v\|_{L^2(H)} \lesssim 1, \quad |\gamma| \leq 2h, \quad (7.17)$$

and

$$\|\mathcal{Z}^\gamma \nabla v\|_{L^2(H)} \lesssim S^{-\frac{1}{2}} T^{\frac{1}{2}}, \quad |\gamma| \leq 2h. \quad (7.18)$$

However, for $(\square + 1)v$, we have

$$\|\mathcal{Z}^\gamma (\square + 1)v\|_{L_t^2 L_x^2} \lesssim T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.19)$$

The next step is to translate the above estimates in the new coordinates. We will use the subscript h to indicate norms evaluated in the spherical hyperbolic coordinates:

$$\|\mathcal{Z}^\gamma v\|_{L_h^2} + \|\mathcal{Z}^\gamma \mathcal{T}v\|_{L_h^2} \lesssim T^{-1}, \quad |\gamma| \leq 2h, \quad (7.20)$$

and

$$\|\mathcal{Z}^\gamma \nabla v\|_{L_h^2} \lesssim S^{-\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq 2h, \quad (7.21)$$

as well as the L^2 bounds on the hyperboloids

$$\|\mathcal{Z}^\gamma v\|_{L_h^2(H)} + \|\mathcal{Z}^\gamma \mathcal{T}v\|_{L_h^2(H)} \lesssim T^{-1}, \quad |\gamma| \leq 2h, \quad (7.22)$$

and

$$\|\mathcal{Z}^\gamma \nabla v\|_{L_h^2(H)} \lesssim S^{-\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq 2h, \quad (7.23)$$

while for $(\square + 1)v$, we have

$$\|\mathcal{Z}^\gamma e^{2\sigma} (\square + 1)v\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.24)$$

We use (7.9) to rewrite

$$-e^{2\sigma} (\square + 1) = -e^{2\sigma} - \left(\partial_\sigma + \frac{1}{2} \right)^2 + \frac{1}{4} + \partial_\phi^2 + \frac{1}{\sinh^2 \phi} \partial_\theta^2 + \frac{\cosh \phi}{\sinh \phi} \partial_\phi,$$

and observe that the bounds (7.22) and (7.24) imply the bound

$$\left\| \mathcal{Z}^\gamma \left(e^{2\sigma} + \left(\partial_\sigma + \frac{1}{2} \right)^2 - \partial_\phi^2 \right) v \right\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.25)$$

This last bound is strictly speaking not needed here, but we have added for completeness; however, its counterpart in the exterior region $\mathcal{C}^-$ will be essential.

Our goal is to estimate v and its derivatives pointwise in C_{TS}^+ . The advantage of working in hyperbolic coordinates is that C_{TS}^+ is a region of size 1. At this point, we have two choices: (i) to use Sobolev embeddings on the hyperboloids or (ii) to use Sobolev embeddings in the full region. Both strategies work; however,

- (i) is more efficient in terms of derivative counting (choice of h);
- (ii) also applies in the regions C_{TS}^- exterior to the cone.

For these reasons, we will alternate between the two strategies from case to case. In the case of the Klein-Gordon problem, it will be convenient to use strategy (i) inside the cone and strategy (ii) outside.

The hyperboloids have dimension two so the simplest Sobolev pointwise inequality is

$$\|v\|_{L_h^\infty(H)} \lesssim \|Z^{\leq 2} v\|_{L_h^2(H)}.$$

One can write this more efficiently as an interpolation inequality

$$\|v\|_{L_h^\infty(H)} \lesssim \|Z^{\leq 2} v\|_{L_h^2(H)}^{\frac{1}{2}} \|v\|_{L_h^2(H)}^{\frac{1}{2}}. \quad (7.26)$$

For higher derivatives, we need a similar bound for derivatives of v :

Lemma 7.1. *The following interpolation estimate holds in C_{TS}^+ :*

$$\sup_\rho \|\partial^{\leq h} w\|_{L_h^\infty(H)} \lesssim \left(\sup_\rho \|Z^{\leq 2} w\|_{L_h^2(H)} \right)^{\frac{1}{2}} \left(\sup_\rho \|\partial^{\leq 2h} w\|_{L_h^2(H)} \right)^{\frac{1}{2}}. \quad (7.27)$$

Proof. Here, we cannot argue on a single hyperboloid because D also contains transversal derivatives. Expressed in spherical hyperbolic coordinates, using the substitute (7.14) for the space-time gradient, this becomes a standard interpolation inequality. The proof is omitted, as it uses the same coordinates and the same principles as the proof of Lemma 1.1 in the appendix. $\square$

The desired pointwise bounds for v and its derivatives follow directly from this interpolation inequality provided that $h \geq 3$.

7.2.1. Extra gain near the cone.

Our goal here is to show that near the cone, the pointwise bounds for v and its derivatives improve to

$$|\partial^j v| \leq T^{-1-\delta} \langle t-r \rangle^\delta, \quad \delta > 0, \quad j = 0, 1, 2, 3. \quad (7.28)$$

Assuming we have no gain when $j = 4$, by interpolation, it suffices to have a gain in the $j = 0$ bound. By the interpolation estimate (7.26), it suffices to improve the L^2 bound on hyperboloids in (7.22) to

$$\|v\|_{L_h^2(H)} \lesssim S^\delta T^{-1-\delta}. \quad (7.29)$$

As a first starting point, we begin with the similar bound without a gain in (7.22) – namely,

$$\|v\|_{L_h^2(H)} \lesssim T^{-1}. \quad (7.30)$$

A second starting point is obtained from (7.22) with exactly one derivative via the relation (7.13), where $\partial_\sigma v$ is a vector field bound and thus better. This yields

$$\|\partial_\sigma v\|_{L_h^2(H)} \lesssim ST^{-1}. \quad (7.31)$$

Finally, our third starting point comes from (7.25) with $\gamma = 0$. In that case, we can use (7.22) with $Z^\gamma = \partial_\phi^2$ as well as (7.30) and (7.31) to simplify (7.25) to

$$\left\| \left(e^{2\sigma} + \partial_\sigma^2 \right) v \right\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}. \quad (7.32)$$

It remains to show that the bounds (7.30), (7.31) and (7.32) imply (7.29). Here, we recall that the set C_{TS} corresponds to an unit interval in σ .

