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Robustness of nonuniform mean-square exponential dichotomies

Hailong Zhu
Anhui University of Finance and Economics, Bengbu 233030, China
(hai-long-zhu@163.com)

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For linear stochastic differential equations with bounded coefficients, we establish the robustness of nonuniform mean-square exponential dichotomy (NMS-ED) on $[t_0, +\infty)$, $(-\infty, t_0]$ and the whole $\mathbb{R}$ separately, in the sense that such an NMS-ED persists under a sufficiently small linear perturbation. The result for the nonuniform mean-square exponential contraction is also discussed. Moreover, in the process of proving the existence of NMS-ED, we use the observation that the projections of the 'exponential growing solutions' and the 'exponential decaying solutions' on $[t_0, +\infty)$, $(-\infty, t_0]$ and $\mathbb{R}$ are different but related. Thus, the relations of three types of projections on $[t_0, +\infty)$, $(-\infty, t_0]$ and $\mathbb{R}$ are discussed.

Keywords: robustness; nonuniform mean-square exponential contraction; nonuniform mean-square exponential dichotomy; stochastic differential equations

2020 *Mathematics Subject Classification:* 60H10; 34D09

1. Introduction

The well-established notion of exponential dichotomy used in the analysis of nonautonomous systems essentially originated from the work of Perron [**41**]. The theory of exponential dichotomy is a powerful tool to describe hyperbolicity of dynamical systems generated by differential equations, especially for the stable and unstable invariant manifolds of time-dependent systems. As mentioned in Coppel [**12**], 'that dichotomies, rather than Lyapunov's characteristic exponents, are the key to questions of asymptotic behaviour for nonautonomous differential equations'.

Over the years, the classical exponential dichotomy and its properties have been established for evolution equations [**24**, **30**, **40**, **47**–**49**], functional differential equations [**11**, **31**, **42**], skew-product flows [**9**, **10**, **29**, **50**] and random systems or stochastic equations [**14**, **53**, **54**, **58**, **59**]. We also refer to the books [**8**, **12**, **36**] for details and further references related to exponential dichotomies.

However, dynamical systems exhibit various different kinds of dichotomic behaviour and the classical notion of exponential dichotomy substantially restricts some dynamics. In order to investigate more general hyperbolicity, many attempts (see e.g. [**37**, **38**, **46**]) have been made to extend the concept of classical dichotomies. Inspired by the work of Barreira and Pesin on the notion of nonuniformly hyperbolic trajectory [**1**, **2**], Barreira and Valls extended the concept of exponential dichotomy to the nonuniform ones and investigated some related problems, see for examples, the works [3–7] and the references therein.

On the other hand, from the point of view of Itô stochastic differential equations (SDE), such properties of mean-square are natural since the Itô stochastic calculus is essentially deterministic in the mean-square setting, and there exist stationary coordinate changes under which flows of nonautonomous random differential equation can be viewed as those of SDE [25]. Some related works on mean-square setting of random systems or stochastic equations can be found in [17, 21–23, 27, 33, 57]. To the best of our knowledge, mean-square exponential dichotomy (MS-ED) was first introduced by Stanzhyts'kyi [51], in which a sufficient condition has been proved to ensure that a linear SDE satisfies an MS-ED. Based on the definition of MS-ED, Stanzhyts'kyi and Krenevych [52] proved the existence of a quadratic form of linear SDE. In [58] the robustness of MS-ED for a linear SDE was established. Stoica [53] studied stochastic cocycles in Hilbert spaces. Recently, Doan *et al.* [14] considered the MS-ED spectrum for random dynamical system.

Now we recall the definition of MS-ED. Consider the following linear n-dimensional Itô stochastic system

$$\mathrm{d}x(t) = A(t)x(t)\mathrm{d}t + G(t)x(t)\mathrm{d}\omega(t), \quad t \in I, \tag{1.1}$$

where I is either the half line $[t_0, +\infty)$, $(-\infty, t_0]$ or the whole $\mathbb{R}$, and $A(t) = (A_{ij}(t))_{n\times n}$, $G(t) = (G_{ij}(t))_{n\times n}$ are continuous functions with real entries. Equation (1.1) is said to possess an *MS-ED* if there exists a linear projection $P(t) : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$ such that

$$\Phi(t)\Phi^{-1}(s)P(s) = P(t)\Phi(t)\Phi^{-1}(s), \quad \forall\, t, s \in I, \tag{1.2}$$

and positive constants K, α such that

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 \leqslant Ke^{-\alpha(t-s)}, \quad \forall\, (t, s) \in I^2_{\geqslant},$$
$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)Q(s)\|^2 \leqslant Ke^{-\alpha(s-t)}, \quad \forall\, (t, s) \in I^2_{\leqslant},$$

where $\Phi(t)$ is a fundamental matrix solution of (1.1), and $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$ for each $t \in I$. $I^2_{\geqslant} := \{(t, s) \in I^2 : t \geqslant s\}$ and $I^2_{\leqslant} := \{(t, s) \in I^2 : t \leqslant s\}$ denotes the relations of s and t on I.

Inspired by the above, this paper is to study the robustness of NMS-ED. (1.1) is said to possess an *NMS-ED* if there exists a linear projection $P(t) : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$ such that (1.2) holds, and some constants M, $\alpha > 0$, $\varepsilon \geqslant 0$ such that

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 \leqslant Me^{-\alpha(t-s)+\varepsilon|s|}, \quad \forall\, (t, s) \in I^2_{\geqslant}, \tag{1.3}$$

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)Q(s)\|^2 \leqslant Me^{-\alpha(s-t)+\varepsilon|s|}, \quad \forall\, (t, s) \in I^2_{\leqslant}, \tag{1.4}$$

where $\Phi(t)$ is a fundamental matrix solution of (1.1), $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection of $P(t)$ for each $t \in I$. $I^2_{\geqslant} := \{(t, s) \in I^2 : t \geqslant s\}$ and $I^2_{\leqslant} := \{(t, s) \in I^2 : t \leqslant s\}$ denotes the relations of s and t on I. For convenience, the constants α and K in (1.3)–(1.4) are called the *exponent* and the *bound* of the NMS-ED, respectively, as in the case of deterministic systems [20]. ε is called the *nonuniform degree* of the NMS-ED. In particular, while $\varepsilon = 0$, we obtain the notion of (uniform) MS-ED. We refer to [**51**–**53**, **57**–**59**] for related results and techniques about this topic.

