BULL. AUSTRAL. MATH. SOC. 06A60, 46A40
VOL. 10 (1974), 371-376.

Countable vector lattices

Paul F. Conrad

In his paper "On the structure of ordered real vector spaces" (*Publ. Math. Debrecen* 4 (1955-56), 334-343), Erdös shows that a totally ordered real vector space of countable dimension is order isomorphic to a lexicographic direct sum of copies of the group of real numbers. Brown, in "Valued vector spaces of countable dimension" (*Publ. Math. Debrecen* 18 (1971), 149-151), extends the result to a valued vector space of countable dimension and greatly simplifies the proof. In this note it is shown that a finite valued vector lattice of countable dimension is order isomorphic to a direct sum of o-simple totally ordered vector spaces. One obtains as corollaries the result of Erdös and the applications that Brown makes to totally ordered spaces.

1. Notation and the statement of the main result

Throughout let G be a vector lattice over a totally ordered division ring D. Then G is an abelian lattice ordered group that is also a left vector space over D, and

$$0 < d \in D \text{ , } 0 < g \in G \text{ imply } 0 < dg \in G \text{ .}$$

In particular, for a fixed $0 < d \in D$, the mapping $g \mapsto dg$ for all $g \in G$ is an l-automorphism of G. For a proof of this and the following assertions, see [3] or [5].

An *l-ideal* is a convex l-subspace of G. If the order on G is archimedean, then each convex l-subgroup of G is a subspace and hence an l-ideal. A *value* of an element $g \in G$ is an l-ideal M that is maximal without g. Let M be a value of g and M^* the intersection of all

Received 30 January 1974.

https://doi.org/10.1017/S000497270004106X Published online by Cambridge University Press

l-ideals of G that properly contain M. Then $g \in M^*\backslash M$ and so M^* covers M. Moreover, G/M is totally ordered with minimal convex subspace M^*/M. In particular, M^*/M is *o-simple* (that is, contains no proper l-ideals). An element of G with only one value is called *special*. An element with a finite number of values has a unique representation as the sum of a finite number of disjoint special elements. G is said to be *finite-valued* if each element has only finitely many values.

Let $\{G_\gamma \mid \gamma \in \Gamma\}$ be the set of all values of elements in G and for each $\gamma \in \Gamma$, let G^γ be the l-ideal of G which covers G_γ. Then Γ has a natural partial order: $\alpha < \beta$ if $G_\alpha \subset G_\beta$. Let $\Sigma = \Sigma\left(\Gamma, G^\gamma/G_\gamma\right)$ be the direct sum of the o-simple ordered vector spaces G^γ/G_γ. A component v_γ of $v \in \Sigma$ is *maximal* if $v_\gamma \neq 0$ and $v_\alpha = 0$ for all $\alpha > \gamma$, $\alpha \in \Gamma$. If $0 \neq v \in \Sigma$, then $v > 0$ if all the maximal components of v are positive in G^γ/G_γ. Then Σ is a vector lattice over D. The following is the main result of this paper.

THEOREM. *If G is a finite valued vector lattice of countable dimension over an ordered division ring D, then G and $\Sigma\left(\Gamma, G^\gamma/G_\gamma\right)$ are isomorphic as vector lattices.*

For a non-archimedean ordered division ring D the spaces G^γ/G_γ can be arbitrary: for each vector space over D can be totally ordered so that it is o-simple (see [2]).

COROLLARY 1. *If, in addition to the above hypotheses, D is archimedean, then $G \cong \Sigma(\Gamma, R_\gamma)$, where each R_γ is a subgroup of the totally ordered group R of reals.*

Proof. Since D is archimedean, we may assume that D is a subfield of the real field. Then each G^γ/G_γ has no proper convex subgroups and so is o-isomorphic to a subgroup R_γ of R. Clearly $\Sigma\left(\Gamma, G^\gamma/G_\gamma\right) \cong \Sigma(\Gamma, R_\gamma)$, and hence $G \cong \Sigma(\Gamma, R_\gamma)$.

https://doi.org/10.1017/S000497270004106X Published online by Cambridge University Press

COROLLARY 2. *If* $D = R$ *, then* $G \cong \Sigma(\Gamma, R)$.

Thus if G is a real vector lattice with countable dimension, then $G \cong \Sigma(\Gamma, R)$ if and only if G is finite valued, and if G is a finite valued real vector lattice with countable Γ , then $G \cong \Sigma(\Gamma, R)$ if and only if G has countable dimension.