We will interpret (7.29) as coming from an energy estimate on hyperboloids, which in the new coordinates are the sets $\sigma = \text{const}$. The natural energy associated to the operator in (7.32) is

$$E(v) = |\partial_\sigma v|^2 + e^{2\sigma} v^2.$$

Here, we omit the ϕ and θ integration as it plays no role at all in the proof of the estimate (7.29). Since in C_{TS} we have $e^{2\sigma} \approx ST$, in order to prove (7.29), it suffices to show that

$$E(v) \lesssim S^{1+\delta} T^{-1-\delta}. \quad (7.33)$$

For this, we will use an energy estimate. We compute

$$\frac{d}{d\sigma} E(v) = 2e^{2\sigma} v^2 + 2\partial_\sigma v \left(e^{2\sigma} + \partial_\sigma^2 \right) v = O(E(v)) + 2\partial_\sigma v \left(e^{2\sigma} + \partial_\sigma^2 \right) v.$$

Initializing at some point $\sigma = \sigma_0$, we use (7.31) and (7.32) to estimate the last term on the right as

$$\|\partial_\sigma v \left(e^{2\sigma} + \partial_\sigma^2 \right) v\|_{L^1} \lesssim S^{\frac{3}{2}} T^{-\frac{3}{2}},$$

and then Gronwall's inequality to arrive at

$$\sup_\sigma E(v)(\sigma) \lesssim E(v)(\sigma_0) + S^{\frac{3}{2}} T^{-\frac{3}{2}}.$$

This suffices provided we find a good initialization. By (7.31), any σ_0 is good for the first, kinetic component of the energy, but we have a problem with the second – namely, the potential energy. To address this difficulty, we use the principle that the two components should be comparable in average. To prove this, we use a compactly supported nonnegative cutoff $\chi(\sigma)$ and integrate by parts

$$\int \chi(\sigma) (|\partial_\sigma v|^2 - e^{2\sigma} v^2) d\sigma = - \int \chi(\sigma) v \left(e^{2\sigma} + \partial_\sigma^2 \right) v d\sigma + \int \frac{1}{2} \chi''(\sigma) v^2 d\sigma = O(S^{\frac{1}{2}} T^{-\frac{3}{2}}),$$

where the integrals on the right were estimated using (7.30) and (7.32). This implies that we also have

$$\int \chi(\sigma) e^{2\sigma} v^2 d\sigma \lesssim S^{\frac{3}{2}} T^{-\frac{3}{2}},$$

and in turn allows us to find a good initialization σ_0 so that

$$E(v)(\sigma_0) \lesssim S^{\frac{3}{2}} T^{-\frac{3}{2}}.$$

Then (7.33) follows with $\delta = \frac{1}{2}$, hence showing that (7.29) holds true with $\delta = \frac{1}{4}$ and (7.28) with $\delta = \frac{1}{8}$.

7.3. Pointwise bounds for the wave equation inside the cone

In the analysis for the wave component, we want to obtain pointwise bounds on C_{TS}^+ for ∇u and its derivatives (first and second) and then for $\mathcal{T}u$ (tangential derivatives) as well as its derivatives (first and second).

Applying (7.1), we will control

$$\|\mathcal{Z}^\gamma \mathcal{T}u\|_{L_t^2 L_x^2} \lesssim S^{\frac{1}{2}}, \quad |\gamma| \leq 2h \quad (7.34)$$

as well as the gradient of u

$$\|\mathcal{Z}^\gamma \nabla u\|_{L_t^2 L_x^2} \lesssim T^{\frac{1}{2}}, \quad |\gamma| \leq 2h. \quad (7.35)$$

On hyperboloids, we have

$$\|\mathcal{Z}^\gamma \mathcal{T}u\|_{L^2(H)} \lesssim 1, \quad |\gamma| \leq 2h, \quad (7.36)$$

$$\|\mathcal{Z}^\gamma \nabla u\|_{L^2(H)} \lesssim S^{-\frac{1}{2}} T^{\frac{1}{2}}, \quad |\gamma| \leq 2h. \quad (7.37)$$

Finally, from the wave equation, we get

$$\|\mathcal{Z}^\gamma \square u\|_{L_t^2 L_x^2} \lesssim T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.38)$$

We translate all the estimates above in spherical hyperbolic coordinates

$$\|\mathcal{Z}^\gamma \mathcal{T}u\|_{L_h^2} \lesssim T^{-1}, \quad |\gamma| \leq 2h \quad (7.39)$$

and

$$\|\mathcal{Z}^\gamma \nabla u\|_{L_h^2} \lesssim (ST)^{-\frac{1}{2}}, \quad |\gamma| \leq 2h. \quad (7.40)$$

On the hyperboloids,

$$\|\mathcal{Z}^\gamma \mathcal{T}u\|_{L_h^2(H)} \lesssim T^{-1}, \quad |\gamma| \leq 2h, \quad (7.41)$$

and

$$\|\mathcal{Z}^\gamma \nabla u\|_{L_h^2(H)} \lesssim (ST)^{-\frac{1}{2}}, \quad |\gamma| \leq 2h. \quad (7.42)$$

Finally, for the wave equation, we have

$$\|\mathcal{Z}^\gamma e^{2\sigma} \square u\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.43)$$

Combining this with (7.9), we obtain

$$\|\mathcal{Z}^\gamma (\partial_\sigma - \partial_\phi)(\partial_\sigma + \partial_\phi + 1)u\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}} \quad |\gamma| \leq h. \quad (7.44)$$

At this point, we will show how to use the bounds on hyperboloids to prove the estimates (7.4), (7.5) as well as (7.6) but with $\delta = 0$.