It is clear that the notion of NMS-ED is a weaker requirement in comparison to the notion of MS-ED. Actually, there exists a linear SDE which has an NMS-ED with nonuniform degree ε cannot be removed. For example, let $a > b > 0$ be real parameters,

$$\begin{cases} \mathrm{d}u &= (-a - bt\sin t)u(t)\mathrm{d}t + \sqrt{2b\cos t}\exp(-at + bt\cos t)\mathrm{d}\omega(t), \\ \mathrm{d}v &= (a + bt\sin t)v(t)\mathrm{d}t - \sqrt{2b\cos t}\exp(at - bt\cos t)\mathrm{d}\omega(t) \end{cases}$$

admits an NMS-ED which is not uniform. See example 6.1 in § 6 for details.

Robustness (also known as roughness , see e.g. [**12**]) here means that an NMS-ED persists under a sufficiently small linear perturbation. More precisely, for small perturbations B, H, the following linear SDE

$$dy(t) = (A(t) + B(t))y(t)\mathrm{d}t + (G(t) + H(t))y(t)\mathrm{d}\omega(t) \tag{1.5}$$

also admits an NMS-ED. As indicated by Coppel [**12**, p. 28], the robustness of exponential dichotomies was first proved by Massera and Schäffer [**36**], which states that all 'neighbouring' linear systems also have the same dichotomy with a similar projection if the same happens for the original system. Robustness is one of the most basic concepts appearing in the theoretical studies of dynamical systems. This topic plays a key role in the stability theory for dynamical systems. Some early papers about robustness (with the exception of [**12**] and [**36**] mentioned above) are Dalec'kiĭ and Kreĭn [**13**] and Palmer [**39**] for ordinary differential equations, Henry [**20**] and Lin [**32**] for parabolic partial differential equations, Hale and Lin [**19**] and Lizana [**34**] for functional differential equations, Pliss and Sell [**43**] and Chow and Leiva [**10**] for skew-product semiflow. For more recent works refer to papers [**5**, **7**, **26**, **44**, **45**, **55**, **56**]. It is worth mentioning that on half line $\mathbb{R}^+$, $\mathbb{R}^-$ as well as the whole $\mathbb{R}$, Ju and Wiggins [**26**] and Popescu [**44**, **45**] considered the case of roughness for exponential dichotomy and analyse their dynamical behaviour; Zhou *et al.* [**55**] discussed the relationship between nonuniform exponential dichotomy and admissibility.

In this study, we extend the results and improve the method of [**58**]. The main differences of our results and those of [**58**] are as follows:

- In contrast to [**58**], we extend the case of robustness of MS-ED to the general nonuniform setting. For this purpose, we need to pass from small bounded perturbations of the coefficient matrix to exponentially decaying perturbations.

- In [**58**], we only consider the case of robustness on the whole line $\mathbb{R}$. In the present paper, we prove the robustness of (1.5) on half line $[t_0, +\infty)$, $(-\infty, t_0]$ and the whole $\mathbb{R}$. The proof is much more delicate than that of MS-ED [**58**]. This is because in different intervals, the different but related explicit expressions of the projections of the 'exponential growing solutions' and the 'exponential decaying solutions' for the perturbed equation (1.5) need to be determined first.

- Furthermore, in contrast to paper [**58**], we analyse and compare the results obtained from operators that make up the projections of (1.1) and (1.5) on different intervals (see theorem 3.10 and remark 5.9), and estimate the distance between the solution of (1.1) and the perturbed solution of (1.5) (see theorem 3.11 and remark 3.12).

The remaining part of this paper is organized as follows. The robustness of nonuniform mean-square exponential contraction (NMS-EC) is established in § 2. Section 3 proves the robustness of NMS-ED on half line $[t_0, +\infty)$ and analyses that the solution of (1.1) and the perturbed solution of (1.5) are forward asymptotic in the mean-square sense. The robustness under the nonuniform setting on half line $(-\infty, t_0]$ is presented in § 4. Section 5 combines the advantages of the projections on half line $[t_0, +\infty)$ and $(-\infty, t_0]$, and proves the robustness of NMS-ED on the whole $\mathbb{R}$. In addition, the relationship of the projections on $[t_0, +\infty)$, $(-\infty, t_0]$ and $\mathbb{R}$ is also discussed in § 5. Finally, an example is given in § 6, which indicates that there exists a linear SDE which admits an NMS-ED but not uniform.

2. Robustness of NMS-EC

In this section we will answer the following question: Does (1.5) admit an NMS-EC if (1.1) admits an NMS-EC while B, H is small? That is to say, we consider the robustness of NMS-EC. The following statement is a particular case of NMS-ED with projection $P(t) = Id$ for every $t \in I$. (1.1) is said to admit an *NMS-EC* if for some constants M, $\alpha > 0$ and $\varepsilon \geqslant 0$ such that

$$\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 \leqslant Me^{-\alpha(t-s)+\varepsilon|s|}, \quad \forall\ (t, s) \in I^2_{\geqslant}. \tag{2.1}$$

In particular, when $\varepsilon = 0$ in (2.1), we obtain the notion of uniform mean-square exponential contraction.

Throughout this paper, we assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability space, $\omega(t) = (\omega_1(t), \dots \omega_n(t))^T$ is an n-dimensional Brownian motion defined on the space $(\Omega, \mathscr{F}, \mathbb{P})$. $\|\cdot\|$ is used to denote both the Euclidean vector norm or the matrix norm as appropriate, and $L^2(\Omega, \mathbb{R}^n)$ stands for the space of all $\mathbb{R}^n$-valued random variables $x : \Omega \to \mathbb{R}^n$ such that

$$\mathbb{E}\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty.$$

In order to describe the robustness in an explicit form, we present the following theorem, which shows that the NMS-EC is robust under sufficiently small linear perturbations. Here we mention that the NMS-EC considered in this section is in an arbitrary interval $I \subset \mathbb{R}$.

THEOREM 2.1. *Let $A(\cdot)$, $B(\cdot)$, $G(\cdot)$, $H(\cdot)$ be $n \times n$-matrix continuous functions with real entries such that (1.1) admits an NMS-EC (2.1) with coefficient matrix bounded and perturbation exponential decaying in I, i.e. there exist constants a, b, g, $h > 0$ such that*

$$\|A(t)\| \leqslant a, \quad \|G(t)\| \leqslant g, \quad \|B(t)\| \leqslant be^{-\frac{\varepsilon|t|}{2}}, \quad \|H(t)\| \leqslant he^{-\frac{\varepsilon|t|}{2}}, \quad t \in I.$$

Let b, h be small enough such that

$$\tilde{M} := 8b^2 + 8g^2h^2 + \alpha h^2 < \frac{\alpha^2}{6M}. \tag{2.2}$$

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Then (1.5) also admits an NMS-EC in I with the bound M replaced by $3M$, and exponent α replaced by $-\frac{\alpha}{2}+\frac{3M\tilde{M}}{\alpha}$, i.e.

$$\mathbb{E}\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\|^2 \leqslant 3Me^{(-\frac{\alpha}{2}+\frac{3M\tilde{M}}{\alpha})(t-s)+\varepsilon|s|}, \quad \forall\ (t,s)\in I^2_{\geqslant}, \tag{2.3}$$

where $\hat{\Phi}(t)$ is a fundamental matrix solution of (1.5).

