

Methods of Microlocal Analysis

19.1 The Hadamard Expansion in Minkowski Space

In Chapter 15, the unregularized kernel of the fermionic projector $P(x, y) = P_-(x, y)$ was constructed abstractly. In Chapter 18, we saw how this kernel can be expanded in a perturbation series in powers of the external potential. In order to gain more explicit information on the form of the unregularized kernel, it is very useful to analyze its singularity structure on the light cone. It turns out that $P(x, y)$ has singularities on the light cone, which can be described by the so-called *Hadamard expansion* of the form

$$P(x, y) = \lim_{\varepsilon \searrow 0} i \not{\partial}_x \left(\frac{U(x, y)}{\Gamma_\varepsilon(x, y)} + V(x, y) \log \Gamma_\varepsilon(x, y) + W(x, y) \right), \quad (19.1)$$

where

$$\Gamma_\varepsilon(x, y) := (y - x)^j (y - x)_j - i\varepsilon (y - x)^0, \quad (19.2)$$

and U , V and W are smooth functions on $\mathcal{M} \times \mathcal{M}$ taking values in the 4×4 -matrices acting on the spinors (we always denote spacetime indices by Latin letters running from $0, \dots, 3$). This local expansion is based on the method of *integration along characteristics*, which will be explained later (see after (19.18) or also [99, 88] or [6]). In Minkowski space, the *light-cone expansion* [39, 40] (see also [45, Section 2.2]) gives an efficient procedure for computing an infinite number of Hadamard coefficients in one step. The Hadamard form (19.1) carries over to curved spacetime. Moreover, there is an interesting connection to the so-called wave front set in microlocal analysis. These generalizations will be briefly outlined in Section 19.3. In all the other sections of this chapter, we restrict attention to Minkowski space.

It turns out that, for an external potential in Minkowski space, the kernel of the fermionic projector is indeed of Hadamard form.

Theorem 19.1.1 *Assume that the external potential $\mathcal{B}$ is smooth and that its time derivatives decay at infinity in the sense that (17.7) holds and in addition that*

$$\int_{-\infty}^{\infty} |\partial_t^p \mathcal{B}(t)|_{C^0} dt < \infty \quad \text{for all } p \in \mathbb{N}, \quad (19.3)$$

with the C^0 -norm as defined in (17.8). Moreover, assume that the potential satisfies the bound

$$\int_{-\infty}^{\infty} |\mathcal{B}(t)|_{C^0} dt < \sqrt{2} - 1. \quad (19.4)$$

Then, the fermionic projector $P(x, y)$ is of Hadamard form.

The proof of this theorem will be given in Section 19.4.

We conclude this section by explaining how the expansion (19.1) comes about and how the involved functions U , V and W , at least in principle, can be computed iteratively using the method of *integration along characteristics*. We begin by computing the unregularized kernel in the Minkowski vacuum. To this end, one rewrites the factor $(\not{k} + m)$ in (5.67) in terms of a differential operator in position space,

$$P(x, y) = (i\not{\partial}_x + m) T_{m^2}(x, y), \quad (19.5)$$

where T_{m^2} is the scalar bi-distribution

$$T_{m^2}(x, y) := \int \frac{d^4 k}{(2\pi)^4} \delta(k^2 - m^2) \Theta(-k^0) e^{-ik(x-y)}. \quad (19.6)$$

We remark that the distribution T_{m^2} solves the Klein–Gordon equation $(-\square - m^2)T_{m^2} = 0$; in quantum field theory, it is sometimes denoted by Δ_- (see, e.g., [147, Section 5.2]). Away from the light cone (i.e., for $\xi^2 \neq 0$), the distribution $T_{m^2}(x, y)$ is a smooth function given by

$$T_{m^2}(x, y) = \begin{cases} \frac{m}{16\pi^2} \frac{Y_1(m\sqrt{\xi^2})}{\sqrt{\xi^2}} + \frac{im}{16\pi^2} \frac{J_1(m\sqrt{\xi^2})}{\sqrt{\xi^2}} \epsilon(\xi^0) & \text{if } \xi \text{ is timelike,} \\ \frac{m}{8\pi^3} \frac{K_1(m\sqrt{-\xi^2})}{\sqrt{-\xi^2}} & \text{if } \xi \text{ is spacelike,} \end{cases} \quad (19.7)$$

where we set

$$\xi := y - x, \quad (19.8)$$

and J_1 , Y_1 and K_1 are Bessel functions. Expanding the Bessel functions in (19.7) in a power series, one obtains (see [124, (10.2.2), (10.8.1) as well as (10.25.2), (10.31.1)])

$$T_{m^2}(x, y) = -\frac{1}{8\pi^3} \left(\frac{\text{PP}}{\xi^2} + i\pi\delta(\xi^2) \epsilon(\xi^0) \right) + \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (j+1)!} \frac{(m^2 \xi^2)^j}{4^j} \left(\log |m^2 \xi^2| + c_j + i\pi \Theta(\xi^2) \epsilon(\xi^0) \right), \quad (19.9)$$

with real coefficients c_j (here Θ and ϵ are again the Heaviside and the sign function, respectively). In particular, one sees that T_{m^2} is a distribution that is singular on the light cone. These singularities can be written in a shorter form using residues as

$$T_{m^2}(x, y) = \lim_{\varepsilon \searrow 0} \left(-\frac{1}{8\pi^3} \frac{1}{\Gamma_\varepsilon(x, y)} + \frac{m^2}{32\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} \frac{(m^2 \Gamma_\varepsilon(x, y))^j}{4^j} \left(\log(m^2 \Gamma_\varepsilon(x, y)) + c_j \right) \right), \quad (19.10)$$

where Γ_ε is again defined by (19.2); for the proof, one uses the distributional relation $\lim_{\varepsilon \searrow 0} (r^2 + (\varepsilon + it)^2)^{-1} = -\text{PP}/\xi^2 - i\pi \delta(\xi^2) \epsilon(\xi^0)$ and similarly the behavior of the logarithm in the complex plane. Noting that the series converge, one obtains a function of the desired form as in the brackets in (19.1). This shows that the term $i\partial_x T_{m^2}(x, y)$ in (19.5) is of Hadamard form. For the term $m T_{m^2}(x, y)$, this can be shown by pulling out one derivative and working with matrix-valued kernels. Indeed,

$$m T_{m^2}(x, y) = -\frac{1}{m} \square_x (T_{m^2}(x, y) - T_0(x, y)) = \partial_x \left\{ -\frac{1}{m} \partial_x (T_{m^2}(x, y) - T_0(x, y)) \right\}, \quad (19.11)$$

and computing the curly brackets by differentiating (19.10) one obtains again an expression of the Hadamard form (19.1).

The summands in (19.10) can be understood by verifying that T_{m^2} satisfies the Klein–Gordon equation. Indeed, using the abbreviation $\xi := y - x$, we obtain

$$\frac{\partial}{\partial x^j} \left(\frac{1}{\Gamma_\varepsilon(x, y)} \right) = -\frac{\partial_j \Gamma_\varepsilon(x, y)}{\Gamma_\varepsilon(x, y)^2} = \frac{1}{\Gamma_\varepsilon(x, y)^2} (2\xi_j - i\varepsilon \delta_{j,0}), \quad (19.12)$$

$$\begin{aligned} \square_x \left(\frac{1}{\Gamma_\varepsilon(x, y)} \right) &= \frac{2}{\Gamma_\varepsilon(x, y)^3} (2\xi_j - i\varepsilon \delta_{j,0}) (2\xi_j - i\varepsilon \delta_0^j) - \frac{8}{\Gamma_\varepsilon(x, y)^2} \\ &= \frac{2}{\Gamma_\varepsilon(x, y)^3} (4\xi^2 - 4i\varepsilon \xi_0 - \varepsilon^2) - \frac{8}{\Gamma_\varepsilon(x, y)^2} = -\frac{2\varepsilon^2}{\Gamma_\varepsilon(x, y)^3}, \end{aligned} \quad (19.13)$$

and this tends to zero as $\varepsilon \searrow 0$. Thus, the leading term in (19.10) satisfies the scalar wave equation. In the Klein–Gordon equation, however, the term involving the mass remains,

$$-m^2 \frac{1}{\Gamma_\varepsilon(x, y)}. \quad (19.14)$$

This term is compensated by the next term in the expansion (19.10) because

$$\frac{\partial}{\partial x^j} \log(\Gamma_\varepsilon(x, y)) = \frac{\partial_j \Gamma_\varepsilon(x, y)}{\Gamma_\varepsilon(x, y)} = -\frac{1}{\Gamma_\varepsilon(x, y)} (2\xi_j - i\varepsilon \delta_{j,0}) \quad (19.15)$$

$$\begin{aligned} \square_x \log(\Gamma_\varepsilon(x, y)) &= -\frac{1}{\Gamma_\varepsilon(x, y)^2} (2\xi_j - i\varepsilon \delta_{j,0}) (2\xi_j - i\varepsilon \delta_0^j) + \frac{8}{\Gamma_\varepsilon(x, y)} \\ &= -\frac{1}{\Gamma_\varepsilon(x, y)^2} (4\xi^2 - 4i\varepsilon \xi_0 - \varepsilon^2) + \frac{8}{\Gamma_\varepsilon(x, y)} \\ &= \frac{4}{\Gamma_\varepsilon(x, y)} + \frac{\varepsilon^2}{\Gamma_\varepsilon(x, y)^2}. \end{aligned} \quad (19.16)$$

Now the first summand in the last line cancels the term (19.14) in the Klein-Gordon equation. Proceeding order by order in powers of $\Gamma_\varepsilon(x, y)$, one can verify all the coefficients in (19.10).