From (7.39) and (7.40), we deduce

$$\|\mathcal{Z}^\gamma \mathcal{Z}u\|_{L_h^2(H)} \lesssim 1, \quad |\gamma| \leq 2h, \quad (7.45)$$

as well as

$$\|\mathcal{Z}^\gamma(\partial_\sigma - \partial_\phi)u\|_{L_h^2(H)} \lesssim S^{\frac{1}{2}}T^{-\frac{1}{2}} \quad |\gamma| \leq 2h. \quad (7.46)$$

The interpolation bound (7.27) yields

$$|\partial^{\leq h} Zu| \lesssim 1,$$

and

$$|\partial^{\leq h}(\partial_\sigma - \partial_\phi)u| \lesssim S^{\frac{1}{2}}T^{-\frac{1}{2}}.$$

Assuming that $h \geq 2$, these last two bounds translated back in terms of regular derivatives give exactly the estimates (7.4), (7.5) as well as (7.6) with $\delta = 0$. In order to obtain the δ gain, we need, however, to return to the L^2 bounds in C_{TS}^+ rather than work on hyperboloids. Since this situation is identical to the one we will encounter in the exterior region C^- , we postpone this proof to Section 7.6.

7.4. Normalized coordinates outside the cone

For the analysis outside the cone, it is very convenient to work with the natural counterpart of the spherical hyperbolic coordinates used in the interior. These are as follows:

$$\begin{cases} t = e^\sigma \sinh(\phi), \\ x_1 = e^\sigma \cosh(\phi) \sin(\theta), \\ x_2 = e^\sigma \cosh(\phi) \cos(\theta), \end{cases} \quad (7.47)$$

where θ and ϕ are the polar coordinates on the single sheeted hyperboloid, ϕ plays the role of the radial coordinate and provides the arch length parametrization of the radial time-like geodesics on the hyperboloid, and θ is the angle from a reference direction. Finally, $e^{2\sigma}$ represents the Minkowski distance to the origin. The Jacobian of the change of variable is

$$J(\sigma, \phi, \theta) = e^{3\sigma} \cosh(\phi).$$

The wave operator in these new variables has the form

$$-\square = e^{-2\sigma} \left(\partial_\sigma^2 - \partial_\phi^2 + \frac{1}{\cosh^2(\phi)} \partial_\theta^2 - \partial_\sigma + \frac{\sinh(\phi)}{\cosh(\phi)} \partial_\phi \right). \quad (7.48)$$

In these coordinates, the regions C_{TS}^- become essentially rectangular regions of size 1. Precisely, in spherical hyperbolic coordinates, the regions C_{TS}^- are represented as follows:

$$C_{TS}^- \longrightarrow D := \{(\sigma, \phi, \theta) : (\sigma, \phi, \theta) \in I_\sigma \times I_\phi \times [0, 2\pi]\},$$

where I_σ and I_ϕ are intervals of size 1. As discussed earlier, we are assuming we are at distance at least one from the cone, which corresponds to $S \geq 1$. Here, we work under the assumption that $S \leq T$, which selects a conical neighborhood of the cone, as the analysis in the outer part is much simpler (see Proposition 2.1). Therefore, in such region C_{TS}^- , we have

$$e^{2\sigma} = |x|^2 - t^2 \approx ST, \quad e^\sigma \cosh(\phi) \approx T, \quad J \approx ST^2. \quad (7.49)$$

7.5. Pointwise bounds for the Klein-Gordon component outside the cone

Here, we prove the pointwise bounds for the Klein-Gordon equation in the exterior region. We harmlessly assume that $S \leq T$, which is the interesting region near the cone. Arguing in the same way as for the

interior region, we move the bounds (7.1), (7.2) in the hypothesis of the theorem to the spherical hyperbolic coordinates; here, there are no hyperboloid bounds. We get

$$\|\mathcal{Z}^\gamma v\|_{L_h^2} + \|\mathcal{Z}^\gamma \mathcal{T}v\|_{L_h^2} \lesssim T^{-1}, \quad |\gamma| \leq 2h, \quad (7.50)$$

and

$$\|\mathcal{Z}^\gamma \nabla v\|_{L_h^2} \lesssim S^{-\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq 2h, \quad (7.51)$$

while for $(\square + 1)v$, we have

$$\|\mathcal{Z}^\gamma e^{2\sigma}(\square + 1)v\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.52)$$

We use (7.48) to rewrite

$$-e^{2\sigma}(\square + 1) = -e^{2\sigma} + \left(\partial_\sigma - \frac{1}{2}\right)^2 - \frac{1}{4} - \partial_\phi^2 + \frac{1}{\cosh^2 \phi} \partial_\theta^2 + \frac{\sinh \phi}{\cosh \phi} \partial_\phi,$$

and observe that the bounds (7.51) and (7.52) give

$$\left\| \mathcal{Z}^\gamma \left(e^{2\sigma} - \left(\partial_\sigma - \frac{1}{2} \right)^2 + \partial_\phi^2 \right) v \right\|_{L_h^2} \lesssim S^{\frac{1}{2}} T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.53)$$

Our goal is to estimate v and its derivatives pointwise in C_{TS}^- . There we can set (see formula (7.14) which still applies)

$$Z = \{\partial_\theta, \partial_\phi\}, \quad \mathcal{T} = T^{-1}Z, \quad \nabla_{t,x} = \{T^{-1}Z, S^{-1}(\partial_\phi - \partial_\sigma)\}, \quad (7.54)$$

so that everything is constant coefficients in θ and ϕ within C_{TS}^- . At this point, we can harmlessly localize on the unit scale in ϕ , then freeze the constants and finally forget about the ϕ localization and assume that ϕ is either on $\mathbb{R}$ or on the circle.