Proof. Write

$$\hat{\Phi}(t,s)=\hat{\Phi}(t)\hat{\Phi}^{-1}(s).$$

One can easily verify that $\hat{\Phi}(t,\,s)$ is a fundamental matrix solution of (1.5) with $\hat{\Phi}(s,\,s)=Id$. $L^2(\Omega,\,\mathbb{R}^n)$ is a Banach space with the norm $(\mathbb{E}\|x\|^2)^{\frac{1}{2}}$. The Banach algebra of bounded linear operators on $L^2(\Omega,\,\mathbb{R}^n)$ is denoted by $\mathfrak{B}(L^2(\Omega,\,\mathbb{R}^n))$. Now we introduce the space

$$\mathscr{L}_c := \{\hat{\Phi} : I^2_{\geqslant} \to \mathfrak{B}(L^2(\Omega,\mathbb{R}^n)) :\ \hat{\Phi} \text{ is continuous and } \|\hat{\Phi}\|_c<\infty\} \tag{2.4}$$

with the norm

$$\|\hat{\Phi}\|_c = \sup\left\{(\mathbb{E}\|\hat{\Phi}(t,s)\|^2)^{\frac{1}{2}}e^{-\frac{\varepsilon}{2}|s|} : (t,s)\in I^2_{\geqslant}\right\}. \tag{2.5}$$

Clearly, $(\mathscr{L}_c,\,\|\cdot\|_c)$ is a Banach space. In order to state our result, we need the following existence and uniqueness lemma.

LEMMA 2.2. *For any given initial value $\xi_0\in\mathbb{R}^n$, (1.5) has a unique solution $\hat{\Phi}(t,\,s)\xi_0$ with $\hat{\Phi}\in(\mathscr{L}_c,\,\|\cdot\|_c)$ such that*

$$\begin{aligned}\hat{\Phi}(t,s) &= \Phi(t)\Phi^{-1}(s)+\int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\omega(\tau)\\ &\quad+\int_s^t \Phi(t)\Phi^{-1}(\tau)\big(B(\tau)-G(\tau)H(\tau)\big)\hat{\Phi}(\tau,s)\mathrm{d}\tau\end{aligned} \tag{2.6}$$

with $\hat{\Phi}(s,\,s)\xi_0=\Phi(s)\Phi^{-1}(s)\xi_0=\xi_0$.

Proof. In what follows (in order to simplify the presentation), write $\tilde{B}(t)=B(t)-G(t)H(t)$. We first prove that the function $\hat{\Phi}(t,\,s)\xi_0$ is a solution of (1.5). Set

$$\begin{aligned}\xi(t) &= \Phi^{-1}(s)\xi_0+\int_s^t \Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\xi_0\mathrm{d}\omega(\tau)\\ &\quad+\int_s^t \Phi^{-1}(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\xi_0\mathrm{d}\tau.\end{aligned}$$

Let $y(t)=\Phi(t)\xi(t)$. Clearly,

$$\hat{\Phi}(t,s)\xi_0=\Phi(t)\xi(t)=y(t).$$

One can easily verify that $\xi(t)$ satisfies the differential

$$d\xi(t)=\Phi^{-1}(t)\big(B(t)-G(t)H(t)\big)y(t)dt+\Phi^{-1}(t)H(t)y(t)\mathrm{d}\omega(t).$$

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Since $\Phi(t)$ is a fundamental matrix solution of (1.1), it follows from Itô product rule that

$$\begin{aligned} dy(t) &= d\Phi(t)\xi(t) + \Phi(t)d\xi(t) + G(t)\Phi(t)\Phi^{-1}(t)H(t)y(t)\mathrm{d}t \\ &= A(t)y(t)\mathrm{d}t + G(t)y(t)\mathrm{d}\omega(t) + \big(B(t) - G(t)H(t)\big)y(t)\mathrm{d}t \\ &\quad + H(t)y(t)\mathrm{d}\omega(t) + G(t)H(t)y(t)\mathrm{d}t \\ &= (A(t) + B(t))y(t)\mathrm{d}t + (G(t) + H(t))y(t)\mathrm{d}\omega(t), \end{aligned}$$

which means that $y(t) = \hat{\Phi}(t,\, s)\xi_0$ is a solution of (1.5). This conclusion is consistent with that in [**35**, theorem 3.3.1] (see also [**28**, section 2.4.2]).

Now we prove that $\hat{\Phi}$ is unique in $(\mathscr{L}_c,\, \|\cdot\|_c)$. Let

$$\begin{aligned} (\Gamma\hat{\Phi})(t,s) = \Phi(t)\Phi^{-1}(s) &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\omega(\tau) \\ &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\tau. \end{aligned}$$

It follows from (2.1), $\mathbb{E}\|x\| \leqslant \sqrt{\mathbb{E}\|x\|^2}$, Cauchy–Schwarz inequality, Itô isometry property of stochastic integrals and the elementary inequality

$$\left\|\sum_{k=1}^m a_k\right\|^2 \leqslant m\sum_{k=1}^m \|a_k\|^2 \tag{2.7}$$

that

$$\begin{aligned} \mathbb{E}\|(\Gamma\hat{\Phi})(t,s)\|^2 &\leqslant 3\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 + 3\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\ &\quad + 3\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\tau\right\|^2 \\ &\leqslant 3Me^{-a(t-s)+\varepsilon|s|} + 3\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2\mathrm{d}\tau \\ &\quad + 3\left(\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|\mathbb{E}\|\tilde{B}(\tau)\|\mathrm{d}\tau\right) \\ &\quad \times\left(\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|\mathbb{E}\|\tilde{B}(\tau)\|\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2\mathrm{d}\tau\right) \\ &\leqslant 3Me^{-\alpha(t-s)+\varepsilon|s|} + 3Me^{\varepsilon|s|}\sup_{(\tau,s)\in I^2_{\geqslant}}\left(\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2e^{-\varepsilon|s|}\right) \\ &\quad \times\left(h^2\int_s^t e^{-\alpha(t-\tau)}\mathrm{d}\tau + 2(b^2+g^2h^2)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathrm{d}\tau\right)^2\right) \\ &\leqslant 3Me^{\varepsilon|s|} + 3Me^{\varepsilon|s|}\left(\frac{\alpha h^2 + 8b^2 + 8g^2h^2}{\alpha^2}\right)\sup_{(\tau,s)\in I^2_{\geqslant}}\left(\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2e^{-\varepsilon|s|}\right), \end{aligned}$$