This method of applying the wave operator term by term is also useful for computing the functions U , V and W in (19.1) in the case that an external potential is present. In fact, these functions can be expressed in terms of line integrals along the light cone. This *method of integration along characteristics* goes back to Hadamard [99] and is described in the classic textbook [88] in curved spacetime. In order to explain the method in the simplest possible context, let us assume that we consider the wave equation with an external scalar potential $a(x)$, that is,

$$(-\square_x - a(x))\tilde{T}(x, y) = 0, \quad (19.17)$$

the Dirac equation will be treated more systematically in Section 19.2. In modification of the series in (19.10), we make the ansatz

$$\tilde{T}(x, y) = \lim_{\varepsilon \searrow 0} \left(\frac{1}{\Gamma_\varepsilon(x, y)} + \sum_{n=1}^{\infty} f_n(x, y) \Gamma_\varepsilon(x, y)^n \log(\Gamma_\varepsilon(x, y)) \right). \quad (19.18)$$

Compared to (19.14), now the error term of the first summand involves the potential $a(x)$,

$$- \frac{a(x)}{\Gamma_\varepsilon(x, y)}. \quad (19.19)$$

The hope is to compensate this term by a suitable choice of $f_1(x, y)$. Indeed,

$$\frac{\partial}{\partial x^j} \left(f_1(x, y) \log \Gamma_\varepsilon(x, y) \right) \quad (19.20)$$

$$= f_1(x, y) \frac{\partial_j \Gamma_\varepsilon(x, y)}{\Gamma_\varepsilon(x, y)} + \partial_j f_1(x, y) \log \Gamma_\varepsilon(x, y), \quad (19.21)$$

$$\begin{aligned} & \square_x \left(f_1(x, y) \log \Gamma_\varepsilon(x, y) \right) \\ &= f_1(x, y) \frac{4}{\Gamma_\varepsilon(x, y)} - 2 \partial_j f_1(x, y) \frac{2\xi_j}{\Gamma_\varepsilon(x, y)} + \cdots, \end{aligned} \quad (19.22)$$

where $\cdots$ stands for all terms that either have a lower-order singularity on the light cone or tend to zero as $\varepsilon \searrow 0$. In order for this contribution to compensate (19.19), the function f_1 must satisfy the equation

$$4 f_1(x, y) - 4 \xi^j \partial_j f_1(x, y) = a(x). \quad (19.23)$$

Such a differential equation of first order can be solved with the method of characteristics (see, e.g., [32, Section I.3.2]). More specifically, the solution is an integral along the straight line $\xi\mathbb{R}$. In order to describe the singular behavior on the light cone, it suffices to consider the case that ξ is tangential to the light cone. Similarly, also to higher order in the expansion parameter n , we obtain transport equations along the light cone, which can be solved iteratively order by order.

19.2 The Light-Cone Expansion

We first give the basic definition of the light-cone expansion and explain it afterward.

Definition 19.2.1 *A distribution $A(x, y)$ on $\mathcal{M} \times \mathcal{M}$ is of the order $\mathcal{O}((y-x)^{2p})$ for $p \in \mathbb{Z}$ if the product*

$$(y-x)^{-2p} A(x, y) \quad (19.24)$$

is a regular distribution (i.e., a locally integrable function). An expansion of the form

$$A(x, y) = \sum_{j=g}^{\infty} A^{[j]}(x, y), \quad (19.25)$$

*with $g \in \mathbb{Z}$, is called **light-cone expansion** if the $A^{[j]}(x, y)$ are distributions of the order $\mathcal{O}((y-x)^{2j})$ and if A is approximated by the partial sums in the sense that for all $p \geq g$,*

$$A(x, y) - \sum_{j=g}^p A^{[j]}(x, y) \quad \text{is of the order } \mathcal{O}((y-x)^{2p+2}). \quad (19.26)$$

The parameter g gives the leading order of the singularity of $A(x, y)$ on the light cone. We point out that we do not demand that the infinite series in (19.25) converges. Thus, similar to a formal Taylor series, the series in (19.25) is defined only via the approximation by the partial sums (19.26). The notion of the light-cone expansion is illustrated in Exercise 19.1.

As a concrete example, due to the factors $\Gamma_\varepsilon(x, y)$, the series (19.10) is a light-cone expansion. The term with the leading singularity becomes integrable after multiplying by $(y-x)^2$, showing that $g = -1$.

Our task is to perform the light-cone expansion of the unregularized kernel of the fermionic projector. Schematically, this construction consists of several steps:

- (1) Perform the light-cone expansion of the causal Green's operators $\tilde{s}_m^\vee$ and $\tilde{s}_m^\wedge$. Here, one proceeds inductively for each summand of the perturbation series (18.13).
- (2) Using the relation (18.16), one obtains a corresponding light-cone expansion for the causal fundamental solution $\tilde{k}_m$.
- (3) The so-called residual argument relates the sought-after light-cone expansion of $\tilde{P}(x, y)$ to that of $\tilde{k}_m$.

This procedure is described in detail in [45, Chapter 2]. In order to avoid an unnecessary overlap, we here focus on the light-cone expansion of the causal Green's operators and only introduce the concepts needed for the basics on the continuum limit in Chapter 21. Before doing so, we illustrate the light-cone expansion by a simple example.

Example 19.2.2 Consider the massless Dirac equation in the presence of an external electromagnetic potential A ,

$$(i\partial + A)\tilde{P}(x, y). \quad (19.27)$$

For simplicity, assume that A is smooth and compactly supported in space-time. Then, to first order in perturbation theory, the light-cone expansion of the unregularized kernel $\tilde{P}(x, y)$ takes the form

$$\tilde{P}(x, y) = \frac{i}{2} \exp \left(-i \int_0^1 A_j|_{\alpha y + (1-\alpha)x} \xi^j \, d\alpha \right) P(x, y) \quad (19.28)$$

$$- \frac{1}{2} \xi_i \xi_j \int_0^1 (\alpha - \alpha^2) j^i|_{\alpha y + (1-\alpha)x} \, d\alpha \, T^{(0)} \quad (19.29)$$

$$+ \frac{1}{4} \xi_i \xi_j \int_0^1 F^{ij}|_{\alpha y + (1-\alpha)x} \gamma_i \gamma_j \, d\alpha \, T^{(0)} \quad (19.30)$$

$$- \xi_i \int_0^1 (1 - \alpha) F^{ij}|_{\alpha y + (1-\alpha)x} \gamma_j \, d\alpha \, T^{(0)} \quad (19.31)$$

$$- \xi_i \int_0^1 (1 - \alpha)(\alpha - \alpha^2) \not{\partial} j^i|_{\alpha y + (1-\alpha)x} \, d\alpha \, T^{(1)} \quad (19.32)$$

$$- \int_0^1 (1 - \alpha)^2 j^i|_{\alpha y + (1-\alpha)x} \gamma_i \, d\alpha \, T^{(1)} \quad (19.33)$$

$$+ \xi (\deg < 1) + (\deg < 0) + \mathcal{O}(A^2),$$

where $F^{jk} = \partial^j A^k - \partial^k A^j$ is the field tensor and $j^k = \partial_j^k A^j - \square A^k$ is the corresponding Maxwell current. Moreover, the factors $T^{(0)}$ and $T^{(1)}$ are the leading summands in (19.10); more precisely,

$$\begin{aligned} T^{(0)}(x, y) &= -\frac{1}{8\pi^3} \lim_{\varepsilon \searrow 0} \frac{1}{\Gamma_\varepsilon(x, y)} \\ T^{(1)}(x, y) &= \frac{1}{32\pi^3} \lim_{\varepsilon \searrow 0} \log \Gamma_\varepsilon(x, y). \end{aligned} \quad (19.34)$$

Each summand has the general structure of being the product of a smooth function and a distribution that is singular on the light cone. The smooth factor is an integral along the straight line segment joining the points x and y . The integrand involves the electromagnetic potential and its partial derivatives. We remark for clarity that the term (19.28) involves a gauge phase as needed for gauge invariance (as already mentioned in (5.74) in Section 5.8). All the other integrands are gauge invariant, as is obvious from the fact that they are expressed in terms of the electromagnetic field tensor and the Maxwell current.