We localize to a frequency λ in θ , ϕ . Based on the symbol of the operator in (7.53), the interesting threshold for λ is $\sqrt{ST}$. For smaller λ , the ∂_ϕ^2 component is controlled by $e^{2\sigma}$ and thus perturbative in (7.53), which can then be treated as an elliptic bound.

Based on this, we distinguish two cases:

(i) Large λ – namely, $\lambda \gtrsim \sqrt{ST}$. Then we disregard (7.53) and work only with (7.50) and (7.51). There we have a second interesting frequency – namely, the one for $\partial_\sigma - \partial_\phi$. We localize this dyadically to the frequency μ . Then from (7.54),

$$\nabla_{t,x} \approx T^{-1}\lambda + S^{-1}\mu.$$

Hence, from (7.50) and (7.51), we have the following L^2 bound for the corresponding component $v_{\lambda\mu}$ of v :

$$\|v_{\lambda\mu}\|_{L^2} \lesssim T^{-1} \frac{1}{\left[\lambda^2 + (T^{-1}\lambda + S^{-1}\mu)^{2h}\right] \left[1 + (T^{-1}\lambda + S^{-1}\mu)S^{\frac{1}{2}}T^{-\frac{1}{2}}\right]}.$$

Applying Bernstein, we arrive at the L^∞ bound

$$\|\nabla^j v_{\lambda\mu}\|_{L^\infty} \lesssim T^{-1} \frac{\lambda\mu^{\frac{1}{2}}(T^{-1}\lambda + S^{-1}\mu)^j}{\left[\lambda^2 + (T^{-1}\lambda + S^{-1}\mu)^{2h}\right] \left[1 + (T^{-1}\lambda + S^{-1}\mu)S^{\frac{1}{2}}T^{-\frac{1}{2}}\right]}.$$

It remains to maximize the right-hand side with respect to λ and μ subject to the constraint $\lambda \geq \sqrt{ST}$.

On μ , we have no constraint, and the maximum is attained when its contribution in the first factor in the denominator balances that of λ . Depending on which λ term gets balanced, we have two scenarios:

(a) $(S^{-1}\mu)^{2h} = \lambda^2 \gtrsim (T^{-1}\lambda)^{2h}$. Then the above expression becomes

$$T^{-1}S^{\frac{1}{2}}\lambda^{-1+\frac{j}{h}+\frac{1}{2h}}\left(1+\lambda^{\frac{1}{h}}S^{\frac{1}{2}}T^{-\frac{1}{2}}\right)^{-1},$$

which is decreasing in λ and hence maximized at $\lambda = \sqrt{ST}$. We get a maximum of

$$T^{-1}S^{\frac{1}{2}}(ST)^{\frac{1}{2}(-1+\frac{j}{h}+\frac{1}{2h})}\left(1+(ST)^{\frac{1}{2h}}S^{\frac{1}{2}}T^{-\frac{1}{2}}\right)^{-1} \lesssim T^{-1}S^{\frac{1}{2}}(ST)^{\frac{1}{2}(-1+\frac{j}{h}+\frac{1}{2h})}.$$

This is favorable (i.e., $\leq T^{-1-\delta}$) if $h > 2j + 1$. Since we need $j \leq 3$, we should have $h \geq 8$.

(b) $(S^{-1}\mu)^{2h} = (T^{-1}\lambda)^{2h} \gtrsim \lambda^2$. We get again a negative power of λ which is maximized if λ is as small as possible within these constraints. But this cannot happen at $\lambda = \sqrt{ST}$, as this is inconsistent with the above constraint. Therefore, it happens when the two cases (b) and (a) are in balance. So the maximum is always attained in case (a).

(ii) Small λ – namely, $\lambda \ll \sqrt{ST}$. The contribution of ∂_ϕ^2 in (7.53) can be estimated by

$$\|\mathcal{Z}^\gamma \partial_\phi^2 v\|_{L_h^2} \ll ST \|\mathcal{Z}^\gamma v\|_{L_h^2}, \quad |\gamma| \leq h,$$

while the remaining operator $e^{2\sigma} - \partial_\sigma^2$ is elliptic, so we have

$$ST \|\mathcal{Z}^\gamma v\|_{L_h^2} + \|\partial_\sigma^2 \mathcal{Z}^\gamma v\|_{L_h^2} \lesssim \|(e^{2\sigma} - \partial_\sigma^2)v\|_{L_h^2}, \quad |\gamma| \leq h.$$

Combining these two estimates with (7.53), we obtain the elliptic bound

$$ST \|\mathcal{Z}^\gamma v\|_{L_h^2} + \|\partial_\sigma^2 \mathcal{Z}^\gamma v\|_{L_h^2} \lesssim S^{\frac{1}{2}}T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.55)$$

We keep the same μ notation but introduce a third dyadic scale γ for the σ frequency.

