ON THE SMOOTHNESS OF GENERAL KERNELS

JOSE BARROS-NETO

In **(3**, §2**)**, the writer and F. E. Browder stated briefly, without proof, some results concerning general distribution kernels. It is our aim here to prove and complete those results.

The terminology and notations are introduced in §1.

In §2 we define the notion of domain of dependence with respect to the kernel $K_{x,y}$ (Definition 1) as well as the notion of smoothness of a distribution kernel at a point (Definition 2). Theorem 1 states that the set of points, where the distribution kernel is smooth, is open and the kernel is a smooth function in this set. Theorems 2 and 3 are the converse of Theorem 1.

In §3 we extend Schwartz's results on very regular kernels (differentiable in our terminology) and our own results on analytic kernels **(1)**.

Throughout this paper, we restrict ourselves to the case of distribution kernels defined on Euclidean spaces. Our results are still valid for kernels defined on paracompact real analytic manifolds.

1. Let $\mathbf{R}^n$ ($\mathbf{R}^m$) denote the Euclidean space of n (of m) dimensions. We shall indicate by x (by y) an element of $\mathbf{R}^n$ (of $\mathbf{R}^m$). $\mathfrak{D}_x$ or $\mathfrak{D}(\mathbf{R}^n)$ will denote the space of C^∞ functions with compact support in $\mathbf{R}^n$ equipped with the inductive limit topology and $\mathfrak{D}'_x$ or $\mathfrak{D}'(\mathbf{R}^n)$ will denote its dual, the space of distributions in $\mathbf{R}^n$, equipped with the strong topology of the dual. $\mathfrak{E}_x$ or $\mathfrak{E}(\mathbf{R}^n)$ will be the space of C^∞ functions in $\mathbf{R}^n$ with its natural topology: uniform convergence, on compact subsets, of the function and all its derivatives; $\mathfrak{E}'_x$ or $\mathfrak{E}'(\mathbf{R}^n)$ will be the dual of $\mathfrak{E}_x$, space of compactly supported distributions. If U is an open subset of $\mathbf{R}^n$, then $\mathfrak{E}(U)$, $\mathfrak{D}(U)$, $\mathfrak{D}'(U)$, and $\mathfrak{E}'(U)$ have the corresponding meanings.

If A is a closed subset of $\mathbf{R}^n$, let $\mathfrak{A}(A)$ be the space of classes of holomorphic functions defined on open sets U of $\mathbf{C}^n$ containing A, two functions being identified if they coincide in some neighbourhood of A. We consider $\mathfrak{A}(A)$ equipped with the inductive limit topology of the spaces of holomorphic functions defined on complex open neighbourhoods of A.

Also, A being a subset of $\mathbf{R}^n$, we denote by $\mathfrak{E}'_d(A;\mathbf{R}^n)$ (by $\mathfrak{E}'_a(A;\mathbf{R}^n)$) the subspace of $\mathfrak{E}'(\mathbf{R}^n)$ of compactly supported distributions that are C^∞ (that are analytic) on an open neighbourhood of A in $\mathbf{R}^n$.

We shall consider distribution kernels in the sense of Schwartz **(6)** $K_{x,y} \in \mathfrak{D}'(\mathbf{R}^n \times \mathbf{R}^m)$ and we shall make use of the following definitions and properties **(6)** and also **(1)**. A kernel $K_{x,y}$ is said to be *semi-regular in y* if the

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natural map $L_K: \mathfrak{D}_y \to \mathfrak{D}'_x$ associated with it **(6)** can be extended continuously to $\mathfrak{E}'_y$; it is *semi-regular in* x if the transpose ${}^tL_k: \mathfrak{D}_x \to \mathfrak{D}'_y$ of L_K can be extended continuously to a mapping of $\mathfrak{E}'_x$ into $\mathfrak{D}'_y$. We shall say that $K_{x,y}$ is *regular* if it is semi-regular in x and in y. The space of kernels that are semi-regular in y is given by the topological tensor product $\mathfrak{D}'_x \hat{\otimes} \mathfrak{E}_y$, while the space of those semi-regular in x is given by $\mathfrak{E}_x \hat{\otimes} \mathfrak{D}'_y$ **(1)**.