To higher order on the light cone or to higher order in the mass or the external potentials, the formulas of the light-cone expansions have a similar structure. More detailed formulas can be found in the original papers [39, 40], in [41, Appendix B] and [45, Appendix B]. $\diamond$

We now explain how to perform the light-cone expansion of the causal Green's operators. In order to get a first idea for how to proceed, we begin by considering the free advanced Green's operator $s_m^\vee$ of the Dirac equation of mass m in position space: Similar to (19.5), it is again convenient to pull the Dirac matrices out of $s_m^\vee$ by setting

$$s_m^\vee(x, y) = (i\not{\partial}_x + m) S_{m^2}^\vee(x, y), \quad (19.35)$$

where $S_{m^2}^\vee$ is the advanced Green's operator of the Klein–Gordon operator,

$$S_{m^2}^\vee(x, y) = \lim_{\nu \searrow 0} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 - m^2 - i\nu p^0} e^{-ip(x-y)}. \quad (19.36)$$

Computing this Fourier integral and expanding the resulting Bessel function in a power series gives (for details, see Exercise 19.3)

$$\begin{aligned} S_{m^2}^\vee(x, y) &= -\frac{1}{2\pi} \delta(\xi^2) \Theta(\xi^0) \\ &\quad + \frac{m^2}{4\pi} \frac{J_1(\sqrt{m^2 \xi^2})}{\sqrt{m^2 \xi^2}} \Theta(\xi^2) \Theta(\xi^0) \end{aligned} \quad (19.37)$$

$$\begin{aligned} &= -\frac{1}{2\pi} \delta(\xi^2) \Theta(\xi^0) \\ &\quad + \frac{m^2}{8\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (j+1)!} \frac{(m^2 \xi^2)^j}{4^j} \Theta(\xi^2) \Theta(\xi^0). \end{aligned} \quad (19.38)$$

This computation shows that $S_{m^2}^\vee(x, y)$ has a $\delta(\xi^2)$ -like singularity on the light cone. Furthermore, one sees that $S_{m^2}^\vee$ is a power series in m^2 . The important point for what follows is that the higher-order contributions in m^2 contain more factors ξ^2 and are thus of higher order on the light cone. More precisely,

$$\left(\frac{d}{da} \right)^n S_a^\vee(x, y) \Big|_{a=0} \quad \text{is of the order } \mathcal{O}(\xi^{2n-2}). \quad (19.39)$$

Here, and in what follows, we often use the abbreviation $a = m^2$. According to (19.35), the Dirac Green's operator is obtained by taking the first partial derivatives of (19.38). Therefore, $s_m^\vee(x, y)$ has a singularity on the light cone that is even $\sim \delta'(\xi^2)$. The higher-order contributions in m are again of increasing order on the light cone. This means that we can view the Taylor expansion of (19.35) in m ,

$$s_m^\vee(x, y) = \sum_{n=0}^{\infty} (i\not{\partial} + m) \frac{m^{2n}}{n!} \left(\frac{d}{da} \right)^n S_a^\vee(x, y) \Big|_{a=0}, \quad (19.40)$$

as a light-cone expansion of the free Green's operator. Our idea is to generalize this formula to the case with interaction. More precisely, we want to express the perturbed Green's operator in the form

$$\tilde{s}^\vee(x, y) = \sum_{n=0}^{\infty} F_n(x, y) \left(\frac{d}{da} \right)^n S_a^\vee(x, y) \Big|_{a=0}, \quad (19.41)$$

with factors F_n that depend on the external potential. We will see that this method is very convenient; especially, we can in this way avoid working with the rather complicated explicit formula (19.38). Apart from giving a motivation for the desired form (19.41) of the formulas of the light-cone expansion, the mass expansion (19.38) leads to the conjecture that even the higher-order contributions in the mass to the *perturbed* Green's operators might be of higher-order on the light cone. If this conjecture were true, it would be a good idea to expand the perturbation expansion of $\tilde{s}$ with respect to the parameter m . Therefore, our

strategy is to first expand (18.13) with respect to the mass and to try to express the contributions to the resulting expansion in a form similar to (19.41).

The expansion of (18.13) with respect to m gives a double sum over the orders in the mass parameter and in the external potential. It is convenient to combine these two expansions in a single perturbation series. To this end, we rewrite the Dirac operator as

$$i\partial + \mathcal{B} - m = i\partial + B \quad \text{with} \quad B := \mathcal{B} - m. \quad (19.42)$$

For the light-cone expansion of the Green's operators, we will always view B as the perturbation of the Dirac operator. This has the advantage that the unperturbed objects are massless. Expanding in powers of B gives the mass expansion and the perturbation expansion in one step. In order to further simplify the notation, for the massless objects, we usually omit the index m . Thus, we write the Green's operator of the massless Dirac equation in the Minkowski vacuum as

$$s^\vee(x, y) = i\partial_x S_{m^2}^\vee(x, y)|_{m=0}, \quad s^\wedge(x, y) = i\partial_x S_{m^2}^\wedge(x, y)|_{m=0}. \quad (19.43)$$

Then, the interacting Green's operators are given by the perturbation series

$$\tilde{s}^\vee = \sum_{k=0}^{\infty} (-s^\vee B)^k s^\vee, \quad \tilde{s}^\wedge = \sum_{k=0}^{\infty} (-s^\wedge B)^k s^\wedge. \quad (19.44)$$

The constructions of the following subsections are exactly the same for the advanced and retarded Green's operators. In order to treat both cases at once, in the remainder of this section, we will omit all superscripts " $\vee$ " and " $\wedge$." The formulas for the advanced and retarded Green's operators are obtained by either adding " $\vee$ " or " $\wedge$ " to all factors s and S .

We now explain how each contribution to the perturbation expansions in (19.44) can be written similar to the right-hand side of (19.41) as a sum of terms of increasing order on the light cone. For the mass expansion of S_{m^2} , we again set $a = m^2$ and use the notation

$$S^{(l)} = \left(\frac{d}{da} \right)^l S_a|_{a=0}. \quad (19.45)$$

In preparation, we derive some computation rules for the $S^{(l)}$: S_a satisfies the defining equation of a Klein–Gordon Green's operator

$$(-\square_x - a) S_a(x, y) = \delta^4(x - y). \quad (19.46)$$

Differentiating with respect to a and setting $a = 0$ gives

$$-\square_x S^{(l)}(x, y) = \delta_{l,0} \delta^4(x - y) + l S^{(l-1)}(x, y), \quad l \geq 0. \quad (19.47)$$

(For $l = 0$, this formula does not seem to make sense because $S^{(-1)}$ is undefined. The expression is meaningful, however, if one keeps in mind that in this case, the factor l is zero, and thus, the whole second summand vanishes. We will also use this convention in the following calculations.) Next, we differentiate the formulas for S_a in momentum space,

$$S_a^\vee(p) = \frac{1}{p^2 - a - i\nu p^0}, \quad S_a^\wedge(p) = \frac{1}{p^2 - a + i\nu p^0}, \quad (19.48)$$

with respect to both p and a . Comparing the results gives the relation

$$\frac{\partial}{\partial p^k} S_a(p) = -2p_k \frac{d}{da} S_a(p), \quad (19.49)$$

or, after expanding in the parameter a ,

$$\frac{\partial}{\partial p^k} S^{(l)}(p) = -2p_k S^{(l+1)}(p), \quad l \geq 0. \quad (19.50)$$

This formula also determines the derivatives of $S^{(l)}$ in position space; namely,

$$\begin{aligned} \frac{\partial}{\partial x^k} S^{(l)}(x, y) &= \int \frac{d^4 p}{(2\pi)^4} S^{(l)}(p) (-ip_k) e^{-ip(x-y)} \\ &\stackrel{(19.50)}{=} \frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{\partial}{\partial p^k} S^{(l-1)}(p) e^{-ip(x-y)} \\ &= -\frac{i}{2} \int \frac{d^4 p}{(2\pi)^4} S^{(l-1)}(p) \frac{\partial}{\partial p^k} e^{-ip(x-y)} \\ &= \frac{1}{2} (y-x)_k S^{(l-1)}(x, y), \quad l \geq 1. \end{aligned} \quad (19.51)$$

We iterate this relation to calculate the Laplacian,

$$\begin{aligned} -\square_x S^{(l)}(x, y) &= -\frac{1}{2} \frac{\partial}{\partial x^k} \left((y-x)^k S^{(l-1)}(x, y) \right) \\ &= 2 S^{(l-1)}(x, y) + \frac{1}{4} (y-x)^2 S^{(l-2)}(x, y), \quad l \geq 2 \end{aligned} \quad (19.52)$$

(in the last step, we used the product rule and applied (19.51) for l replaced by $l-1$). After comparing with (19.47), we conclude that

$$(y-x)^2 S^{(l)}(x, y) = -4l S^{(l+1)}(x, y), \quad l \geq 0. \quad (19.53)$$

Finally, $S^{(l)}(x, y)$ is only a function of $(y-x)$, which implies that

$$\frac{\partial}{\partial x^k} S^{(l)}(x, y) = -\frac{\partial}{\partial y^k} S^{(l)}(x, y), \quad l \geq 0. \quad (19.54)$$

The following lemma gives the light-cone expansion of an operator product that is linear in the external potential. It can be used iteratively to perform the light-cone expansion of more complicated operator products; in this case, the potential is a composite expression in B and its partial derivatives. With this in mind, in the next lemma, we denote the external potential by V .