Combining the L^2 bounds (7.50) and (7.55), as above, we get the L^∞ bound

$$\|\nabla^j v_{\lambda\mu\gamma}\|_{L^\infty} \lesssim T^{-1} \frac{\lambda \min\{\mu, \gamma\}^{\frac{1}{2}} (S^{-1}\mu + T^{-1}\lambda)^j}{(\lambda^2 + (S^{-1}\mu + T^{-1}\lambda)^{2h}) + (ST)^{-\frac{1}{2}}(ST + \gamma^2)(\lambda + (S^{-1}\mu + T^{-1}\lambda)^h)}.$$

Now we harmlessly replace $(S^{-1}\mu + T^{-1}\lambda)^j$ by $\lambda^{\frac{j}{h}} + (S^{-1}\mu + T^{-1}\lambda)^j$ at the numerator. Then we can drop the $T^{-1}\lambda$ term to get

$$\|\nabla^j v_{\lambda\mu\gamma}\|_{L^\infty} \lesssim T^{-1} \frac{\lambda \min\{\mu, \gamma\}^{\frac{1}{2}} (\lambda^{\frac{j}{h}} + (S^{-1}\mu)^j)}{(\lambda^2 + (S^{-1}\mu)^{2h}) + (ST)^{-\frac{1}{2}}(ST + \gamma^2)(\lambda + (S^{-1}\mu)^h)}.$$

We need to maximize the right-hand side. The μ maximum is attained when $\lambda = (S^{-1}\mu)^h$, in which case the above expression simplifies to

$$T^{-1} \frac{\min\{\mu, \gamma\}^{\frac{1}{2}} \lambda^{\frac{j}{h}}}{\lambda + (ST)^{-\frac{1}{2}}(ST + \gamma^2)}.$$

Since $\lambda < \sqrt{ST}$, we can drop it from the first term from the denominator and then maximize it at the numerator. Finally, the γ maximum is attained if $\gamma = \sqrt{ST}$. We get a maximum of

$$T^{-1} \frac{\min\{S^{\frac{1}{2}}(ST)^{\frac{1}{4h}}, (ST)^{\frac{1}{4}}\} (ST)^{\frac{j}{2h}}}{(ST)^{\frac{1}{2}}}.$$

Using the second term in the min, this is favorable if $2j < h$ (i.e., the same as in case (i)).

Remark 7.2. Here, we gain a better bound of $T^{-1-\delta}$ which shows that outside the cone, we have better decay for the Klein-Gordon component.

7.6. Pointwise bounds for the wave component outside the cone

Here, we prove the pointwise bounds for the wave equation in the exterior region. As before, we harmlessly assume that $S \leq T$. Arguing in the same way as for the interior region, we move the bounds (7.1), (7.2) in the hypothesis of the theorem to the spherical hyperbolic coordinates; here, there are no hyperboloid bounds. In each region C_{TS}^- , we get

$$\|\mathcal{Z}^\gamma Zu\|_{L_h^2} \lesssim 1, \quad |\gamma| \leq 2h, \quad (7.56)$$

and

$$\|\mathcal{Z}^\gamma (\partial_\sigma - \partial_\phi)u\|_{L_h^2} \lesssim S^{\frac{1}{2}}T^{-\frac{1}{2}}, \quad |\gamma| \leq 2h, \quad (7.57)$$

as well as the bounds

$$\|\mathcal{Z}^\gamma (\partial_\sigma - \partial_\phi)(\partial_\sigma + \partial_\phi + 1)u\|_{L_h^2} \lesssim S^{\frac{1}{2}}T^{-\frac{1}{2}}, \quad |\gamma| \leq h. \quad (7.58)$$

All the analysis below applies equally not only to C_{TS}^- but also to C_{TS}^+ , providing an alternative approach in the latter case. For this reason, we drop the $\pm$ superscripts below. We are allowed to freely localize in ϕ on the unit scale in all of the C_{TS} , as well as localize on the unit scale in σ if $S \approx T$. For the purpose of the proofs below, we assume these localizations.

Relative to the variable σ , we are not allowed to localize directly on the unit scale in the full set of inequalities (7.56), (7.57) and (7.58). However, we can finesse this minor difficulty if we agree to use only the pair of bounds (7.56), (7.57)³ to prove (7.3) and (7.4), and the pair of bounds (7.57) and (7.58) to prove (7.6) and (7.5) (modulo (7.3), can be recast as bounds for $(\partial_\sigma - \partial_\phi)u$). Then in the first case, we can localize the function u on the unit scale in σ , whereas in the second case, we can instead localize the function $(\partial_\sigma - \partial_\phi)u$ on the unit scale in σ .

Our main tool will be the following Sobolev pointwise inequality:

Lemma 7.3. *For functions w compactly supported in C_{TS} , we have the following pointwise bound (interpolation inequality):*

$$\|\partial^{\leq h/2} w\|_{L^\infty(C_{TS})} \lesssim \|\mathcal{Z}^{\leq 2h} w\|_{L_h^2(C_{TS})} + \|\mathcal{Z}^{\leq h} (\partial_\sigma \pm \partial_\phi)w\|_{L_h^2(C_{TS})}. \quad (7.59)$$

As in previously discussed interpolation inequalities, in hyperbolic coordinates, this can be viewed as a standard constant coefficient bound which is obtained from Bernstein type inequalities. For instance, if $h = 0$, the inequality becomes

$$\|w\|_{L_h^\infty} \lesssim \|\partial_{\theta,\phi}^{\leq 2} w\|_{L_h^2} + \|\partial_{\theta,\phi}^{\leq 1} (\partial_\sigma \pm \partial_\phi)w\|_{L_h^2}.$$

Denoting by λ the $\partial_{\phi,\theta}$ frequency and by μ the $\partial_\sigma \pm \partial_\phi$ frequency, by Bernstein's inequality, the above bound reduces to

$$\lambda\mu^{\frac{1}{2}} \lesssim (1 + \lambda^2) + \mu(1 + \lambda),$$

which is straightforward, also with room for dyadic summation. The case $h > 0$ is similar but with more cases, interpreting regular derivatives as $\partial_{x,t} \approx (T^{-1}\partial_{\theta,\phi}, S^{-1}(\partial_\sigma \pm \partial_\phi))$.

We will use these Sobolev embeddings to estimate in L^∞ the following two functions – namely, Zu (Z is either ∂_θ or ∂_ϕ or any combinations of them) and $(\partial_\sigma - \partial_\phi)u$ – as well as their derivatives.

