Canad. Math. Bull. Vol. **22** (3), 1979

EXTENSION OF A RESULT OF S. MANDELBROJT

BY

T. PHAM-GIA(*)

ABSTRACT. We extend the following result to several variables: For any sequence $\{N_j\}$, we have $C\{N_j\}=C\{M_j\}$, with $\{M_j\}$ logarithmically convex i.e. $M_j^2 \le M_{j-1}M_{j+1}$ $j=1,2,\ldots$.

Let $\{N_j\}_{j=0}^{\infty}$ be a sequence of positive numbers and $C\{N_j\}$ the class of complex-valued infinitely differentiable functions on R, verifying $\|D^n f\| \le \alpha_f \beta_f^n N_n$, $n=0,1,2,\ldots$ where $D^n f = d^n f/dx^n$, $\|f\| = \sup_{x\in R}|f(x)|$ and α_f and β_f are positive constants depending only on f.

It is known that $C\{N_j\}=C\{\bar{N}_j\}$ where $\{\bar{N}_j\}$ is the largest logarithmically convex minorant of $\{N_j\}$ ($\{\log \bar{N}_j\}$ is then convex or $\bar{N}_j^2 \le \bar{N}_{j-1}\bar{N}_{j+1}$, $j=1,2,\ldots$, see e.g. [3]).

We wish to extend the above result to several variables.

Let $m \ge 1$ be a positive integer, $(j)=(j_1, j_2, \ldots, j_m)$, $0 \le j_k < \infty$, $1 \le k \le m$, be a multi-index and $C\{N_{(j)}\}$ be the class of complex-valued functions in $C^\infty(R^m)$ s.t.

$$\|D^{(j)}f\| \le \alpha_f \beta_f^{\|(j)\|} N_{(j)} \quad \text{where} \quad \|(j)\| = \sum_{k=1}^{m} j_k,$$

$$D^{(j)}f = \frac{\partial^{\|(j)\|} f}{\partial^{j_1}x_1 \cdots \partial^{j_m}x_m}, \quad \|f\| = \sup_{x\in R^m} |f(x)| \quad \text{and} \quad \alpha_f, \beta_f > 0$$

depend only on f.

We define $N_{(j)}$ to be log-convex if it is so componentwise i.e.

$$\forall (j), \forall k \ 1 \le k \le m, \ N^2_{j_1,\ldots,j_k,\ldots,j_m} \le N_{j_1,\ldots,j_k-1,\ldots,j_m} \cdot N_{j_1,\ldots,j_k+1,\ldots,j_m}$$

To a multisequence $N_{(j)}$, we associate the m marginal sequences $\{N_{1,\ell}\}_{\ell=0}^{\infty} = \{N_{\ell,0,\ldots,0}\}_{\ell=0}^{\infty}, \ldots, \{N_{m,\ell}\}_{\ell=0}^{\infty} = \{N_{0,0,\ldots,\ell}\}_{\ell=0}^{\infty}$ and the product marginal multisequence

$$N^*_{(j)} = N_{1,j_1} N_{2,j_2} \cdots N_{m,j_m} \ \forall (j).$$

Received by the editors February 6, 1978 and, in revised form, June 7, 1978.

(*) The author wishes to thank the referee for helpful comments and for remarks on the same problem on a bounded interval.

We prove the following

THEOREM. *Let $N_{(j)}$ be a multisequence s.t. $N_{(j)} \geq N^*_{(j)}$, $\forall(j)$. Then $C\{N_{(j)}\}$ contains an algebra $C\{M_{(j)}\}$ with $\{M_{(j)}\}$ log-convex. Moreover $C\{N_{(j)}\} = C\{M_{(j)}\}$ if $N_{(j)} = N^*_{(j)}, \forall(j)$.*

Proof. As for the one-dimensional case, the proof relies on the Kolmogoroff-Gorny inequality on successive derivatives. For $m=1$, this inequality can be written as:

$$(1) \quad \|D^n f\| \leq 2\,\|D^p f\|^{(r-n)/(r-p)}\,\|D^r p\|^{(n-p)/(r-p)}$$

$$\text{for} \quad f \in C^r(R) \quad \text{and} \quad 0 \leq p \leq n < r.$$

(This is rather a simplified form of the inequality, where 2 has replaced a constant $t(p, n, r)$ with $1 \leq t \leq 2$ (see e.g. [1] p. 216).)