Lemma 19.2.3 (Light-cone expansion to first order) *For any $l, r \geq 0$, the operator product $S^{(l)} V S^{(r)}$ has the light-cone expansion*

$$\begin{aligned} (S^{(l)} V S^{(r)})(x, y) &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n (\square_\alpha V)|_{\alpha y + (1-\alpha)x} d\alpha \\ &\quad \times S^{(n+l+r+1)}(x, y). \end{aligned} \quad (19.55)$$

Before coming to the proof, we briefly explain this lemma. We first recall that, according to (19.39), the higher a -derivatives of $S_a(x, y)$ are of higher order on the light cone. Thus, the summands in (19.55) are of increasing order on the light cone, and the infinite sum is mathematically well defined in the sense of Definition 19.2.1 via the approximation by the partial sums (19.26).

The second point that requires an explanation is related to the arbitrariness in choosing the potential V in the case $l = 0$ (and analogously in the case $r = 0$). In this case, the distribution $S^{(l)} = S_0$ is supported on the light cone (see (19.37)). Therefore, the function V enters the operator product on the left-hand side of (19.55) only evaluated on the light cone $L_x = \{z | (x - z)^2 = 0\}$. This means that we may modify the function V arbitrarily outside this light cone. When doing so, the argument $\square^n V$ in the integrand on the right-hand side of (19.55) does in general change. Therefore, the individual summands in (19.55) do in general change. But, clearly, in order for the identity (19.55) to remain valid, the whole series must remain unchanged. This is indeed the case due to cancellations in the series (this is illustrated in Exercise 19.4). With this in mind, one can sometimes simplify the application of Lemma 19.2.3 in the case $l = 0$ by choosing V outside the light cone L_x in such a way that the computation of the right-hand side simplifies.

Proof of Lemma 19.2.3. The method of proof is to compute the Laplacian of both sides of (19.55). The resulting formulas will have a similar structure, making it possible to proceed inductively.

On the left-hand side of (19.55), we calculate the Laplacian with the help of (19.47) to

$$-\square_x(S^{(l)} V S^{(r)})(x, y) = \delta_{l,0} V(x) S^{(r)}(x, y) + l(S^{(l-1)} V S^{(r)})(x, y). \quad (19.56)$$

The Laplacian of the integral on the right-hand side of (19.55) can be computed with the help of (19.51) and (19.47),

$$\begin{aligned} & -\square_x \int_0^1 \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)_{|\alpha y + (1-\alpha)x} d\alpha \\ & \quad \times S^{(n+l+r+1)}(x, y) \\ &= -\int_0^1 \alpha^l (1-\alpha)^{r+2} (\alpha - \alpha^2)^n (\square^{n+1} V)_{|\alpha y + (1-\alpha)x} d\alpha S^{(n+l+r+1)}(x, y) \\ & \quad - \int_0^1 \alpha^l (1-\alpha)^{r+1} (\alpha - \alpha^2)^n (\partial_k \square^n V)_{|\alpha y + (1-\alpha)x} d\alpha \\ & \quad \times (y-x)^k S^{(n+l+r)}(x, y) \\ & \quad + (n+l+r+1) \int_0^1 \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)_{|\alpha y + (1-\alpha)x} d\alpha \\ & \quad \times S^{(n+l+r)}(x, y). \end{aligned} \quad (19.57)$$

In the second summand, we rewrite the partial derivative as a derivative with respect to α ,

$$(y-x)^k (\partial_k \square^n V)_{|\alpha y + (1-\alpha)x} = \frac{d}{d\alpha} (\square^n V)_{|\alpha y + (1-\alpha)x} \quad (19.59)$$

(as is verified immediately by computing the right-hand side with the chain rule). This makes it possible to integrate in α by parts. We thus obtain

$$\begin{aligned}
 & \int_0^1 \alpha^l (1-\alpha)^{r+1} (\alpha - \alpha^2)^n (\partial_k \square^n V)|_{\alpha y + (1-\alpha)x} d\alpha (y-x)^k \\
 &= \int_0^1 \alpha^l (1-\alpha)^{r+1} (\alpha - \alpha^2)^n \frac{d}{d\alpha} \left((\square^n V)|_{\alpha y + (1-\alpha)x} \right) d\alpha \\
 &= -\delta_{n,0} \delta_{l,0} V(x) \\
 &\quad - (n+l) \int_0^1 \alpha^l (1-\alpha)^{r+2} (\alpha - \alpha^2)^{n-1} (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha \\
 &\quad + (n+r+1) \int_0^1 \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha \\
 &= -\delta_{n,0} \delta_{l,0} V(x) \\
 &\quad - n \int_0^1 \alpha^l (1-\alpha)^{r+2} (\alpha - \alpha^2)^{n-1} (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha \\
 &\quad + (n+l+r+1) \int_0^1 \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha \\
 &\quad - l \int_0^1 \alpha^{l-1} (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha. \tag{19.60}
 \end{aligned}$$

We substitute back into the original equation to obtain

$$\begin{aligned}
 (19.57) &= \delta_{n,0} \delta_{l,0} V(x) S^{(r)}(x, y) \\
 &\quad + l \int_0^1 \alpha^{l-1} (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha S^{(n+l+r)}(x, y) \\
 &\quad - \int_0^1 \alpha^l (1-\alpha)^{r+2} (\alpha - \alpha^2)^n (\square^{n+1} V)|_{\alpha y + (1-\alpha)x} d\alpha S^{(n+l+r+1)}(x, y) \\
 &\quad + n \int_0^1 \alpha^l (1-\alpha)^{r+2} (\alpha - \alpha^2)^{n-1} (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha S^{(n+l+r)}(x, y). \tag{19.61}
 \end{aligned}$$

After dividing by $n!$ and summation over n , the last two summands are telescopic and cancel each other. Thus one gets

$$\begin{aligned}
 & -\square \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^l (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha S^{(n+l+r+1)}(x, y) \\
 &= \delta_{l,0} V(x) S^{(r)}(x, y) \\
 &\quad + l \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \alpha^{l-1} (1-\alpha)^r (\alpha - \alpha^2)^n (\square^n V)|_{\alpha y + (1-\alpha)x} d\alpha S^{(n+l+r)}(x, y). \tag{19.62}
 \end{aligned}$$

We now compare the formulas (19.56) and (19.62) for the Laplacian of both sides of (19.55). In the special case $l = 0$, these formulas coincide, and we can use a uniqueness argument for the solutions of the wave equation to prove (19.55): We

assume that we consider the advanced Green's operator (for the retarded Green's operator, the argument is analogous). For given y , we denote the difference of both sides of (19.55) by $F(x)$. Since the support of $F(x)$ is in the past light cone $x \in L_y^\wedge$, F vanishes in a neighborhood of the hypersurface $\mathcal{H} = \{z \in \mathbb{R}^4 \mid z^0 = y^0 + 1\}$. Moreover, the Laplacian of F is identically zero according to (19.56) and (19.62). We conclude that

$$\square F = 0 \quad \text{and} \quad F|_{\mathcal{H}} = \partial_k F|_{\mathcal{H}} = 0. \quad (19.63)$$

Since the wave equation has a unique solution for given initial data on the Cauchy surface $\mathcal{H}$, F vanishes identically.

The general case follows by induction in l : Suppose that (19.55) holds for given $\hat{l}$ (and arbitrary r). Then, according to (19.56), (19.62), and the induction hypothesis, the Laplacian of both sides of (19.55) coincides for $l = \hat{l} + 1$. The above uniqueness argument for the solutions of the wave equation again gives (19.55). $\square$

We finally remark that the method of the previous lemma generalizes to other operator products. In particular, in [52, Appendix C] light-cone expansions are derived, which involve unbounded line integrals.

19.3 The Hadamard Form in Curved Spacetime and the Wave Front Set

The Hadamard expansion (19.1) can also be formulated in curved spacetime. To this end, one simply replaces the function (19.2) by

$$\Gamma_\varepsilon(x, y) := \Gamma(x, y) - i\varepsilon (\mathfrak{t}(y) - \mathfrak{t}(x)), \quad (19.64)$$

where $\mathfrak{t}$ is a time function and $\Gamma(x, y)$ is the geodesic distance squared, with the sign convention that Γ is positive in timelike and negative in spacelike directions. It turns out that if a bi-distribution is of Hadamard form in one chart, it is also of Hadamard form in any other chart. More details on the Hadamard expansion for Dirac fields can be found in [138, 98] or [55, Appendix A].

The Hadamard form can be formulated alternatively in terms of the wave front set, as we now briefly mention. We work in an open subset $U \subset \mathbb{R}^n$. We denote the distributions in U by $\mathcal{D}'(U)$ (being the dual space of $C^\infty(U, \mathbb{C})$ with the topology induced by the C^k -norms). An *open conic neighborhood* of a point $\xi \in \mathbb{R}^n$ is defined to be an open neighborhood that is invariant under the action of $\mathbb{R}^+$ by multiplication. Thus, an open conic neighborhood can be written in the form

$$\{\lambda x \mid x \in S, \lambda \in \mathbb{R}^+\}, \quad (19.65)$$

where S is an open subset of $S^{n-1} \subset \mathbb{R}^n$.