As in **(3)** we introduce the following

DEFINITION 1. *Let $y \in \mathbf{R}^m$ and let A be a subset of $\mathbf{R}^n$. We shall say that A is a domain of differentiable (of analytic) dependence for y with respect to the kernel $K_{x,y}$ if there exists an open neighbourhood V of y in $\mathbf{R}^m$ such that:*

(i) *${}^tL_K$ can be extended to a continuous linear mapping of $\mathfrak{E}'_d(A; \mathbf{R}^n)$ (of $\mathfrak{E}'_a(A; \mathbf{R}^n)$) into $\mathfrak{D}'(V)$ (restricting the values of ${}^tL_K$ to V);*

(ii) *the image of $\mathfrak{E}'_d(A; \mathbf{R}^n)$ (of $\mathfrak{E}'_a(A; \mathbf{R}^n)$) by ${}^tL_K$ is contained in $\mathfrak{E}(V)$ (in $\mathfrak{A}(V)$).*

A similar definition can be given for a domain B of differentiable (analytic) dependence for $x \in \mathbf{R}^n$. (Differentiable always means C^∞, while analytic means real analytic.)

Clearly, if A is a domain of dependence for y, then every $A' \supset A$ is a domain of dependence for y. Also, if A is a domain of dependence for y and V is the open neighbourhood of y associated with A, by Definition 1, then A is a domain of dependence for each $y' \in V$. Furthermore, if A is open, it suffices in conditions (i) and (ii) to consider those compactly supported distributions that are differentiable (analytic) in A.

As an example, if $n = m$ and $K_{x,y}$ is a regular kernel differentiable outside the diagonal of $\mathbf{R}^n \times \mathbf{R}^n$, then any closed or open neighbourhood of y is a domain of differentiable dependence for y **(6)**. More generally, if $K_{x,y}$ is differentiable outside the strip

$$\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : |x - y| \leqslant \epsilon\},$$

then it can be seen that any closed or open ball with centre y and radius $r > \epsilon$ is a domain of differentiable dependence for y; in this case, V can be taken as the open ball of centre y and radius $r - \epsilon$.

DEFINITION 2. *A kernel $K_{x,y}$ is differentiably (analytically) smooth at a point $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ if there exists a domain of differentiable (of analytic) dependence A for y which does not intersect a neighbourhood of x and a domain of differentiable (of analytic) dependence B for x which does not intersect a neighbourhood of y.*

THEOREM 1. *Suppose that $K_{x,y}$ is a regular kernel and let R be the set of points $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ where the kernel is differentiably (is analytically) smooth. Then R is an open subset of $\mathbf{R}^n \times \mathbf{R}^m$ and $K_{x,y}$ is a differentiable (an analytic) function in R.*

Proof. Let $(x, y) \in R$, let A be a domain of dependence (differentiable or analytic) for y not containing an open neighbourhood of x, and let B be a domain of dependence for x not containing an open neighbourhood of y. Corresponding to A (to B) there is an open neighbourhood V (U) of y (of x); cf. Definition 1. Clearly, we can suppose that V does not intersect B and that U does not intersect A. As we have already remarked, A (B) is a domain of dependence (differentiable or analytic) for all $y' \in V$ (for all $x' \in U$). Hence $U \times V \subset R$; thus R is open.

The differentiable case. Let $(x, y) \in R$ and let A, B, U and V be as above. Let T be a distribution in $\mathbf{R}^m$ with compact support in V, i.e., $T \in \mathfrak{E}'(V)$. T being prolonged to be equal to zero outside its support, we have it differentiable in a neighbourhood of B, since B does not intersect V. Since B is a domain of dependence for x, Definition 1 (ii) implies that $L_K(T) \in \mathfrak{E}(U)$, hence that L_K maps $\mathfrak{E}'(V)$ into $\mathfrak{E}(U)$. By the closed graph theorem **(5)**, L_K is continuous, i.e.,

$$L_K \in \mathfrak{L}(\mathfrak{E}'(V), \mathfrak{E}(U)) = \mathfrak{E}(U) \hat{\otimes} \mathfrak{E}(V) = \mathfrak{E}(U \times V).$$

Consequently, $K_{x,y} \in \mathfrak{E}(U \times V)$.