COROLLARY 3. *If* H *is a countable finite valued abelian* l*-group then its divisible hull* G *is* l*-isomorphic to* $\Sigma(\Gamma, R_\gamma)$ *, where each* R_γ *is a countable divisible subgroup of* R .

Proof. G is a countable rational vector lattice.

Note that if $\{H_\gamma \mid \gamma \in \Gamma\}$ is the set of convex l-subgroups of H that are maximal without elements of H , then G^γ/G_γ is the divisible hull of H^γ/H_γ for each $\gamma \in \Gamma$. Thus if each H^γ/H_γ is a rank one group, then $G \cong \Sigma(\Gamma, Q)$, where Q is the group of rationals.

2. The concept of an *I*-set

A subset S of G consisting of special elements is called an *I-set* if $s_1, s_2, \ldots, s_n \in S$ have the same value γ and $\sum_{i=1}^{n} d_i s_i \in G_\gamma$ for $d_1, \ldots, d_n \in D$ implies $d_1 = d_2 = \ldots = d_n = 0$. Thus $\{G_\gamma + s_i \mid 1 \le i \le n\}$ is an independent subset of G^γ/G_γ .

LEMMA 1. *An* I*-set* S *is an independent subset of* G .

Proof. Suppose that $s_1, \ldots, s_n \in S$ and $d_1, \ldots, d_n \in D$ with $\sum_{i=1}^{n} d_i s_i = 0$. We may assume that $d_i \neq 0$, $1 \le i \le n$ and that the value γ of s_1 is maximal among the values of $s_1, \ldots, s_n$. Then if α is a value of s_i , $\alpha \nsucc \gamma$. Thus if $s_1, \ldots, s_k$ are the elements with value γ , $\sum_{i=1}^{k} d_i s_i \in G_\gamma$ and hence $d_1 = \ldots = d_k = 0$.

https://doi.org/10.1017/S000497270004106X Published online by Cambridge University Press

LEMMA 2. *For a vector lattice* G *the following are equivalent:*

(a) G *has an* I*-set for a basis;*

(b) $G \cong \Sigma\left(\Gamma, G^{\gamma}/G_{\gamma}\right)$.

Proof. *(b)* → *(a)* Clearly Σ has an I-set for a basis and hence so does G .

(a) → *(b)* Let S be an I-set and a basis of G . If $s \in S$ with value γ , then define $\pi : S \to \Sigma\left(\Gamma, G^{\alpha}/G_{\alpha}\right)$ by $(s\pi)_{\alpha} = 0$ if $\alpha \neq \gamma$; $(s\pi)_{\gamma} = G_{\gamma} + s$. By linearity π can be extended to a D-homomorphism σ of G into $\Sigma = \Sigma\left(\Gamma, G^{\gamma}/G_{\gamma}\right)$. We shall show that σ is an l-isomorphism of G onto Σ .

If $g \in G$ is a special element with value γ , then g has a unique representation, $g = \sum_{i=1}^{n} d_i s_i$, with $0 \neq d_i \in D$, $s_i \in S$. Since S is an I-set, it follows that each $s_i \in G^{\gamma}$ and hence $g\sigma \in \Sigma^{\gamma} = \{v \in \Sigma \mid v_{\alpha} = 0 \text{ if } a > \gamma\}$. Without loss of generality, $s_1, \ldots, s_t \in G^{\gamma}\backslash G_{\gamma}$, $s_{t+1}, \ldots, s_n \in G_{\gamma}$. Thus

$$(g\sigma)_{\gamma} = \sum_{i=1}^{t} G_{\gamma} + d_i s_i = G_{\gamma} + g .$$

Therefore $g\sigma$ is special with maximal component $(g\sigma)_{\gamma} = G_{\gamma} + g$, and if g is positive, so is $g\sigma$. Also note that

$$\left[\left(\sum_{i=1}^{t} d_i s_i\right)\sigma\right]_{\alpha} = \begin{cases} G_{\gamma}+g & \text{if } \alpha = \gamma , \\ G_{\alpha} & \text{if } \alpha \neq \gamma . \end{cases}$$

Thus it follows that σ is a homomorphism of G onto Σ .