Definition 19.3.1 *Let $\phi \in \mathcal{D}'(U)$. The **wave front set** $\text{WF}(\phi)$ is the complement in $U \times \mathbb{R}^n \setminus \{0\}$ of all points $(x, \xi) \in U \times \mathbb{R}^n \setminus \{0\}$ with the following property: There exists a function $f \in C^\infty(U, \mathbb{R})$ with $f(x) = 1$ and an open conic neighborhood V of ξ such that*

$$\sup_{\zeta \in V} (1 + |\zeta|)^N \left| (\widehat{f\phi})(\zeta) \right| < \infty \quad \text{for all } N \in \mathbb{N}. \quad (19.66)$$

In simple terms, the wave front set consists of all points $x \in U$ where the distribution is singular, together with the directions ξ into which the singularity points. More precisely, the above definition can be understood as follows. First, in view of taking the complement, the condition (19.66) ensures that the point (x, ξ) does *not* lie in the wave front set. With the help of the cutoff function f , one can disregard the behavior of ϕ away from x . In other words, the condition 19.66 only depends on the behavior of ϕ in an arbitrarily small neighborhood of x . This condition states that the Fourier transform has rapid decay in a cone around ξ . Since decay properties of the Fourier transform correspond to smoothness properties in position space, we obtain a smoothness statement for ϕ at x , but only along the “wave front” described by ξ .

Definition 19.3.1 readily extends to a distribution ϕ on a manifold $\mathcal{M}$, in which case the wave front set is a subset of the cotangent bundle,

$$\text{WF}(\phi) \subset T^*\mathcal{M} \setminus 0, \quad (19.67)$$

where 0 is the zero section. The wave front set can also be defined for bundle-valued distributions by choosing a local trivialization and taking the wave front sets of the component functions. The unregularized kernel of the fermionic projector P is a bi-distribution on $\mathcal{M} \times \mathcal{M}$. Therefore, its wave front set takes values in the product of the cotangent bundles,

$$\text{WF}(P) \subset (T^*\mathcal{M} \setminus 0) \times (T^*\mathcal{M} \setminus 0). \quad (19.68)$$

Definition 19.3.2 *The unregularized kernel $P \in \mathcal{D}'(\mathcal{M} \times \mathcal{M})$ is said to be of Hadamard form if its wave front set has the property*

$$\begin{aligned} \text{WF}(P) \subset \left\{ (x_1, \xi_1, x_2, -\xi_2) \right. & \left| \text{there is a null geodesic } \gamma : I \rightarrow \mathcal{M} \right. \\ & \text{with } a, b \in I, \gamma(a) = x_1, \gamma(b) = x_2, \\ & \left. \text{and } \xi_1 = \dot{\gamma}(a), \xi_2 = \dot{\gamma}(b) \text{ past-directed} \right\}. \end{aligned} \quad (19.69)$$

In words, this definition means that there are singularities only on the light cone and that these singularities are formed only of negative frequencies. The equivalence of this definition with the local Hadamard expansion (19.1) has been established in [129]. Physically, the Hadamard condition can be understood as a microlocal formulation of an energy condition, noting that “frequencies” can also be interpreted as “energies.” Good references on microlocal analysis and the wave front set are [105] and [4, Chapter 4].

19.4 Proof of the Hadamard Property in an External Potential

In this section, we give the proof of Theorem 19.1.1. We closely follow the presentation in [81]. In preparation, we derive so-called frequency splitting estimates that give control of the “mixing” of the positive and negative frequencies in the

solutions of the Dirac equation as generated by the time-dependent external potential (Theorem 19.4.1). Based on these estimates, we will complete the proof of Theorem 19.1.1 at the end of Section 19.4.2.

19.4.1 Frequency Mixing Estimates

For the following constructions, we again choose the hypersurface $\mathcal{N} := \mathcal{N}_{t_0}$ at some given time t_0 . Moreover, we always fix the mass parameter $m > 0$. Since we are no longer considering families of solutions, for ease of notation, we omit the index m at the Dirac wave functions, the scalar products and the corresponding norms. We also identify the solution space $\mathcal{H}_m$ with the Hilbert space $\mathcal{H}_{t_0}$ of square integrable wave functions on $\mathcal{N}$. On $\mathcal{H}_{t_0}$, we can act with the Hamiltonian H of the vacuum, and using the above identification, the operator H becomes an operator on $\mathcal{H}_m$ (which clearly depends on the choice of t_0).

We work with a so-called *frequency splitting* with respect to the vacuum dynamics. To this end, we decompose the Hilbert space $\mathcal{H}_m$ as

$$\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^- \quad \text{with} \quad \mathcal{H}^\pm = \chi^\pm(H)\mathcal{H}_m, \quad (19.70)$$

where $\chi^\pm$ are the characteristic functions

$$\chi^+ := \chi_{[0,\infty)} \quad \text{and} \quad \chi^- := \chi_{(-\infty,0)}. \quad (19.71)$$

For convenience, we write this decomposition in components and use a block matrix notation for operators, that is,

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_+^+ & A_+^- \\ A_-^+ & A_-^- \end{pmatrix}, \quad (19.72)$$

where $A_{s'}^s = \chi^s(H)A\chi^{s'}(H)$ and $s, s' \in \{\pm\}$.

The representation in Proposition 17.2.5 makes it possible to let the fermionic signature operator $\tilde{\mathcal{S}}_m$ act on the Hilbert space $\mathcal{H}_m$ (for fixed m). We decompose this operator with respect to the above frequency splitting,

$$\begin{aligned} \tilde{\mathcal{S}}_m &= S^\mathbb{D} + \Delta\tilde{\mathcal{S}}, \quad \text{where} \\ S^\mathbb{D} &:= \tilde{\mathcal{S}}_+^+ + \tilde{\mathcal{S}}_-^- \quad \text{and} \quad \Delta\tilde{\mathcal{S}} := \tilde{\mathcal{S}}_-^+ + \tilde{\mathcal{S}}_+^-. \end{aligned} \quad (19.73)$$

Thus, the operator $S^\mathbb{D}$ maps positive to positive and negative to negative frequencies. The operator $\Delta\tilde{\mathcal{S}}$, on the other hand, mixes positive and negative frequencies. In the next theorem, it is shown under a suitable smallness assumption on $\mathcal{B}$ that the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ coincide with the projections $\chi^\pm(H)$, up to smooth contributions. The main task in the proof is to control the “frequency mixing” as described by the operator $\Delta\tilde{\mathcal{S}}$.

Theorem 19.4.1 *Under the assumptions of Theorem 19.1.1, the operators $\chi^\pm(\tilde{\mathcal{S}}_m)$ have the representations*

$$\chi^\pm(\tilde{\mathcal{S}}_m) = \chi^\pm(H) + \frac{1}{2\pi i} \oint_{\partial B_{\frac{1}{2}}(\pm 1)} (\tilde{\mathcal{S}}_m - \lambda)^{-1} \Delta\tilde{\mathcal{S}} (S^\mathbb{D} - \lambda)^{-1} d\lambda, \quad (19.74)$$

where the contour integral is an integral operator with a smooth integral kernel.

Here, $B_{\frac{1}{2}}$ denotes the open ball of radius $1/2$. The operator $(\tilde{\mathcal{S}}_m - \lambda)^{-1}$ is also referred to as the *resolvent* of $\tilde{\mathcal{S}}_m$.

This theorem will be proved in several steps. We begin with a preparatory lemma.

Lemma 19.4.2 *Under the assumptions (17.7) and (19.4), the spectrum of $S^{\mathbb{D}}$ is located in the set*

$$\sigma(S^{\mathbb{D}}) \subset \left[-\frac{3}{2}, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \frac{3}{2}\right]. \quad (19.75)$$

Moreover,

$$\chi^{\pm}(S^{\mathbb{D}}) = \chi^{\pm}(H), \quad (19.76)$$

and the operators $\chi^{\pm}(\tilde{\mathcal{S}}_m)$ have the representations (19.74).

Proof Since the subspaces $\mathcal{H}^{\pm}$ are invariant under the action of $S^{\mathbb{D}}$, our task is to show that the spectrum of $S^{\mathbb{D}}|_{\mathcal{H}^{\pm}}$ is positive and negative, respectively. This statement would certainly be true if we replaced $S^{\mathbb{D}}$ with $\mathcal{S}_m$ because the operator $\mathcal{S}_m$ has the eigenvalues ± 1 with $\mathcal{H}^{\pm}$ as the corresponding eigenspaces. Estimating the representation in Proposition 17.2.5 with the Schwarz inequality, we obtain

$$|(\psi|S^{\mathbb{D}}\phi) - (\psi|\mathcal{S}_m\phi)| \leq \left(c + \frac{c^2}{2}\right) \|\psi\| \|\phi\| \quad \text{with} \quad c := \int_{-\infty}^{\infty} |\mathcal{B}(\tau)|_{C^0} d\tau. \quad (19.77)$$

Using the assumption (19.4), we conclude that

$$|(\psi|S^{\mathbb{D}}\phi) - (\psi|\mathcal{S}_m\phi)| < \frac{1}{2} \|\psi\| \|\phi\| \quad \text{for all } \psi, \phi \in \mathcal{H}_m. \quad (19.78)$$

Standard estimates on the continuity of the spectrum (see, e.g., [108, §IV.3]) yield that the spectrum of $S^{\mathbb{D}}$ differs by that of the operator $\mathcal{S}_m$ at most by $1/2$. This gives (19.75) and (19.76).