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and this implies that

$$\mathbb{E}\|(\Gamma\hat{\Phi})(t,s)\|^2 e^{-\varepsilon s} \leqslant 3M + \frac{3\tilde{M}M}{\alpha^2} \sup_{(\tau,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2 e^{-\varepsilon|s|}\right) < \infty$$

with $\tilde{M} = 8b^2 + 8g^2h^2 + \alpha h^2$. Following the same procedure above, for any $\hat{\Phi}_1,\ \hat{\Phi}_2 \in \mathscr{L}_c$, we have

$$\|\Gamma\hat{\Phi}_1 - \Gamma\hat{\Phi}_2\|_c^2 \leqslant \frac{3\tilde{M}M}{\alpha^2} \sup_{(\tau,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|\hat{\Phi}_1(\tau,s) - \hat{\Phi}_1(\tau,s)\|^2 e^{-\varepsilon|s|}\right). \tag{2.8}$$

Note that

$$\begin{aligned}
&\sup_{(t,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|\hat{\Phi}_1(t,s) - \hat{\Phi}_1(t,s)\|^2 e^{-\varepsilon|s|}\right) \\
&\quad = \sup_{(t,s)\in I^2_{\geqslant}} \left((\mathbb{E}\|\hat{\Phi}_1(t,s) - \hat{\Phi}_1(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon|s|}{2}}\right)^2 \\
&\quad \leqslant \left(\sup_{(t,s)\in I^2_{\geqslant}} (\mathbb{E}\|\hat{\Phi}_1(t,s) - \hat{\Phi}_1(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon|s|}{2}}\right)^2 \\
&\quad = \|\hat{\Phi}_1 - \hat{\Phi}_2\|_c^2,
\end{aligned}$$

which together with (2.8) implies

$$\|\Gamma\hat{\Phi}_1 - \Gamma\hat{\Phi}_2\|_c \leqslant \sqrt{\frac{3\tilde{M}M}{\alpha^2}} \|\hat{\Phi}_1 - \hat{\Phi}_2\|_c.$$

Since $\tilde{M} < \frac{\alpha^2}{3M}$, Γ is a contraction operator. Hence, there exists a unique $\hat{\Phi} \in \mathscr{L}_c$ such that $\Gamma\hat{\Phi} = \hat{\Phi}$, which satisfies the identity (2.6). This completes the proof of the lemma. □

We proceed with the proof of the theorem. Squaring both sides of (2.6), and taking expectations, it follows from (2.7) that

$$\begin{aligned}
\mathbb{E}\|\hat{\Phi}(t,s)\|^2 &\leqslant 3\mathbb{E}\|\Phi(t)\Phi^{-1}(s)\|^2 + 3\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\
&\quad + 3\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\tau\right\|^2.
\end{aligned} \tag{2.9}$$

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By using Itô isometry property and inequalities (2.1), the second term of the right-hand side in (2.9) can be deduced as follows:

$$\begin{aligned}\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\ &= \int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2\mathrm{d}\tau \\ &\leqslant Mh^2\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2\mathrm{d}\tau.\end{aligned}$$

As to the third term in (2.9), it follows from $\mathbb{E}\|x\| \leqslant \sqrt{\mathbb{E}\|x\|^2}$, Cauchy–Schwarz inequality and the inequalities (2.1) that

$$\begin{aligned}&\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\tau\right\|^2 \\ &\quad\left\|\int_s^t \left(\Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\right)^{\frac{1}{2}}\left(\left(\Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\right)^{\frac{1}{2}}\hat{\Phi}(\tau,s)\right)\mathrm{d}\tau\right\|^2 \\ &\quad\leqslant \left(\int_s^t \mathbb{E}\left\|\Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\right\|\mathrm{d}\tau\right) \\ &\qquad\times\left(\int_s^t \mathbb{E}\left\|\Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\right\|\mathbb{E}\left\|\hat{\Phi}(\tau,s)\right\|^2\mathrm{d}\tau\right) \\ &\quad\leqslant 2M(b^2+g^2h^2)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathrm{d}\tau\right)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\right\|^2\mathrm{d}\tau\right) \\ &\quad\leqslant \frac{4M(b^2+g^2h^2)}{\alpha}\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\right\|^2\mathrm{d}\tau.\end{aligned}$$

Since $\alpha > 0$, we can rewrite inequality (2.9) as

$$\begin{aligned}\|\hat{\Phi}(t,s)\|^2 &\leqslant 3Me^{-\alpha(t-s)+\varepsilon|s|} + 3Mh^2\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\right\|^2\mathrm{d}\tau \\ &\quad+\frac{12M(b^2+g^2h^2)}{\alpha}\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\right\|^2\mathrm{d}\tau \\ &\leqslant 3Me^{-\frac{\alpha}{2}(t-s)+\varepsilon|s|} + 3M\left(\frac{\alpha h^2+8b^2+8g^2h^2}{\alpha}\right)\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\right\|^2\mathrm{d}\tau.\end{aligned} \tag{2.10}$$

Let

$$x(t) = \mathbb{E}\|\hat{\Phi}(t,s)\|^2, \quad X(t) = 3Me^{-\frac{\alpha}{2}(t-s)+\varepsilon|s|} + \frac{3M\tilde{M}}{\alpha}\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}x(\tau)\mathrm{d}\tau \tag{2.11}$$

for any fixed $s \in I$ with $\tilde{M} = \alpha h^2 + 8b^2 + 8g^2h^2$. Clearly, inequality (2.10) can be rewritten as

$$x(t) \leqslant X(t), \quad \text{for all } (t,s) \in I^2_{\geqslant}.$$

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On the contrary,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = -\frac{\alpha}{2}X(t) + \frac{3M\tilde{M}}{\alpha}x(t),$$

and therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) \leqslant \left(\frac{3M\tilde{M}}{\alpha} - \frac{\alpha}{2}\right) X(t).$$

Integrating the above inequality from s to t and note that $X(s) = 3Me^{\varepsilon|s|}$, we obtain

$$x(t) \leqslant X(t) \leqslant 3Me^{\varepsilon|s|}e^{(-\frac{\alpha}{2}+\frac{3M\tilde{M}}{\alpha})(t-s)}, \quad \text{for all } (t,s) \in I^2_{\geqslant}. \tag{2.12}$$

By (2.12), using (2.11), we obtain the desired inequality (2.3), and this completes the proof of the theorem. □

REMARK 2.3. Since the nonuniform degree $\varepsilon > 0$ exists for $(t, s) \in I^2_{\geqslant}$, the perturbations B and H should be chosen with exponential decaying to eliminate the effect caused by the nonuniform degree. For the uniform case, it suffices to consider the bounded condition instead of exponential decaying. See [**58**] for details about the case of $\varepsilon = 0$, which generalizes (and imitates) the notion of robustness of exponential dichotomy for ODE (see e.g. [**12**, **36**]).