Now consider an arbitrary element $0 \neq g \in G$. Then $g = g_1 + \ldots + g_n$ where the g_i's are disjoint and special and hence $g\sigma = g_1\sigma + \ldots + g_n\sigma$ where the $g_i\sigma$'s are disjoint and special. In

https://doi.org/10.1017/S000497270004106X Published online by Cambridge University Press

particular, σ is an isomorphism. But $g \vee 0$ is the sum of the positive g_i's and $g\sigma \vee 0$ is the sum of the positive $g_i\sigma$'s . Therefore $(g\vee 0)\sigma = g\sigma \vee 0$ and so σ is an l-isomorphism of G onto Σ .

3. Proof of the theorem

Let G be a finite valued vector lattice with countable dimension. For a subset T of G , $\langle T\rangle$ will denote the subspace of G generated by T . Let $\{b_1, b_2, \ldots\}$ be a basis of G consisting of special elements.

Suppose that $A = \{a_1, \ldots, a_m\}$ is an I-set such that $\langle a_1, \ldots, a_m\rangle \supseteq \langle b_1, \ldots, b_n\rangle$. We show that A can be extended to an I-set $A' = \{a_1, \ldots, a_m, a_{m+1}, \ldots, a_{m+t}\}$ so that $\langle a_1, \ldots, a_{m+t}\rangle \supseteq \langle b_1, \ldots, b_n, b_{n+1}\rangle$. Thus it will follow that G has an I-set for a basis and so by Lemma 2, $G \cong \Sigma(\Gamma, G^\gamma/G)$.

If $b_{n+1} \in \langle a_1, \ldots, a_m\rangle$, or if $\{a_1, \ldots, a_m, b_{n+1}\}$ is an I-set then the result follows. Suppose that this is not the case and let $Y = \langle a_1, \ldots, a_m, b_{n+1}\rangle$. Since $\{a_1, \ldots, a_m\}$ is an I-set and $\{a_1, \ldots, a_m, b_{n+1}\}$ is not, we may assume that $a_1, \ldots, a_s, b_{n+1}$ have the same value, G_γ . And there exist $d_1, \ldots, d_s$, $d \in D$ with $d \neq 0$ such that $\sum_{i=1}^{s} d_i a_i + db_{n+1} \in G_\gamma$. Now $\sum_{i=1}^{s} (d_i/d)a_i + b_{n+1} = x$ can be written as $x = x_1 + \ldots + x_k$, where x_i's are disjoint special elements. Then the values of x are the values of the x_i's and so the value of x_i is contained in G_γ , for each i , $1 \leq i \leq k$. If $\{a_1, \ldots, a_m, x_1, \ldots, x_k\}$ is an I-set, then

$$\langle a_1, \ldots, a_m, x_1, \ldots, x_k\rangle \supseteq Y \supseteq \langle b_1, \ldots, b_n, b_{n+1}\rangle$$

and so we are done. If not, then again without loss of generality, $a_{s+1}, \ldots, a_{s+r}, x_1$ have the same value α , and there exist $d'_1, \ldots, d'_r$, $d' \in D$, $d' \neq 0$ such that $\sum_{i=1}^{r} d'_i a_{s+i} + d'x_1 \in G_\alpha$. Let

https://doi.org/10.1017/S000497270004106X Published online by Cambridge University Press

$x' = \Sigma(d_i'/d')a_{s+i} + x_1$. Repeating the above argument, several times as necessary, there exists $y \in Y$ such that $y = y_1 + \ldots + y_t$, where $y_1, \ldots, y_t$ are disjoint special, $\{a_1, \ldots, a_m, y_1, \ldots, y_t\}$ is an I-set, and

$$\langle a_1, \ldots, a_m, y_1, \ldots, y_t\rangle \supseteq \langle a_1, \ldots, a_m, b_{n+1}\rangle \supseteq \langle b_1, \ldots, b_n, b_{n+1}\rangle .$$

References

[1] Ron Brown, "Valued vector spaces of countable dimension", *Publ. Math. Debrecen* 18 (1971), 149-151.

[2] Paul Conrad, "Methods of ordering a vector space", *J. Indian Math. Soc.* 22 (1958), 1-25.

[3] P. Conrad, *Lattice ordered groups* (Lecture Notes, Tulane University, Louisiana, 1970).

[4] J. Erdős, "On the structure of ordered real vector spaces", *Publ. Math. Debrecen* 4 (1955-56), 334-343.

[5] L. Fuchs, *Partially ordered algebraic systems* (Pergamon Press, Oxford, London, New York, Paris, 1963).

Department of Mathematics,
University of Kansas,
Lawrence, Kansas,
USA.