In order to prove the representation (19.74), we take the resolvent identity

$$(\tilde{\mathcal{S}}_m - \lambda)^{-1} = (S^{\mathbb{D}} - \lambda)^{-1} - (\tilde{\mathcal{S}}_m - \lambda)^{-1} \Delta \tilde{\mathcal{S}} (S^{\mathbb{D}} - \lambda)^{-1}, \quad (19.79)$$

form the contour integral and apply (19.76). This gives the result. $\square$

The next lemma relates the smoothness of an integral kernel to the boundedness of the product of the operator with powers of the vacuum Hamiltonian.

Lemma 19.4.3 *Let $A \in \mathcal{L}(\mathcal{H}_m)$ be an operator that maps smooth functions to smooth functions and has the property that for all $p, q \in \mathbb{N}$, the operator product*

$$H^q A H^p : C_0^{\infty}(\mathcal{N}, S\mathcal{M}) \rightarrow C^{\infty}(\mathcal{N}, S\mathcal{M}) \quad (19.80)$$

extends to a bounded linear operator on $\mathcal{H}_m$. Then, considering A as an operator on $\mathcal{H}_m$, this operator can be represented as an integral operator with a smooth integral kernel, that is,

$$(A\psi)(x) = \int_{\mathcal{N}} \mathcal{A}(x, (t_0, \vec{y})) \gamma^0 \psi(t_0, \vec{y}) d^3y \quad \text{with} \quad \mathcal{A} \in C^{\infty}(\mathcal{M} \times \mathcal{M}). \quad (19.81)$$

Proof Since in momentum space, the square of the Hamiltonian takes the form

$$H(\vec{k})^2 = \left(\gamma^0 (\vec{\gamma} \vec{k} + m) \right)^2 = (-\vec{\gamma} \vec{k} + m)(\vec{\gamma} \vec{k} + m) = |\vec{k}|^2 + m^2, \quad (19.82)$$

the wave function $\hat{\psi}$ defined by

$$\hat{\psi}(\vec{k}) := \frac{1}{|\vec{k}|^2 + m^2} e^{i\vec{k}\vec{x}_0} \Xi, \quad (19.83)$$

for a constant spinor Ξ and $\vec{x}_0 \in \mathbb{R}^3$, satisfies the equation

$$H^2 \psi(\vec{x}) = \delta^3(\vec{x} - \vec{x}_0) \Xi. \quad (19.84)$$

Moreover, one verifies immediately that $\psi \in \mathcal{H}_{t_0}$ is square-integrable. Using the last equation together with (19.80), we conclude that

$$H^q A(\delta^3(\vec{x} - \vec{x}_0) \Xi) = H^q A H^2 \psi \in \mathcal{H}_{t_0}. \quad (19.85)$$

Since q is arbitrary, it follows that A has an integral representation in the spatial variables,

$$(A\phi)(\vec{x}) = \int_{\mathcal{N}} \mathcal{A}(\vec{x}, \vec{y}) \gamma^0 \phi(\vec{y}) \, d^3y \quad \text{with} \quad \mathcal{A} \in C^\infty(\mathcal{N} \times \mathcal{N}). \quad (19.86)$$

We now extend this integral kernel to $\mathcal{M} \times \mathcal{M}$ by solving the Cauchy problem in the variables x and y . This preserves smoothness by the global existence and regularity results for linear hyperbolic equations, giving the result. $\square$

Lemma 19.4.4 *Under the assumptions of Theorem 19.1.1, for all $p \in \mathbb{N}$ the iterated commutator*

$$\mathcal{S}^{(p)} := \underbrace{\left[H, [H, \dots, [H, \tilde{S}_m] \dots] \right]}_{p \text{ factors}} \quad (19.87)$$

is a bounded operator on $\mathcal{H}_m$.

Proof In the vacuum, the Hamiltonian clearly commutes with the time evolution operator,

$$[H, U_m^{t,t'}] = 0. \quad (19.88)$$

In order to derive a corresponding commutator relation in the presence of the external potential, one must take into account that $\tilde{H}$ is time-dependent. For ease of notation, we do not write out this dependence but instead understand that the Hamiltonian is to be evaluated at the correct time, that is,

$$\tilde{U}_m^{t,t'} \tilde{H} \equiv \tilde{U}_m^{t,t'} \tilde{H}(t') \quad \text{and} \quad \tilde{H} \tilde{U}_m^{t,t'} \equiv \tilde{H}(t) \tilde{U}_m^{t,t'}. \quad (19.89)$$

Then

$$\begin{aligned} (i\partial_t - \tilde{H})(\tilde{H} \tilde{U}_m^{t,t'} - \tilde{U}_m^{t,t'} \tilde{H}) &= i\dot{\tilde{H}} \tilde{U}_m^{t,t'} \quad \text{and} \\ \tilde{H} \tilde{U}_m^{t,t'} - \tilde{U}_m^{t,t'} \tilde{H} \Big|_{t=t'} &= 0. \end{aligned} \quad (19.90)$$

Here, and in what follows, the dot denotes the partial derivative with respect to t . Solving the corresponding Cauchy problem gives

$$[\tilde{H}, \tilde{U}_m^{t,t'}] = \int_{t'}^t \tilde{U}_m^{t,\tau} \dot{\tilde{H}} \tilde{U}_m^{\tau,t'} d\tau. \quad (19.91)$$

In order to compute the commutator of H with the operator products in (17.23) and (17.24), we first differentiate the expression $U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}$ with respect to t ,

$$i\partial_t(U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) = iU_m^{t'',t} \dot{\mathcal{V}} \tilde{U}_m^{t,t'} + U_m^{t'',t} \mathcal{V} \tilde{H} \tilde{U}_m^{t,t'} - U_m^{t'',t} H \mathcal{V} \tilde{U}_m^{t,t'}. \quad (19.92)$$

Moreover, using the commutation relations (19.88) and (19.91), we obtain

$$\begin{aligned} & H(U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) - (U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) \tilde{H} \\ &= U_m^{t'',t} H \mathcal{V} \tilde{U}_m^{t,t'} - U_m^{t'',t} \mathcal{V} \tilde{H} \tilde{U}_m^{t,t'} + U_m^{t'',t} \mathcal{V} [\tilde{H}, \tilde{U}_m^{t,t'}] \\ &= iU_m^{t'',t} \dot{\mathcal{V}} \tilde{U}_m^{t,t'} - i\partial_t(U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) + \int_{t'}^t U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,\tau} \dot{\tilde{H}} \tilde{U}_m^{\tau,t'} d\tau, \end{aligned} \quad (19.93)$$

where in the last step, we applied (19.92). It follows that

$$\begin{aligned} [H, U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}] &= H(U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) - (U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) \tilde{H} + (U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) \mathcal{V} \\ &= iU_m^{t'',t} \dot{\mathcal{V}} \tilde{U}_m^{t,t'} + (U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) \mathcal{V} \\ &\quad - i\partial_t(U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,t'}) + \int_{t'}^t U_m^{t'',t} \mathcal{V} \tilde{U}_m^{t,\tau} \dot{\tilde{H}} \tilde{U}_m^{\tau,t'} d\tau. \end{aligned} \quad (19.94)$$

Proceeding in this way, one can calculate the commutator of H with all the terms in (17.23) and (17.24). We write the result symbolically as

$$[H, \tilde{\mathcal{S}}_m] = \mathcal{S}^{(1)}, \quad (19.95)$$

where $\mathcal{S}^{(1)}$ is a bounded operator. Higher commutators can be computed inductively, giving the result. $\square$

We point out that this lemma only makes a statement on the iterative commutators. Expressions like $[H^p, \tilde{\mathcal{S}}_m]$ or $H^q \tilde{\mathcal{S}}_m H^p$ will not be bounded operators in general. However, the next lemma shows that the operator $\Delta \tilde{\mathcal{S}}$ has the remarkable property that multiplying by powers of H from the left and/or right again gives a bounded operator.