We need to extend this inequality to several variables first.

Let (i) and (j) be multi-indices. We write $(j) \leq (i)$ if $j_k \leq i_k$, $1 \leq k \leq m$. By $(i)-(j)$ we mean the multi-index $(i_1 - j_1, i_2 - j_2, \ldots, i_m - j_m)$ and by $|(i)-(j)|$ the product $\prod_{k=1}^m |i_k - j_k|$, $\forall(i), (j)$.

Let (p), (n) and (r) be s.t. $(0) \leq (p) \leq (n) < (r)$. We associate to these multi-indices 2^m couples $\{(\xi)_\ell, (\theta)_\ell\}$, $\ell = 1, 2, \ldots, 2^m$, defined as follows: $(\xi)_\ell = (\xi_{\ell,1}, \xi_{\ell,2}, \ldots, \xi_{\ell,m})$, $(\theta) = (\theta_{\ell,1}, \theta_{\ell,2}, \ldots, \theta_{\ell,m})$.

$\xi_{\ell,k}$ and $\theta_{\ell,k}$ being either p_k or r_k, $1 \leq k \leq m$ but $\xi_{\ell,k} \neq \theta_{\ell,k}$. (There are obviously 2^m such couples).

If $C^{(r)}(R^m)$ denotes the class of functions f defined on R^m s.t. $D^{(i)} f$ is continuous for any $(i) \leq (r)$ and $D^{i_k} f$ the partial derivative $\partial^{i_k} f / \partial x_k^{i_k}$, we have the following

LEMMA. *Let $(0) \leq (p) \leq (n) < (r)$ and let $f \in C^{(r)}(R^m)$ be such that $\|D^{(j)} f\| < \infty$, $(j) \leq (r)$. Then we have:*

$$(2) \qquad \|D^{(n)} f\| \leq 2^m \prod_{j=1}^{2^m} \|D^{(\xi)_j} f\|^{|(n)-(\theta)_j|/|(r)-(p)|}$$

Proof. Suppose we have inequality (2) for m and let's prove it for $m+1$, $m \geq 1$.

If $(n) = (n_1, n_2, \ldots, n_m, n_{m+1})$, we denote by (n') the restriction of (n) to its first m components. Similarly, $(\xi')_j$ and $(\theta')_j$ are restrictions of $(\xi)_j$ and $(\theta)_j$ to their first m components.

By (1), we have

$$\sup_{x_{m+1}} |D^{n+1}(D^{(n')} f)| \leq 2 \sup_{x_{m+1}} |D^{p_{m+1}}(D^{(n')} f)|^{(r_{m+1}-n_{m+1})/(r_{m+1}-p_{m+1})}$$

$$\times \sup_{x_{m+1}} |D^{r_{m+1}}(D^{(n')} f)|^{(n_{m+1}-p_{m+1})/(r_{m+1}-p_{m+1})}$$

https://doi.org/10.4153/CMB-1979-050-2 Published online by Cambridge University Press

Hence,

$$\|D^{n+1}(D^{(n')}f)\| \leq 2\|D^{p_{m+1}}(D^{(n')}f)\|^{(r_{m+1}-n_{m+1})/(r_{m+1}-p_{m+1})} \|D^{r_{m+1}}(D^{(n')}f)\|^{(n_{m+1}-p_{m+1})/(r_{m+1}-p_{m+1})}$$

By (2),

$$\|D^{(n')}(D^{p_{m+1}}f)\| \leq 2^m \prod_{j=1}^{2^m} \|D^{(\xi')_j}(D^{p_{m+1}}f)\|^{|(n')-(\theta')_j|/|(r')-(p')|}$$

Hence,

$$\|D^{(n)}f\| \leq 2\left(2^m \prod_{j=1}^{2^m} \|D^{(\xi')_j}(D^{p_{m+1}}f)\|^{|(n')-(\theta')_j|/|(r')-(p')|}\right)^{(r_{m+1}-n_{m+1})/(r_{m+1}-p_{m+1})} \left(2^m \prod_{j=1}^{2^m} \|D^{(\xi')_j}(D^{r_{m+1}}f)\|^{|(n')-(\theta')_j|/|(r')-(p')|}\right)^{(n_{m+1}-p_{m+1})/(r_{m+1}-p_{m+1})}$$

and

$$\|D^{(n)}f\| \leq 2^{m+1} \prod_{j=1}^{2^{m+1}} \|D^{(\xi)_j}f\|^{|(n)-(\theta)_j|/|(r)-(p)|}$$

This completes the proof of the lemma.