As a special case of (1.5), if we consider the system

$$dy(t) = (A(t) + B(t))y(t)\mathrm{d}t + G(t)y(t)\mathrm{d}\omega(t), \tag{2.13}$$

in which the linear perturbed term only appears in the 'drift'. Of course, theorem 2.1 can also be applied to (2.13) but merely with the development of slightly better estimation (with the bound and the exponent replaced by smaller constants) than the one in theorem 2.1, since there is no perturbation in the 'volatility'. Actually, for any given initial value $\xi_0 \in \mathbb{R}^n$, (2.13) has a unique solution $\hat{\Phi}(t, s)\xi_0$ with $\hat{\Phi} \in (\mathscr{L}_c, \|\cdot\|_c)$ such that

$$\hat{\Phi}(t,s) = \Phi(t)\Phi^{-1}(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)B(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\tau$$

instead of (2.6), which is more similar to solutions of the classical ordinary differential equations (see e.g. [**18**]).

THEOREM 2.4. *Let $A(\cdot)$, $B(\cdot)$, $G(\cdot)$ be $n \times n$-matrix continuous functions with real entries such that (1.1) admits an NMS-EC (2.1) with coefficient matrix bounded and perturbation exponential decaying in I, i.e. there exist constants a, b, $g > 0$ such that*

$$\|A(t)\| \leqslant a, \quad \|G(t)\| \leqslant g, \quad \|B(t)\| \leqslant be^{-\frac{\varepsilon|t|}{2}}, \quad t \in I.$$

If $b < \alpha/(2\sqrt{2M})$, then (2.13) also admits an NMS-EC in I with the bound M replaced by $2M$, and exponent α replaced by $-\frac{\alpha}{2} + \frac{4Mb^2}{\alpha}$, i.e.

$$\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\|^2 \leqslant 2Me^{(-\frac{\alpha}{2}+\frac{4Mb^2}{\alpha})(t-s)+\varepsilon|s|}, \quad \forall\ (t,s) \in I^2_{\geqslant}.$$

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3. Robustness of NMS-ED on the half line $[t_0, +\infty)$

In this section, we state and prove our main result on the robustness of NMS-ED on $I = [t_0, +\infty)$. The case of the interval $I = (-\infty, t_0]$ and the whole $\mathbb{R}$ will be discussed in § 4 and § 5, respectively.

The following theorem is on the robustness of NMS-ED of (1.1) on $[t_0, +\infty)$, and its proof is more general and complicated than that of theorem 2.1, because we need to find out the explicit expressions of the 'exponential growing solutions' and the 'exponential decaying solutions' for the perturbed equation (1.5) along with the stable and unstable directions, respectively. To do this, we rewrite the unique solution of (1.5) along the stable direction under a natural condition: boundedness. It is also worth mentioning that the following theorem is also valid for NMS-EC. Indeed, a contraction is a dichotomy with $P(t) = Id$ for every $t \in I$.

THEOREM 3.1. *Let $A(\cdot)$, $B(\cdot)$, $G(\cdot)$, $H(\cdot)$ be $n \times n$-matrix continuous functions with real entries such that (1.1) admits an NMS-ED (1.3)–(1.4) with $\varepsilon < \alpha$, and assume that coefficient matrices of (1.5) satisfy*

$$\|A(t)\| \leqslant a, \quad \|G(t)\| \leqslant g, \quad \|B(t)\| \leqslant be^{-\varepsilon|t|}, \quad \|H(t)\| \leqslant he^{-\varepsilon|t|}, \quad t \in I \tag{3.1}$$

with constants a, b, g, $h > 0$. Let b, h be small enough such that

$$\tilde{M} := 8b^2 + 8g^2h^2 + \alpha h^2 < \frac{\alpha^2}{20M}.$$

Then (1.5) admits an NMS-ED in I with linear projections $\hat{P}(t) : L^2(\Omega, \mathbb{R}^n) \to L^2(\Omega, \mathbb{R}^n)$ such that

$$\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{P}(s) = \hat{P}(t)\hat{\Phi}(t)\hat{\Phi}^{-1}(s), \quad \forall\ t, s \in I, \tag{3.2}$$

and

$$\mathbb{E}\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{P}(t)\|^2 \leqslant \hat{M}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}, \quad \forall\ (t, s) \in I^2_{\geqslant}, \tag{3.3}$$

$$\mathbb{E}\|\hat{\Phi}(t)\hat{\Phi}^{-1}(s)\hat{Q}(t)\|^2 \leqslant \hat{M}e^{-\hat{\alpha}(s-t)+\hat{\varepsilon}|s|}, \quad \forall\ (t, s) \in I^2_{\leqslant}, \tag{3.4}$$

where bound $\hat{M} := 40M$, exponent $\hat{\alpha} := \frac{\alpha}{2} - \frac{10M\tilde{M}}{\alpha}$ and nonuniform degree $\hat{\varepsilon} := 2\varepsilon$.

Proof of theorem 3.1. We first prove several lemmas which are essential in proving the theorem. The first one is the existence and uniqueness lemma, which is slightly different from lemma 2.2 since $U(s, s)\xi_0$ is not necessarily equal to ξ_0 in (3.5). We will explain the reason after lemma 3.7 under which condition there exists an equivalence between (2.6) and (3.5) below.