³with a weaker bound of 1 in the second case

To estimate Zu , w will be replaced with Zu . Then we can use the estimates (7.56) and (7.57) to bound the ‘-’ version of the right-hand side in (7.59) by 1. This implies that in C_{TS} , we have the pointwise bound for Zu

$$\|\partial^{\leq h/2} Zu\|_{L^\infty(C_{TS})} \lesssim 1.$$

For $(\partial_\sigma - \partial_\phi)u$, we replace w with $(\partial_\sigma - \partial_\phi)u$. Then we can use the estimates (7.57) and (7.58) to bound the ‘+’ version of the right-hand side in (7.59) by $S^{\frac{1}{2}}T^{-\frac{1}{2}}$.

This implies that in C_{TS} , we have the pointwise bound for $(\partial_\sigma - \partial_\phi)u$

$$\|\partial^{\leq h/2}(\partial_\sigma - \partial_\phi)u\|_{L^\infty(C_{TS})} \lesssim S^{\frac{1}{2}}T^{-\frac{1}{2}}.$$

At this point, we can rephrase the last two bounds as bounds for ∇u and $\mathcal{T}u$:

$$\begin{aligned}\|\partial^{\leq h/2}\mathcal{T}u\|_{L^\infty(C_{TS})} &\lesssim T^{-1}, \\ \|\partial^{\leq h/2}\nabla u\|_{L^\infty(C_{TS})} &\lesssim S^{-\frac{1}{2}}T^{-\frac{1}{2}}.\end{aligned}$$

This suffices for the bounds (7.4), (7.5) and (7.6) with $\delta = 0$ in our theorem provided that $h/2 > 2$ (i.e., $h \geq 5$).

Extra gain away from the cone. The remaining step in the proof of the theorem is to obtain the δ improvement in the bound (7.6), which is needed only for derivatives of u of second and third order. As a byproduct, we will also obtain a similar improvement in (7.4). Precisely, we will prove that in C_{TS} , we have

$$|\partial^j \nabla u| \lesssim T^{-\frac{1}{2}}S^{-\frac{1}{2}-\delta}, \quad j = 1, 2, \quad (7.60)$$

respectively

$$|\partial^j \mathcal{T}u| \lesssim T^{-1}S^{-\delta} \quad j = 1, 2. \quad (7.61)$$

For this, we need an improvement in (7.59) when on the left we put $\partial^j w$ with $0 < j < h/2$ – namely, with some $\delta > 0$,

$$\|\partial^j w\|_{L^\infty(C_{TS})} \lesssim S^{-\delta} \left(\|\mathcal{Z}^{\leq 2h} w\|_{L_h^2(C_{TS})} + \|\mathcal{Z}^{\leq h}(\partial_\sigma \pm \partial_\phi)w\|_{L_h^2(C_{TS})} \right). \quad (7.62)$$

To prove (7.62), we separate into two cases:

Case I. We consider first the slightly simpler case of the – sign in (7.59), which corresponds to (7.60). To obtain a better than $T^{-\frac{1}{2}}S^{-\frac{1}{2}}$ bound for $\partial^j \nabla u$ in (7.60) or equivalently a better than 1 bound for $\partial^j w$ in (7.59), we consider the balance of frequencies there. Taking a Littlewood-Paley decomposition, denote by λ the Z frequency and by μ the $\partial_\sigma - \partial_\phi$ frequency. We can take both $\lambda, \mu \geq 1$ since we work in a unit size region; all frequencies below 1 can be combined in the frequency 1 case. As before, for the gradient, we can think of the vector fields

$$\nabla = \{T^{-1}Z, S^{-1}(\partial_\sigma - \partial_\phi)\}, \quad \mathcal{T} = T^{-1}Z.$$

The L^2 bound for $w_{\lambda\mu}$ given by the right-hand side of (7.59) is

$$\|w_{\lambda\mu}\|_{L^2} \lesssim \left(\lambda^2 + (T^{-1}\lambda + S^{-1}\mu)^{2h} + \mu(\lambda + (T^{-1}\lambda + S^{-1}\mu)^h) \right)^{-1}.$$

To estimate $\partial^j w_{\lambda\mu}$ in L^∞ , we use Bernstein's inequality,

$$\|\partial^j w_{\lambda\mu}\|_{L^\infty} \lesssim \frac{\lambda\mu^{\frac{1}{2}}(T^{-1}\lambda + S^{-1}\mu)^j}{\lambda^2 + (T^{-1}\lambda + S^{-1}\mu)^{2h} + \mu(\lambda + (T^{-1}\lambda + S^{-1}\mu)^h)}.$$

Now we maximize over λ and μ on the right. One also needs to sum with respect to λ and μ , but this is straightforward as there is dyadic exponential decay away from the maximum. We first consider homogeneous variations in λ, μ where we keep the ratio fixed but vary the size. The maximum will be attained exactly when⁴

$$\lambda = (T^{-1}\lambda + S^{-1}\mu)^h. \quad (7.63)$$

Below that, we have a positive power of the size parameter, and above that, a negative one. Here, it is important that $0 < j < 2h - \frac{3}{2}$. Otherwise, if $j = 0$, the power in the denominator always dominates and the maximum is exactly at $\lambda = \mu = 1$. If j is too large, then the above expression is unbounded. The above bound from above is not too important, as it gets tighter later on.

Assuming (7.63), the expression above simplifies to

$$\frac{\lambda^{\frac{j}{h}}\mu^{\frac{1}{2}}}{\lambda + \mu}.$$

We distinguish two cases:

(i) Large λ ,

$$\lambda = (T^{-1}\lambda)^h, \quad T^{-1}\lambda \geq S^{-1}\mu.$$

Here, we have $\lambda > \mu$ so the denominator becomes λ . Then the μ in the numerator must be maximal. This leads to

$$\lambda = T^{1+\frac{1}{h-1}}, \quad \mu = \frac{S\lambda}{T},$$

and the above expression becomes

$$[S^{\frac{1}{2}}T^{-\frac{1}{2}}\lambda^{\frac{j}{h}-\frac{1}{2}}].$$

which gains a power of T provided that $2j < h$.