Lemma 19.4.5 *Under the assumptions of Theorem 19.1.1, for all $p, q \in \mathbb{N} \cup \{0\}$, the product $H^q \Delta \tilde{\mathcal{S}} H^p$ is a bounded operator on $\mathcal{H}_m$.*

Proof We only consider the products $H^q \mathcal{S}_+^- H^p$ because the operator $\mathcal{S}_-^+$ can be treated similarly. Multiplying (19.91) from the left and right by the resolvent of H , we obtain

$$[(H - \mu)^{-1}, \tilde{\mathcal{S}}_m] = -(H - \mu)^{-1} \mathcal{S}^{(1)} (H - \mu)^{-1}. \quad (19.96)$$

Writing the result of Lemma 19.4.4 as

$$[H, \mathcal{S}^{(p)}] = \mathcal{S}^{(p+1)} \quad \text{with} \quad \mathcal{S}^{(p+1)} \in \mathcal{L}(\mathcal{H}), \quad (19.97)$$

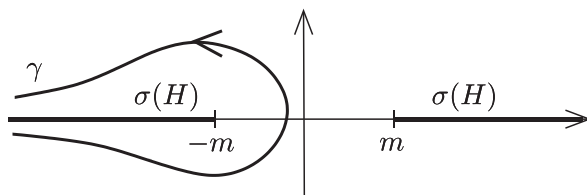


Figure 19.1 The contour γ . Reprinted from [81] with the permission of AIP Publishing.

yields more generally the commutation relations

$$[(H - \mu)^{-1}, \mathcal{S}^{(p)}] = -(H - \mu)^{-1} \mathcal{S}^{(p+1)} (H - \mu)^{-1} \quad \text{for } p \in \mathbb{N}. \quad (19.98)$$

Choosing a contour γ that encloses the interval $(-\infty, -m]$ as shown in Figure 19.1, one finds

$$\begin{aligned} H \mathcal{S}_+^- &= -\frac{1}{2\pi i} \int_{\gamma} \mu (H - \mu)^{-1} \tilde{\mathcal{S}}_m \chi^+(H) d\mu \\ &= \mathcal{S} H \chi^-(H) \chi^+(H) + \frac{1}{2\pi i} \int_{\gamma} \mu (H - \mu)^{-1} \mathcal{S}^{(1)} (H - \mu)^{-1} \chi^+(H) d\mu \\ &= \frac{1}{2\pi i} \int_{\gamma} \mu (H - \mu)^{-1} \mathcal{S}^{(1)} (H - \mu)^{-1} \chi^+(H) d\mu, \end{aligned} \quad (19.99)$$

where in the last step, we used that $\chi^-(H) \chi^+(H) = 0$. In order to show that this operator product is bounded, it is useful to employ the spectral theorem for H , which we write as

$$f(H) = \int_{\mathbb{R} \setminus [-m, m]} f(\lambda) dE_{\lambda}, \quad (19.100)$$

where dE_{λ} is the spectral measure of H . This gives

$$\begin{aligned} H \mathcal{S}_+^- &= \iint_{\mathbb{R} \times \mathbb{R}} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{\mu}{\lambda - \mu} \frac{1}{\lambda' - \mu} \chi^+(\lambda') dE_{\lambda'} \right) \mathcal{S}^{(1)} dE_{\lambda'} d\mu \\ &= - \iint_{\mathbb{R} \times \mathbb{R}} \frac{\lambda}{\lambda - \lambda'} \chi^-(\lambda) \chi^+(\lambda') dE_{\lambda} \mathcal{S}^{(1)} dE_{\lambda'}. \end{aligned} \quad (19.101)$$

Note that the term $\lambda - \lambda'$ is bounded away from zero. Thus, the factor $\lambda/(\lambda - \lambda')$ is bounded, showing that the operator $H \mathcal{S}_+^-$ is in $L(\mathcal{H}_m)$.

This method can be iterated. To this end, we first rewrite the product with commutators,

$$\begin{aligned} H^q \mathcal{S}_+^- &= \chi^-(H) (H^- \chi^-(H))^p \tilde{\mathcal{S}}_m \chi^+(H) \\ &= \chi^-(H) [H^-, [H^-, \dots, [H^-, \mathcal{S}] \dots]] \chi^+(H), \end{aligned} \quad (19.102)$$

where we used the abbreviation $H^- := H \chi^-(H)$. Multiplying from the right by H^p , we can commute factors $H^+ := H \chi^+(H)$ to the left to obtain

$$\begin{aligned} H^q \mathcal{S}_+^- H^p &= (-1)^p \chi^-(H) \\ &\quad \times \underbrace{[H^+, \dots, [H^+, [H^-, \dots, [H^-, \tilde{\mathcal{S}}_m] \dots]]}_{p \text{ factors}} \dots \chi^+(H). \end{aligned} \quad (19.103)$$

Representing each factor $H^\pm$ by a contour integral, one can compute the commutators inductively with the help (19.98). Applying the spectral theorem (19.100) to the left and right of the resulting factor $\mathcal{S}^{(p+q)}$ yields a constant times the expression

$$\iint_{\mathbb{R} \times \mathbb{R}} \chi^-(\lambda) \chi^+(\lambda') \, dE_\lambda \mathcal{S}^{(p+q)} \, dE_{\lambda'} \\ \times \oint_{\gamma_1} \frac{\mu_1 \, d\mu_1}{(\lambda - \mu_1)(\lambda' - \mu_1)} \cdots \oint_{\gamma_{p+q}} \frac{\mu_{p+q} \, d\mu_{p+q}}{(\lambda - \mu_{p+q})(\lambda' - \mu_{p+q})} . \quad (19.104)$$

Carrying out the contour integrals with residues, we obtain similar to (19.101) an expression of the form

$$H^q \mathcal{S}_+^{-1} H^p = \iint_{\mathbb{R} \times \mathbb{R}} f(\lambda, \lambda') \chi^-(\lambda) \chi^+(\lambda') \, dE_\lambda \mathcal{S}^{(p+q)} \, dE_{\lambda'} \quad (19.105)$$

with a bounded function f . This concludes the proof. $\square$

Proof of Theorem 19.4.1. It remains to show that the contour integral in (19.74) has a smooth integral kernel. To this end, we multiply the integrand from the left by H^q and from the right by H^p and commute the factors H iteratively to the inside. More precisely, we use the formula

$$H^q (\tilde{\mathcal{S}}_m - \lambda)^{-1} = \sum_{a=0}^q \underbrace{[H, \dots, [H, (\tilde{\mathcal{S}}_m - \lambda)^{-1}] \cdots]}_{a \text{ factors}} H^{q-a}, \quad (19.106)$$

note that the sum is telescopic; here, we use the convention that the summand for $a = 0$ is simply $(\tilde{\mathcal{S}}_m - \lambda)^{-1} H^q$. Hence,

$$H^q (\tilde{\mathcal{S}}_m - \lambda)^{-1} \Delta \tilde{\mathcal{S}} (S^\mathbb{D} - \lambda)^{-1} H^p \\ = \sum_{a=0}^q \sum_{b=0}^p \underbrace{[H, \dots, [H, (\tilde{\mathcal{S}}_m - \lambda)^{-1}] \cdots]}_{a \text{ factors}} \\ \times H^{q-a} \Delta \tilde{\mathcal{S}} H^{p-b} \underbrace{\left[\cdots [(S^\mathbb{D} - \lambda)^{-1}, H], \dots, H \right]}_{b \text{ factors}}. \quad (19.107)$$

According to Lemma 19.4.5, the intermediate product $H^{q-a} \Delta \tilde{\mathcal{S}} H^{p-b}$ is a bounded operator. Moreover, the commutators can be computed inductively by applying Lemma 19.4.4 and the formula

$$[H, (\tilde{\mathcal{S}}_m - \lambda^{-1})] = -(\tilde{\mathcal{S}}_m - \lambda^{-1}) [H, \tilde{\mathcal{S}}_m] (\tilde{\mathcal{S}}_m - \lambda^{-1}), \quad (19.108)$$

and similarly for $S^\mathbb{D}$. This gives operators that are all bounded for $\lambda \in \partial B_{\frac{1}{2}}(\pm 1)$. Since the integration contour is compact, the result follows. $\square$

19.4.2 Proof of the Hadamard Form

Relying on the frequency mixing estimates of the previous section, we can now give the proof of Theorem 19.1.1. Recall that the fermionic projector is given by (see (15.61))

$$P = -\chi^-(\tilde{\mathcal{S}}_m) \tilde{k}_m, \quad (19.109)$$

where we again used the short notation (19.71). Here again the operator $\chi^-(\tilde{\mathcal{S}}_m)$ acts on the solution space $\mathcal{H}_m$ of the Dirac equation, which can be identified with the space $\mathcal{H}_{t_0}$ of square integrable wave functions at time t_0 (see the beginning of Section 19.4.1). For the following arguments, it is important to note that this identification can be made at any time t_0 .

In order to prove that the bi-distribution corresponding to P is of Hadamard form, we compare the fermionic projectors for three different Dirac operators and use the theorem on the propagation of singularities in [138]. More precisely, we consider the following three fermionic projectors:

- (1) The fermionic projector P^{vac} in the Minkowski vacuum.
- (2) The fermionic projector $\check{P}$ in the presence of the external potential

$$\check{\mathcal{B}}(x) := \eta(x^0) \mathcal{B}(x), \quad (19.110)$$

where $\eta \geq 0$ is a smooth function with $\eta|_{(-\infty, 0)} \equiv 0$ and $\eta|_{(1, \infty)} \equiv 1$.