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LEMMA 3.2. *For any given initial value* $\xi_0 \in \mathbb{R}^n$, (1.5) *has a unique solution* $U(t, s)\xi_0$ *with* $U \in (\mathscr{L}_c, \|\cdot\|_c)$ *such that*

$$\begin{aligned} U(t,s) = {}& \Phi(t)\Phi^{-1}(s)P(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau) \\ & + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau \\ & - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau) \\ & - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau. \end{aligned} \tag{3.5}$$

Proof. We first prove that the function $U(t, s)\xi_0$ is a solution of (1.5). Set

$$\begin{aligned} \xi(t) = {}& \Phi^{-1}(s)P(s)\xi_0 + \int_s^t \Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,s)\xi_0\mathrm{d}\omega(\tau) \\ & + \int_s^t \Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)U(\tau,s)\xi_0\mathrm{d}\tau - \int_t^\infty \Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,s)\xi_0\mathrm{d}\omega(\tau) \\ & - \int_t^\infty \Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,s)\xi_0\mathrm{d}\tau. \end{aligned}$$

Let $y(t) = \Phi(t)\xi(t)$. Clearly,

$$U(t,s)\xi_0 = \Phi(t)\xi(t) = y(t),$$

and then $\xi(t)$ satisfies the differential

$$d\xi(t) = \Phi^{-1}(t)\big(B(t) - G(t)H(t)\big)y(t)dt + \Phi^{-1}(t)H(t)y(t)\mathrm{d}\omega(t).$$

Since $\Phi(t)$ is a fundamental matrix solution of (1.1). it follows from Itô product rule that

$$\begin{aligned} dy(t) &= d\Phi(t)\xi(t) + \Phi(t)d\xi(t) + G(t)\Phi(t)\Phi^{-1}(t)H(t)y(t)\mathrm{d}t \\ &= A(t)y(t)\mathrm{d}t + G(t)y(t)\mathrm{d}\omega(t) + \big(B(t) - G(t)H(t)\big)y(t)\mathrm{d}t \\ &\quad + H(t)y(t)\mathrm{d}\omega(t) + G(t)H(t)y(t)\mathrm{d}t \\ &= (A(t) + B(t))y(t)\mathrm{d}t + (G(t) + H(t))y(t)\mathrm{d}\omega(t), \end{aligned}$$

which means that $y(t) = U(t, s)\xi_0$ is a solution of (1.5).

Now we prove that U is unique in $(\mathscr{L}_c, \|\cdot\|_c)$. Let

$$\begin{aligned} (\Gamma U)(t,s) = {}& \Phi(t)\Phi^{-1}(s)P(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau) \\ & + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau \end{aligned}$$

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$$- \int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau)$$
$$- \int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau.$$

The same idea as in lemma 2.2 can be applied to prove the uniqueness of the solution to (3.5). Squaring both sides of (3.5), and taking expectations, we have

$$\begin{aligned}
&\mathbb{E}\|(\Gamma U)(t,s)\|^2 \\
&\leqslant 5\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\|^2 + 5\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\
&\quad + 5\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau\right\|^2 \\
&\quad + 5\mathbb{E}\left\|\int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\
&\quad + 5\mathbb{E}\left\|\int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau\right\|^2 \\
&\leqslant 5Me^{\varepsilon|s|} + 10Me^{\varepsilon|s|}(\frac{\alpha h^2 + 8b^2 + 8g^2h^2}{\alpha^2}) \sup_{(\tau,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|U(\tau,s)\|^2 e^{-\varepsilon|s|}\right),
\end{aligned}$$

and this implies that

$$\mathbb{E}\|(\Gamma U)(t,s)\|^2 e^{-\varepsilon s} \leqslant 5M + \frac{10M\tilde{M}}{\alpha^2} \sup_{(\tau,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|\hat{\Phi}(\tau,s)\|^2 e^{-\varepsilon|s|}\right) < \infty$$

with $\tilde{M} = 8b^2 + 8g^2h^2 + \alpha h^2$. Following the same procedure as above, for any U_1, $U_2 \in \mathscr{L}_c$, we have

$$\|\Gamma U_1 - \Gamma U_2\|_c^2 \leqslant \frac{10M\tilde{M}}{\alpha^2} \sup_{(\tau,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|U_1(\tau,s) - U_2(\tau,s)\|^2 e^{-\varepsilon|s|}\right). \tag{3.6}$$

Note that

$$\begin{aligned}
\sup_{(t,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|U_1(t,s) - U_2(t,s)\|^2 e^{-\varepsilon s}\right) &\leqslant \sup_{(t,s)\in I^2_{\geqslant}} \left((\mathbb{E}\|U_1(t,s) - U_2(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon|s|}{2}}\right)^2 \\
&\leqslant \left(\sup_{(t,s)\in I^2_{\geqslant}} (\mathbb{E}\|U_1(t,s) - U_2(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon|s|}{2}}\right)^2 \\
&= \|U_1 - U_2\|_c^2,
\end{aligned}$$

which together with (3.6) implies

$$\|\Gamma U_1 - \Gamma U_2\|_c \leqslant \sqrt{\frac{10M\tilde{M}}{\alpha^2}}\|U_1 - U_2\|_c.$$

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Since $\tilde{M} < \frac{\alpha^2}{10M}$, Γ is a contraction operator. Hence, there exists a unique $U \in \mathscr{L}_c$ such that $\Gamma U = U$, which satisfies identity (3.5). This completes the proof of the lemma.

LEMMA 3.3. *For any $u \in (s,\, t)$ in I, we have*

$$U(t,s) = U(t,u)U(u,s)$$

in the sense of $(\mathscr{L}_c,\, \|\cdot\|_c)$.

Proof. By (1.2) and (3.5) with any $t \geqslant u \geqslant s$ in I, we have

$$\begin{aligned} U(t,u)U(u,s) = \Phi(t)\Phi^{-1}(s)P(s) &+ \int_s^u \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau) \\ &+ \int_s^u \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau \\ &+ \bigg(\int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,u)\mathrm{d}\omega(\tau) \\ &+ \int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)U(\tau,u)\mathrm{d}\tau \\ &- \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,u)\mathrm{d}\omega(\tau) \\ &- \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,u)\mathrm{d}\tau \bigg) U(u,s). \end{aligned} \tag{3.7}$$

Subtracting (3.5) from (3.7) we obtain

$$\begin{aligned} &U(t,s) - U(t,u)U(u,s) \\ &\quad = \int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\left(U(\tau,s) - U(\tau,u)U(u,s)\right)\mathrm{d}\omega(\tau) \\ &\quad + \int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\left(U(\tau,s) - U(\tau,u)U(u,s)\right)\mathrm{d}\tau \\ &\quad - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\left(U(\tau,s) - U(\tau,u)U(u,s)\right)\mathrm{d}\omega(\tau) \\ &\quad - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\left(U(\tau,s) - U(\tau,u)U(u,s)\right)\mathrm{d}\tau. \end{aligned}$$