(ii) Large μ ,

$$\lambda = (S^{-1}\mu)^h, \quad T^{-1}\lambda \leq S^{-1}\mu.$$

We substitute this expression for λ to get

$$\frac{S^{-j}\mu^{j+\frac{1}{2}}}{(S^{-1}\mu)^h + \mu}.$$

This balances when the two terms in the denominator are equal. Then

$$\mu = S^{1+\frac{1}{h-1}},$$

and we get

$$S^{-j+(j-\frac{1}{2})(1+\frac{1}{h-1})} = S^{(\frac{j}{h}-\frac{1}{2})(1+\frac{1}{h-1})},$$

which gains a power of S provided again that $2j < h$.

⁴Strictly speaking, one should also separately consider the cases when $\mu = 1$ or $\lambda = 1$. These are simpler and are omitted.

Case II. Here, we consider the case of the + sign in (7.59), which corresponds to (7.60). Here, we have three relevant dyadic frequencies, denoted by λ for Z , μ for $\partial_\sigma - \partial_\phi$ and ν for $\partial_\sigma + \partial_\phi$. Bernstein's inequality now yields the bound

$$\|\partial^j w_{\lambda\mu}\|_{L^\infty} \lesssim \frac{\lambda \min\{\mu, \nu\}^{\frac{1}{2}} (T^{-1}\lambda + S^{-1}\mu)^j}{\lambda^2 + (T^{-1}\lambda + S^{-1}\mu)^{2h} + \nu(\lambda + (T^{-1}\lambda + S^{-1}\mu)^h)}.$$

Here, we need to maximize the right-hand side above with respect to the three parameters $\lambda, \mu, \nu \geq 1$.

As before, after excluding the cases $\lambda = 1$ and $\mu = 1$, one sees that at the maximum point, we must have the relation (7.63) in which case the expression above simplifies to

$$\frac{\lambda^{\frac{j}{h}} \min\{\mu, \nu\}^{\frac{1}{2}}}{\lambda + \nu}.$$

The ν maximum is at $\nu = \lambda$, so we are left with

$$\frac{\lambda^{\frac{j}{h}} \min\{\mu, \lambda\}^{\frac{1}{2}}}{\lambda}.$$

We consider the same two cases (i) and (ii) as in Case I. Part (i) is identical, whereas in part (ii), we get

$$\frac{S^{-j} \mu^j \min\{\mu, (S^{-1}\mu)^h\}^{\frac{1}{2}}}{(S^{-1}\mu)^h}.$$

The μ maximum is attained when the two terms in the min are equal, which again gives the same outcome as in Case I (ii).

Appendix A. An interpolation lemma

Here, we prove the following interpolation Lemma:

Lemma 1.1. *Assume that $n \geq 0$ and*

$$\frac{2}{p} = \frac{1}{2} + \frac{1}{q}, \quad 2 \leq q \leq \infty.$$

Then we have

$$\|\partial^{n+1} Z\phi\|_{L^p(C_{TS})} \lesssim \|Z^{\leq 2}\phi\|_{L^2(C_{TS})}^{\frac{1}{2}} (\|\partial^{\leq 2n}\partial^2\phi\|_{L^q(C_{TS})} + S^{-1}\|\partial\phi\|_{L^q(C_{TS})})^{\frac{1}{2}}. \quad (\text{A.1})$$

The same holds in C_T^{int} .

Proof. The case of C_T^{int} is similar and is omitted. We prove the result in several modular steps:

Step 1: Reduction to the case of a cube. Here, we use hyperbolic polar coordinates adapted to C_{TS} to view C_{TS} as a unit cube Q , which in turn we can view as a product $Q = Q_1 \times Q_2$, with coordinates denoted by (s, y) .

The differentiation operators in the unit cube, translated to the C_{TS} setting, are

$$(\partial_s, \partial_y) \approx (S\partial_r, Z).$$

Then we can represent the differentiation operators in the Minkowski space as

$$(\partial_t, \partial_x) \approx (S^{-1}\partial_s, T^{-1}\partial_y),$$

where we have the slight difficulty that the connection between the two bases has variable coefficients, which are smooth on the T scale. Hence, when we represent $\partial_{x,t}^j$ in the basis on the right, we also get lower-order terms,

$$|\partial^j \phi| \lesssim \sum_{l=1}^j T^{l-j} |(S^{-1} \partial_s, T^{-1} \partial_y)^l \phi|.$$

Hence, we can estimate the left-hand side in (A.1) by

$$\|\partial^{n+1} Z \phi\|_{L^p(C_{TS})} \lesssim \sum_{j=0}^{n+1} T^{j-n-1} \|(S^{-1} \partial_s, T^{-1} \partial_y)^j \partial_y \phi\|_{L^p(C_{TS})}.$$

However, for the terms on the right, we have

$$\|Z^{\leq 2} \phi\|_{L^2(C_{TS})} \approx \|\partial_y^{\leq 2} \phi\|_{L^2(C_{TS})},$$

respectively

$$\begin{aligned} & \|\partial^{\leq 2n} \partial^2 \phi\|_{L^q(C_{TS})} + S^{-1} \|\partial \phi\|_{L^q(C_{TS})} \\ & \approx \|(S^{-1} \partial_s, T^{-1} \partial_y)^{\leq 2n} (S^{-1} \partial_s, T^{-1} \partial_y)^2 \phi\|_{L^q(C_{TS})} + S^{-1} \|(S^{-1} \partial_s, T^{-1} \partial_y) \phi\|_{L^q(C_{TS})}. \end{aligned}$$