The analytic case. Proceeding as in the differentiable case, we obtain that L_K maps $\mathfrak{E}'(V)$ continuously into $\mathfrak{A}(U)$ and that ${}^tL_K$ maps $\mathfrak{E}'(U)$ continuously into $\mathfrak{A}(V)$. The first conclusion yields

$$K_{x,y} \in \mathfrak{A}(U) \hat{\otimes} \mathfrak{E}(V),$$

while the second yields

$$K_{x,y} \in \mathfrak{E}(U) \hat{\otimes} \mathfrak{A}(V)$$

(1, Theorem 1**)**. From here, using Browder's result **(4)**, we obtain

$$K_{x,y} \in \mathfrak{A}(U \times V).$$

This completes our proof.

In the differentiable case, the following converse of Theorem 1 holds.

THEOREM 2. *Suppose that $K_{x,y}$ is a regular kernel that is differentiable in an open set $R \subset \mathbf{R}^n \times \mathbf{R}^m$. Then, each $(x, y) \in R$ verifies the conditions of Definition 2.*

Proof. Let $(x, y) \in R$, U and V be two relatively compact open neighbourhoods of x and y respectively, such that $U \times V \subset R$. Let K_1 (K_2) be a compact neighbourhood of x (of y) contained in U (in V). Let Ω_1 (Ω_2) be the complement of K_1 (of K_2). To conclude the theorem, it suffices to show that Ω_1 (Ω_2) is a domain of differentiable dependence for y (for x). We shall prove it for Ω_1, the proof being the same for Ω_2.

Since $K_{x,y}$ is regular, it suffices to prove condition (ii) of Definition 1. Let

$T \in \mathfrak{E}'(\mathbf{R}^n)$ be differentiable in Ω_1. Let $\alpha \in \mathfrak{D}(U)$ be equal to 1 in a neighbourhood of K and write

$$T = \alpha T + (1 - \alpha)T.$$

Because $\alpha T \in \mathfrak{E}'\ (U)$ and, by assumption, $K_{x,y} \in (U \times V)$, it follows that ${}^t L_K(\alpha T) \in \mathfrak{E}(V)$. On the other hand, from our hypothesis on T and our choice of α, $(1 - \alpha)T \in \mathfrak{D}(\Omega_1)$. Since $K_{x,y}$ is regular, then

$${}^t L_K((1 - \alpha)T) \in \mathfrak{E}(\mathbf{R}^m);$$

thus ${}^t L_K(T) \in (V)$.

In the analytic case, the analogue of Theorem 2 is not true, in general. The type of difficulties encountered are the same as those appearing in our study of analytic kernels (**1**, p. 437, Theorem 3 and Corollary).

Consider the following hypothesis:

(H) *For each* $(x, y) \in R$, U *relatively compact open neighbourhood of* x, V *relatively compact open neighbourhood of* y *such that* $U \times V \subset R$, *then any compact contained in* U (*in* V) *is a domain of analytic dependence for* y (*for* x).

Under this hypothesis the conclusion of Theorem 2 (differentiability being replaced by analyticity) holds.

In fact, proceeding in the same way as in the previous proof, we obtain $\alpha T \in \mathfrak{E}'(U)$ and $(1 - \alpha)T \in \mathfrak{D}(\Omega_1)$. Since $K_{x,y} \in \mathfrak{A}(U \times V)$, then ${}^t L_K(\alpha T) \in \mathfrak{A}(V)$ (**1**, p. 437, Theorem 3). On the other hand, $(1 - \alpha)\ T$ is equal to zero in a neighbourhood of K_1 which is a domain of analytic dependence for y. There exists, then, an open V' containing y such that

$${}^t L_K((1 - \alpha)T) \in \mathfrak{A}(V').$$

It follows that ${}^t L_K(T) \in \mathfrak{A}(V \cap V')$.

We summarize these results in the following

THEOREM 3. *Suppose that* $K_{x,y}$ *is a regular kernel analytic in an open set* $R \subset \mathbf{R}^n \times \mathbf{R}^m$ *and suppose that* (H) *holds. Then, each* $(x, y) \in R$ *satisfies the conditions of Definition* 2.

We remark that Condition (H) is verified, for example, if $K_{x,y}$ is a composition kernel in $\mathbf{R}^n \times \mathbf{R}^n$, analytic outside the diagonal (**7** and also **2**).

3. In this section we shall extend Schwartz's results, which characterize kernels defined on $\mathbf{R}^n \times \mathbf{R}^n$ and differentiable off the diagonal, as well as the results of (**1**) concerning analytic kernels.

The following theorem gives us a sufficient condition in order that the complement of a compact be a domain of dependence for a given y.