Write $\tilde{U}(t,\, s) = U(t,\, s) - U(t,\, u)U(u,\, s)$. Now we prove $\tilde{U}$ is unique in $(\mathscr{L}_c,\, \|\cdot\|_c)$. Let

$$\begin{aligned} (\mathcal{T}\tilde{U})(t,s) = &\int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\tilde{U}(\tau,s)\mathrm{d}\omega(\tau) \\ &+ \int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\tilde{U}(\tau,s)\mathrm{d}\tau \end{aligned}$$

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$$
\begin{aligned}
&- \int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\tilde{U}(\tau,s)\mathrm{d}\omega(\tau) \\
&- \int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\tilde{U}(\tau,s). \qquad (3.8)
\end{aligned}
$$

Squaring both sides of (3.8), and taking expectations, it follows from (2.7) that

$$
\begin{aligned}
\mathbb{E}\|(\mathcal{T}\tilde{U})(t,s)\|^2 \leqslant{}& 4\mathbb{E}\left\|\int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\tilde{U}(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\tilde{U}(\tau,s)\mathrm{d}\tau\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\tilde{U}(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\
&+ 4\mathbb{E}\left\|\int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\tilde{U}(\tau,s)\mathrm{d}\tau\right\|^2. \qquad (3.9)
\end{aligned}
$$

By using the Itô isometry property and inequalities (1.3), the first term on the right-hand side in (3.9) can be deduced as follows:

$$
\begin{aligned}
&\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\tilde{U}(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\
&= \int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|\tilde{U}(\tau,s)\|^2\mathrm{d}\tau \\
&\leqslant Mh^2\int_s^t e^{-\alpha(t-\tau)}\mathbb{E}\|\tilde{U}(\tau,s)\|^2\mathrm{d}\tau \\
&\leqslant \frac{Mh^2}{\alpha}e^{\varepsilon|s|}\sup_{(\tau,s)\in I^2_{\geqslant}}\left(\mathbb{E}\|\tilde{U}(\tau,s)\|^2e^{-\varepsilon|s|}\right).
\end{aligned}
$$

As for the second term in (3.9), it follows from $\mathbb{E}\|x\| \leqslant \sqrt{\mathbb{E}\|x\|^2}$, Cauchy–Schwarz inequality, Itô isometry property of stochastic integrals and (1.3) that

$$
\begin{aligned}
&\mathbb{E}\left\|\int_u^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\tilde{U}(\tau,s)\mathrm{d}\tau\right\|^2 \\
&\leqslant \left(\int_u^t \mathbb{E}\left\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\right\|\mathrm{d}\tau\right) \\
&\quad\times\left(\int_u^t \mathbb{E}\left\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\right\|\mathbb{E}\left\|\tilde{U}(\tau,s)\right\|^2\mathrm{d}\tau\right) \\
&\leqslant 2M(b^2+g^2h^2)\left(\int_u^t e^{-\frac{\alpha}{2}(t-\tau)}\mathrm{d}\tau\right)\left(\int_u^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\tilde{U}(\tau,s)\right\|^2\mathrm{d}\tau\right) \\
&\leqslant \frac{8M(b^2+g^2h^2)}{\alpha^2}e^{\varepsilon|s|}\sup_{(\tau,s)\in I^2_{\geqslant}}\left(\mathbb{E}\|\tilde{U}(\tau,s)\|^2e^{-\varepsilon|s|}\right).
\end{aligned}
$$

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Clearly, the proof above is also valid for proving the other terms on the right-hand side in (3.9). Thus, we can rewrite inequality (3.9) as

$$\mathbb{E}\|(\mathcal{T}\tilde{U})(t,s)\|^2 \leqslant \frac{8M\tilde{M}}{\alpha^2} e^{\varepsilon|s|} \sup_{(\tau,s)\in I^2_{\geqslant}} \left(\mathbb{E}\|\tilde{U}(\tau,s)\|^2 e^{-\varepsilon|s|}\right),$$

and

$$\|\mathcal{T}\tilde{U}\|_c \leqslant \sqrt{\frac{8M\tilde{M}}{\alpha^2}}\|\tilde{U}\|_c$$

with $\tilde{M} = 8b^2 + 8g^2h^2 + \alpha h^2$. Following the same procedure as above, for any $\tilde{U}_1, \tilde{U}_2 \in \mathscr{L}_c$, we have

$$\|\mathcal{T}\tilde{U}_1 - \mathcal{T}\tilde{U}_2\|_c \leqslant \sqrt{\frac{8M\tilde{M}}{\alpha^2}}\|\tilde{U}_1 - \tilde{U}_2\|_c.$$

Since $\tilde{M} < \frac{\alpha^2}{8M}$, this implies $\mathcal{T}$ is a contraction. Hence, there is a unique $\tilde{U} \in (\mathscr{L}_c, \|\cdot\|_c)$. Besides, $0 \in (\mathscr{L}_c, \|\cdot\|_c)$ also satisfies (3.8). Hence, we must have

$$U(t,s) - U(t,u)U(u,s) = 0$$

in $\mathscr{L}_c$. Therefore, $U(t, s) = U(t, u)U(u, s)$ with $U \in (\mathscr{L}_c, \|\cdot\|_c)$. This completes the proof of the lemma.

LEMMA 3.4. *Given $s \in I$, if $y(t) := \Lambda(t, s)\xi : [s, +\infty) \to L^2(\Omega, \mathbb{R}^n)$ is a solution of (1.5) with $y(s) = \Lambda(s, s)\xi = \xi$ such that Λ is bounded in $(\mathscr{L}_c, \|\cdot\|_c)$, then*

$$\begin{aligned} y(t) = {} & \Phi(t)\Phi^{-1}(s)P(s)\xi + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ & + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ & - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau. \end{aligned} \tag{3.10}$$

Proof. It is easy to see from (2.6) that

$$\begin{aligned} P(t)y(t) = {} & \Phi(t)\Phi^{-1}(s)P(s)\xi + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ & + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} Q(t)y(t) = {} & \Phi(t)\Phi^{-1}(s)Q(s)\xi + \int_s^t \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ & + \int_s^t \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau \end{aligned} \tag{3.12}$$

for each $(t,\, s) \in I^2_{\geqslant}$. Equality (3.12) can be rewritten in the equivalent form

$$Q(s)\xi = \Phi(s)\Phi^{-1}(t)Q(t)y(t) - \int_s^t \Phi(s)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ - \int_s^t \Phi(s)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau. \tag{3.13}$$