Changing also the measure of integration, the bound (A.1) is now reduced to

$$\begin{aligned} \sum_{j=0}^{n+1} T^{2(j-n-1)} \|(S^{-1} \partial_s, T^{-1} \partial_y)^j \partial_y \phi\|_{L^p(Q)}^2 & \lesssim \|\partial_y^{\leq 2} \phi\|_{L^2(Q)} (\|(S^{-1} \partial_s, T^{-1} \partial_y)^{\leq 2n} (S^{-1} \partial_s, T^{-1} \partial_y)^2 \phi\|_{L^q(Q)} \\ & + S^{-1} \|(S^{-1} \partial_s, T^{-1} \partial_y) \phi\|_{L^q(Q)}). \end{aligned}$$

Here, we are in a fixed unit size cube, so S and T simply play the role of large parameters with the only constraint $1 \leq S \leq T$. We need to estimate all components of the norm on the left. By interpolating solely in y , it suffices to consider on the left only the cases when either $j = n + 1$ or when there are no y derivatives in the j -th power. In the latter situation, the case $j = 0$ is trivial, while if $j \geq 1$, we may redefine n to a lower value $n := j - 1$ and thus assume that $j = n + 1$. Then we discard all $T^{-1} \partial_y$ on the right, arriving at

$$\|\partial_s^{n+1} \partial_y \phi\|_{L^p(Q)}^2 \lesssim \|\partial_y^{\leq 2} \phi\|_{L^2(Q)} (\|\partial_s^{\leq 2n} \partial_s^2 \phi\|_{L^q(Q)} + \|\partial_s \phi\|_{L^q(Q)}). \quad (\text{A.2})$$

It remains to consider the case $j = n + 1$ with at least one $T^{-1} \partial_y$ factor on the left. Then we can rewrite this as a bound for $\psi = \partial_y^2 \phi$ – namely,

$$\|(S^{-1} \partial_s, T^{-1} \partial_y)^n \psi\|_{L^p}^2 \lesssim \|\psi\|_{L^2} \|(S^{-1} \partial_s, T^{-1} \partial_y)^{\leq 2n} \psi\|_{L^q(Q)}.$$

But after rescaling back to the original size, this is just the classical Gagliardo-Nirenberg inequality in a cube of size $S \times T \times T$. It remains to prove (A.2) in a unit sized cube.

Step 2: Reduction to a bound in $\mathbb{R}^3$. Here, we harmlessly subtract the s average of ϕ from ϕ . By Poincaré's inequality, this allows us to reduce to the case when ϕ has zero average in s , where the $L^q(Q)$ norm of ϕ is also under control. Thus, (A.2) reduces to

$$\|\partial_s^{n+1} \partial_y \phi\|_{L^p(Q)}^2 \lesssim \|\partial_y^{\leq 2} \phi\|_{L^2(Q)} \|\partial_s^{\leq 2n+2} \phi\|_{L^q(Q)}. \quad (\text{A.3})$$

Having inhomogeneous norms on the right allows us to extend ϕ to a double cube and then truncate, thus reducing to proving (A.3) in all of $\mathbb{R}^3$, for a compactly supported function ϕ .

Step 3: Interpolation in $\mathbb{R}^3$. Here, we use Stein's interpolation theorem for the holomorphic family of operators

$$T_z \phi = e^{(z-\frac{1}{2})^2} |D_y|^{2(1-z)} |D_s|^{(2n+2)z} \phi$$

for z in the strip

$$S = \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}.$$

For this family, we have the interpolation inequality

$$\|\partial_s^{n+1} \partial_y \phi\|_{L^p}^2 \lesssim \| |D_s|^{n+1} |D_y| \phi \|_{L^p}^2 = \|T_{\frac{1}{2}} \phi\|_{L^p}^2 \lesssim \sup_{\Re z=0} \|T_z \phi\|_{L^2} \sup_{\Re z=1} \|T_z \phi\|_{L^q}, \quad (\text{A.4})$$

where for the first step, we use that the Hilbert and Riesz transforms are bounded from $L^p \rightarrow L^p$, with $1 < p < \infty$. Hence, it suffices to show that

$$\sup_{\Re z=0} \|T_z \phi\|_{L^2} \lesssim \|\partial_y^2 \phi\|_{L^2}, \quad (\text{A.5})$$

respectively

$$\sup_{\Re z=1} \|T_z \phi\|_{L^q} \lesssim \|\partial_s^{2n+2} \phi\|_{L^q}. \quad (\text{A.6})$$

The first bound is straightforward by Plancherel's theorem. For the second bound, we need to verify that

$$|D_s|^{i\theta} |D_y|^{i\sigma} : L^q \rightarrow L^q,$$

with sub-Gaussian norm growth in θ and σ at $\pm\infty$.

If $1 < q < \infty$, then we are in a special case of the Hormander-Mikhlin theorem applied separately with respect to the two variables.

It remains to consider the special case $q = \infty$, where we show instead that

$$|D_s|^{i\theta} |D_y|^{i\sigma} : L^\infty \rightarrow BMO.$$

This is true separately for each factor, but not immediately true for the product. To address this difficulty, we use a 0-homogeneous cutoff function χ with smooth symbol $\chi(\xi, \eta)$, where ξ and η are the Fourier variables for s , respectively y . This is chosen so that $\chi = 1$ near $\xi = 0$, respectively $\chi = 0$ near $\eta = 0$. We separate the above product into two parts,

$$|D_s|^{i\theta} |D_y|^{i\sigma} = (\chi(D) |D_y|^{i\sigma}) |D_s|^{i\theta} + ((1 - \chi(D)) |D_s|^{i\theta}) |D_y|^{i\sigma}.$$

These are similar, so we estimate the first one. Here,

$$|D_s|^{i\theta} : L_{s,y}^\infty \rightarrow L_y^\infty BMO_s \subset BMO_{s,y},$$

while $\chi(D) |D_y|^{i\sigma}$ is a Hormander-Mikhlin multiplier so it maps BMO into BMO. $\square$

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