Proof of the theorem. For each marginal sequence $\{N_{k,\ell}\}_{\ell=0}^{\infty}$, $1 \leq k \leq m$, we consider $\liminf_{n\to\infty}(N_{k,\ell})^{1/\ell}$ and call that sequence an α, β or γ- sequence if the value of this limit is respectively finite, zero or infinite. The proof then follows Mandelbrojt ([1], p. 226) using properties of the convex regularized sequences $\{N^c_{k,\ell}\}_{\ell=0}^{\infty}$ of $\{N_{k,\ell}\}_{\ell=0}^{\infty}$, $1 \leq k \leq m$ and inequality (2) of the lemma.

Distinguishing between different cases, we show that if one of the marginal sequences is β, then $C\{N^*_{(j)}\} = C\{0\}$ while for other cases $C\{N^*_{(j)}\} = C\{M_{(j)}\}$ with $M_{(j)} = M_{1_{j_1}} M_{2_{j_2}} \cdots M_{mj_m}$, where M_{kj_k} is either 1 or $N^c_{kj_k}$, depending on whether $\{N_{k,\ell}\}_{\ell=0}^{\infty}$ is α or γ, $1 \leq k \leq m$. Without loss of generality, we can suppose $M_{k,0} = 1\ \forall k$.

To see that $C\{M_{(j)}\}$ is an algebra, let f and g be in $C\{M_{(j)}\}$. By Leibniz's rule

$$D^{(j)}(fg) = \sum_{n_1=0}^{j_1} \sum_{n_2=0}^{j_2} \cdots \sum_{n_m=0}^{j_m} \binom{j_1}{n_1}\binom{j_2}{n_2} \cdots \binom{j_m}{n_m} D^{(n)}f \cdot D^{(j)-(n)}g$$

where $n = (n_1, n_2, \ldots, n_m)$. Hence,

$$|D^{(j)}(fg)\| \leq \beta_f\beta_g\left(\sum_{n_1=0}^{j_1} \sum_{n_2=0}^{j_2} \cdots \sum_{n_m=0}^{j_m} \binom{j_1}{n_1}\binom{j_2}{n_2} \cdots \binom{j_m}{n_m} B_f^{\|(n)\|} M_{(n)} B_g^{\|(j)-(n)\|} M_{(j)-(n)}\right.$$

or by commutativity,

$$\|D^{(j)}(fg)\| \leq \beta_f\beta_g\left(\sum_{n_1=0}^{j_1} \binom{j_1}{n_1} B_f^{n_1} B_g^{j_1-n_1} M_{1,n_1} M_{1,j_1-n_1}\right) \cdots \times\left(\sum_{n_m=0}^{j_m} \binom{j_m}{n_m} B_f^{n_m} B_g^{j_m-n_m} M_{m,n_m} M_{m,j_m-n_m}\right)$$

https://doi.org/10.4153/CMB-1979-050-2 Published online by Cambridge University Press

The convexity of $\{\log M_{k,\ell}\}_{\ell=0}^{\infty}$ combined with $M_{k,0}=1$ shows that $M_{k,n_k}M_{k,j_k-n_k} \le M_{k,j_k}$, $1 \le k \le m$. Hence, we have:

$$\|D^{(j)}(fg)\| \le \beta_f\beta_g(B_f+B_g)^{\|(j)\|}M_{(j)}$$

which shows that $C\{M_{(j)}\}$ is an algebra under pointwise addition and multiplication.

If $N_{(j)}=N^*_{(j)}\ \forall(j)$, we see immediately that $C\{N_{(j)}\}=C\{M_{(j)}\}$. This completes the proof of the theorem.

References

1. S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Gauthier-Villars, Paris, 1952.

2. T. Pham-Gia, *On a theorem of Lelong*, Canad. Math. Bull., Vol. **19** (4), 1976, 505–506.

3. W. Rudin, *Division in algebras of infinitely differentiable functions*, Journ. Math. Mech., Vol. **II,** 5 (1962), 797–809.

4. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.

Department of Mathematics
Université de Moncton
Moncton, N.B. E1A 3E9