- (3) The fermionic projector P in the presence of the external potential $\mathcal{B}(x)$.

The potential $\check{\mathcal{B}}$ vanishes for negative times, whereas for times $x^0 > 1$ it coincides with $\mathcal{B}$. Thus, it smoothly interpolates between the dynamics with and without external potential. The specific form of the potential $\check{\mathcal{B}}$ in the transition region $0 \leq x^0 \leq 1$ is of no relevance for our arguments.

In the Minkowski vacuum, the relation (19.109) gives the usual two-point function composed of all negative-frequency solutions of the Dirac equation. It is therefore obvious that the bi-distribution $P^{\text{vac}}(x, y)$ is of Hadamard form.

We now compare P^{vac} with $\check{P}$. To this end, we choose an arbitrary time $t_0 < 0$. Then, applying the result of Theorem 19.4.1 to (19.109), we get

$$P^{\text{vac}} = -\chi^-(H) k_m \quad \text{and} \quad \check{P} = -\chi^-(H) \check{k}_m + (\text{smooth}), \quad (19.111)$$

where $\check{k}_m$ is the causal fundamental solution in the presence of the potential $\check{\mathcal{B}}$. Since $\check{\mathcal{B}}$ vanishes in a neighborhood of the Cauchy surface at time t_0 , we conclude that P^{vac} and $\check{P}$ coincide in this neighborhood up to a smooth contribution. It follows also that $\check{P}(x, y)$ is of Hadamard form in this neighborhood. Using the theorem on the propagation of singularities [138, Theorem 5.5], we conclude that $\check{P}(x, y)$ is of Hadamard form for all $x, y \in \mathcal{M}$.

Next, we compare $\check{P}$ with P . Thus, we choose an arbitrary time $t_0 > 1$. Using again the result of Theorem 19.4.1 in (19.109), we obtain

$$\begin{aligned} \check{P} &= -\chi^-(H) \check{k}_m + (\text{smooth}) & \text{and} \\ P &= -\chi^-(H) \tilde{k}_m + (\text{smooth}), \end{aligned} \quad (19.112)$$

where the smooth contributions may of course be different. Since $\check{\mathcal{B}}$ and $\mathcal{B}$ coincide in a neighborhood of the Cauchy surface at time t_0 , we infer that $\check{P}$ and P coincide in this neighborhood up to a smooth contribution. As a consequence, $P(x, y)$ is

of Hadamard form in this neighborhood. Again applying [138, Theorem 5.5], it follows that $P(x, y)$ is of Hadamard form for all $x, y \in \mathcal{M}$. This concludes the proof of Theorem 19.1.1.

19.5 Exercises

Exercise 19.1 This exercise explains the notion of the *light-cone expansion* in simple examples.

- (a) What is the light-cone expansion of a smooth function on $\mathcal{M} \times \mathcal{M}$? In which sense is it trivial? In which sense is it nonunique?
- (b) Show that $A(x, y) = \log(|y - x|^2)$ is a well-defined distribution on $\mathcal{M} \times \mathcal{M}$. What is the order on the light cone? Write down a light-cone expansion.
- (c) Now consider the distributional derivatives

$$\left(\frac{\partial}{\partial x^0}\right)^p A(x, y) \quad \text{with} \quad p \in \mathbb{N}, \quad (19.113)$$

and $A(x, y)$ as in part (b). What is the order on the light cone? Write down a light-cone expansion.

- (d) Consider the function

$$E(x, y) = \sin((y - x)^2) \log(|y - x|^2). \quad (19.114)$$

Determine the order on the light cone and give a light-cone expansion.

- (e) Consider the function

$$E(x, y) = \begin{cases} e^{-\frac{1}{(y-x)^2}} & \text{if } (y - x)^2 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (19.115)$$

Determine the order on the light cone and give a light-cone expansion.

- (f) Show that the expression

$$\lim_{\varepsilon \searrow 0} \frac{\log(|y - x|^2)}{(y - x)^4 + i\varepsilon} \quad (19.116)$$

is a well-defined distribution on $\mathcal{M} \times \mathcal{M}$. Derive its light-cone expansion.

Exercise 19.2 (*Understanding the light-cone expansion*) This exercise aims to familiarize you with some of the particularities of the light-cone expansion.

- (a) Let $A(x, y) := (x - y)^{2k_0}$ with $k_0 \in \mathbb{Z}$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $A(x, y)$ and prove that it is one.
- (b) Let $B(x, y) := (x - y)^{2k_0} + (x - y)^{2k_1}$, where $k_0, k_1 \in \mathbb{Z}$ and $k_0 < k_1$. Which order(s) on the light cone is this? (Prove your answer.) Construct a light-cone expansion of $B(x, y)$ and prove that it is one.
- (c) Let $C(x, y) := (x - y)^{2k_0} f(x, y) + (x - y)^{2k_1} g(x, y)$, where f and g are smooth functions in x and y and k_0, k_1 as above. Construct a light-cone expansion of $C(x, y)$ and prove that it is one.

- (d) Let $D(x, y) := \sin((x - y)^2) (x - y)^2$. Use your results from (b) and (c) to construct two different light-cone expansions of $D(x, y)$. Why might this nonuniqueness not be a problem for the scope of this book?
- (e) Finally, consider the function

$$E(x, y) = \sin((y - x)^2) + \begin{cases} e^{-\frac{1}{(y-x)^2}} & \text{if } (y - x)^2 \geq 0, \\ 0 & \text{else.} \end{cases} \quad (19.117)$$

Determine its order on the light cone and derive a light-cone expansion.

Hint: For (d) and (e): Expand the sine function.

Exercise 19.3 This exercise is devoted to computing the Fourier transform of the *advanced Green's operator* (19.36) and deriving the series expansion (19.38).

- (a) We again set $\xi = y - x$ and $\xi = (t, \vec{\xi})$ with $t > 0$. Moreover, we choose polar coordinates $r = (|\vec{\xi}|, \vartheta, \varphi)$. Carry out the ω -integration with residues and compute the angular integrals to obtain

$$S_{m^2}^\vee(x, y) = \frac{i}{8\pi r} \int_0^\infty \frac{p}{\omega(p)} (e^{-ipr} - e^{ipr}) (e^{i\omega(p)t} - e^{-i\omega(p)t}) dp, \quad (19.118)$$

where $p = |\vec{p}|$ and $\omega(p) := \sqrt{|\vec{p}|^2 + m^2}$. Justify this integral as the Fourier transform of a distribution and show that

$$S_{m^2}^\vee(x, y) = \frac{i}{8\pi r} \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-\varepsilon p} \frac{p}{\omega(p)} (e^{-ipr} - e^{ipr}) (e^{i\omega(p)t} - e^{-i\omega(p)t}) dp, \quad (19.119)$$

with convergence as a distribution.

- (b) Verify (19.37) in the case $m = 0$ by setting $\omega(p) = p$ and using (16.61).
- (c) In order to analyze the behavior away from the light cone, it is most convenient to take the limit $r \searrow 0$ and use Lorentz invariance. Show that in this limit,

$$S_{m^2}^\vee(x, y) = \frac{1}{4\pi} \lim_{\varepsilon \searrow 0} \int_0^\infty e^{-\varepsilon p} \frac{p^2}{\omega(p)} (e^{i\omega(p)t} - e^{-i\omega(p)t}) dp \quad (19.120)$$

$$= \frac{1}{4\pi} \lim_{\varepsilon \searrow 0} \int_m^\infty e^{-\varepsilon p} \sqrt{\omega^2 - m^2} (e^{i\omega t} - e^{-i\omega t}) d\omega. \quad (19.121)$$

Compute this integral using [96, formula (3.961.1)]. Use the relations between Bessel functions [124, (10.27.6), (10.27.11)] to obtain (19.37) away from the light cone.

As an alternative method for computing the Fourier integral, one can begin from the integral representation for J_0 in [124, (10.9.12)], differentiate with respect to x and use [124, (10.6.3)].

- (d) Combine the results of (b) and (c) to prove (19.37). Why is there no additional contribution at $\xi = 0$?
- (e) Use the series expansion [124, (10.2.2)] to derive (19.38).
- (f) The series expansion (19.38) can also be derived without using Bessel functions. To this end, one expands (19.120) in powers of m^2 and computes the

Fourier transform term by term. Verify explicitly that this procedure really gives (19.38).

Exercise 19.4 In this exercise, we illustrate the dependence of the light-cone expansion (19.55) on the function V . We choose $l = 0$, $r \in \mathbb{N}_0$ arbitrary, $x = 0$ and $V(z) = z^2$.

- (a) Show that the left-hand side of (19.55) vanishes for all $y \in M$. *Hint:* Use the causal structure of S_a as given in (19.37) in the massless case.
- (b) Show that all the summands in (19.55) for $n \geq 2$ vanish.
- (c) Show that the summands in (19.55) for $n = 0$ and $n = 1$ are both nonzero and cancel each other. *Hint:* Compute $\square_z z^2$. Moreover, make use of the relation between the coefficients of the power series (19.38) for $j = r + 1$ and $j = r + 2$. It might be a good idea to begin with the case $r = 0$.