THEOREM 4. *Suppose the kernel* $K_{x,y}$ *is regular and differentiable* (*analtyic and verifies* (H)) *in an open set* $R \subset \mathbf{R}^n \times \mathbf{R}^m$. *Suppose* K *is a compact of* $\mathbf{R}^n$

such that there exists a relatively compact open neighbourhood L *of* K *such that* $\bar{L} \times \{y\} \subset R$. *Then* $\Omega = {}^cK$ *is a domain of dependence for* y.

Proof. Since $\bar{L}$ is compact, we can find an open set $U \supset \bar{L}$ and an open V containing y such that $U \times V \subset R$. Now, if $T \in \mathfrak{E}'(\mathbf{R}^n)$ and is differentiable (analytic) on Ω, by taking $\alpha \in \mathfrak{D}(U)$ equal to 1 on K and by decomposing T, the proof will follow as in Theorems 2 and 3.

For regular kernels defined on $\mathbf{R}^n \times \mathbf{R}^n$ and differentiable off the diagonal, the following property is well known: if $T \in \mathfrak{E}'(\mathbf{R}^n)$ is differentiable in an open set Ω, then ${}^tL_K(T)$ is differentiable in the same Ω. The next theorem extends this result.

THEOREM 5. *Let* $K_{x,y}$ *be a regular kernel, differentiable* (*analytic and satisfying* (H)) *in an open set* $R \subset \mathbf{R}^n \times \mathbf{R}^m$. *Let* K *be a compact in* $\mathbf{R}^n$, $\Omega = {}^cK$ *and let* Ω' *be the set of all* $y \in \mathbf{R}^m$ *such that there exists a relatively compact open neighbourhood* L *of* K *such that* $\bar{L} \times \{y\} \subset R$. *Then, if* $T \in \mathfrak{E}'(\mathbf{R}^n)$ *is differentiable* (*analytic*) *in* Ω, ${}^tL_K(T)$ *is differentiable* (*analytic*) *in* Ω'.

Proof. Theorem 5 is an easy consequence of Theorem 4.

Suppose $K_{x,y}$ is a regular kernel in $\mathbf{R}^n \times \mathbf{R}^n$ satisfying the property: for each $T \in \mathfrak{E}'(\mathbf{R}^n)$, ${}^tL_K(T)$ is differentiable on each open set where T is differentiable. Then it is well known **(6)** that $K_{x,y}$ is differentiable off the diagonal. The analogous property in the analytic case, i.e. the analyticity of $K_{x,y}$ off the diagonal, was proved in **(1; 3)**. The following theorem extends these results.

THEOREM 6. *Let* $K_{x,y}$ *be a regular kernel and let* R *be an open subset of* $\mathbf{R}^n \times \mathbf{R}^m$. *Let* K_1 *and* K_2 *be two arbitrarily given compact subsets of* $\mathbf{R}^n$ *and* $\mathbf{R}^m$, *respectively; let* $\Omega_1 = {}^cK_1$, $\Omega_2 = {}^cK_2$, *and let* Ω'_1, Ω'_2 *be defined as in Theorem* 5. *Suppose that for each distribution* $T_1 \in \mathfrak{E}'(\mathbf{R}^n)$ *differentiable* (*analytic*) *in* Ω_1 *and for each distribution* $T_2 \in \mathfrak{E}'(\mathbf{R}^m)$ *differentiable* (*analytic*) *in* Ω_2 *we have* ${}^tL_K(T_1)$ *differentiable* (*analytic*) *in* Ω'_1 *and* $L_K(T_2)$ *differentiable* (*analytic*) *in* Ω'_2. *Then* $K_{x,y}$ *is differentiable* (*analytic*) *in* R.

Proof. According to Theorem 1, it suffices to prove that $K_{x,y}$ is differentiably (is analytically) smooth at each $(x, y) \in R$. We choose K_1 and K_2 to be two compact neighbourhoods of x and y respectively and we choose L_1 and L_2 two relatively compact open neighbourhoods of K_1 and K_2, respectively, such that $\bar{L}_1 \times \bar{L}_2 \subset R$. Then, by Theorem 4, $\Omega_1 = {}^cK_1$ is a differentiable (an analytic) domain of dependence for y not intersecting a neighbourhood of x, while $\Omega_2 = {}^cK_2$ is a differentiable (an analytic) domain of dependence for x not intersecting a neighbourhood of y. Our conclusion follows from Theorem 1.

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Université de Montréal and
The University of Rochester