For convenience we can assume that $D = \|\Lambda\|_c < \infty$, since Λ is bounded in $(\mathscr{L}_c,\, \|\cdot\|_c)$. Then it follows from (2.5) and (1.4) that

$$\mathbb{E}\|\Phi(s)\Phi^{-1}(t)Q(t)y(t)\|^2 \leqslant MD^2\|\xi\|^2 e^{-\alpha(t-s)+\varepsilon(|t|+|s|)}.$$

Since $\alpha > \varepsilon$, the right-hand side of this inequality goes to zero as $t \to +\infty$. Furthermore, we have

$$\begin{aligned}\mathbb{E}&\left\|\int_s^\infty \Phi(s)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau)\right\|^2 \\ &= \int_s^\infty \mathbb{E}\left\|\Phi(s)\Phi^{-1}(\tau)Q(\tau)\right\|^2 \mathbb{E}\left\|H(\tau)\right\|^2 \mathbb{E}\left\|y(\tau)\right\|^2 \mathrm{d}\tau \\ &\leqslant \frac{h^2D^2M}{\alpha}e^{\varepsilon|s|},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}&\left\|\int_s^\infty \Phi(s)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau\right\|^2 \\ &\leqslant \left(\int_s^\infty \mathbb{E}\left\|\Phi(s)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\right\|\mathrm{d}\tau\right) \\ &\quad\times \left(\int_s^\infty \mathbb{E}\left\|\Phi(s)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\right\|\mathbb{E}\left\|y(\tau)\right\|^2 \mathrm{d}\tau\right) \\ &\leqslant 2M(b^2+g^2h^2)\left(\int_s^\infty e^{-\frac{\alpha}{2}(\tau-s)}\mathrm{d}\tau\right)\left(\int_s^\infty e^{-\frac{\alpha}{2}(\tau-s)}\mathbb{E}\left\|y(\tau)\right\|^2\mathrm{d}\tau\right) \\ &\leqslant \frac{8MD^2(b^2+g^2h^2)}{\alpha^2}e^{\varepsilon|s|}.\end{aligned}$$

Taking limits as $t \to +\infty$ in (3.13), we obtain

$$Q(s)\xi = -\int_s^\infty \Phi(s)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ - \int_s^\infty \Phi(s)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau,$$

and substituting it into (3.12) yields

$$Q(t)y(t) = -\int_s^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ + \int_s^t \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau)$$

$$
\begin{aligned}
&- \int_s^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau \\
&+ \int_s^{t} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau \\
= &- \int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\
&- \int_t^{\infty} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau.
\end{aligned}
$$

Since ξ is an arbitrary one in $\mathbb{R}^n$, then by adding this identity to (3.11) yields the desired equation (3.10).

Recall that $\hat{\Phi}(t, s) = \hat{\Phi}(t)\hat{\Phi}^{-1}(s)$ denotes the fundamental matrix solution of (1.5) with $\hat{\Phi}(s, s) = Id$. For each $t \in I$, define linear operators as

$$
\hat{P}(t) = \hat{\Phi}(t, t_0)U(t_0, t_0)\hat{\Phi}(t_0, t) \quad \text{and} \quad \hat{Q}(t) = Id - \hat{P}(t), \tag{3.14}
$$

where t_0 is the left boundary point of the interval I. After presenting that $\hat{P}(t)$ are projections, we prove relationship (3.2), show the explicit expressions of the fundamental matrix solution $\hat{\Phi}(t, s)$ under the projections $\hat{P}(t)$, $\hat{Q}(t)$, and then deduce inequalities (3.3) and (3.4).

LEMMA 3.5. *The operator $\hat{P}(t)$ is a linear projection for $t \in I$, and (3.2) holds.*

Proof. By lemma 3.3, we have $U(t_0, t_0)U(t_0, t_0) = U(t_0, t_0)$. Thus,

$$
\hat{P}(t)\hat{P}(t) = \hat{\Phi}(t, t_0)U(t_0, t_0)\hat{\Phi}(t_0, t)\hat{\Phi}(t, t_0)U(t_0, t_0)\hat{\Phi}(t_0, t) = \hat{P}(t).
$$

Furthermore, for any $t, s \in I$, we obtain

$$
\begin{aligned}
\hat{P}(t)\hat{\Phi}(t, s) &= \hat{\Phi}(t, t_0)U(t_0, t_0)\hat{\Phi}(t_0, t)\hat{\Phi}(t, s) \\
&= \hat{\Phi}(t, s)\hat{\Phi}(s, t_0)U(t_0, t_0)\hat{\Phi}(t_0, s) = \hat{\Phi}(t, s)\hat{P}(s),
\end{aligned}
$$

and this completes the proof of the lemma.

LEMMA 3.6. *For any given initial value $\xi_0 \in \mathbb{R}^n$, the function $\hat{P}(t)\hat{\Phi}(t, s)\xi_0$ is a solution of (1.5) with $\hat{P}(t)\hat{\Phi}(t, s)$ bounded in $(\mathscr{L}_c, \|\cdot\|_c)$.*

Proof. By lemma 3.2, the function $U(t, t_0)\xi_0$ is a solution of (1.5) with initial value $U(t_0, t_0)\xi_0$ at time t_0. Clearly, $U(t, t_0) = \hat{\Phi}(t, t_0)U(t_0, t_0)$. Thus, it is easy to see that

$$
\hat{P}(t)\hat{\Phi}(t, s) = \hat{\Phi}(t, t_0)U(t_0, t_0)\hat{\Phi}(t_0, t)\hat{\Phi}(t, s) = U(t, t_0)\hat{\Phi}(t_0, s).
$$

Therefore, it follows again from lemma 3.2 that $\hat{P}(t)\hat{\Phi}(t, s)\xi_0 = U(t, t_0)\hat{\Phi}(t_0, s)\xi_0$ is a solution of (1.5) with initial value $\hat{\Phi}(t_0, s)\xi_0 \in \mathbb{R}^n$. Moreover, from $U \in (\mathscr{L}_c, \|\cdot\|_c)$ and definition (2.4)–(2.5) of the space $(\mathscr{L}_c, \|\cdot\|_c)$, we can see that $\hat{P}(t)\hat{\Phi}(t, s)$ is bounded in $(\mathscr{L}_c, \|\cdot\|_c)$.

Lemma 3.7. *For any given initial value* $\xi_0 \in \mathbb{R}^n$*, the function* $\hat{P}(t)\hat{\Phi}(t,\, s)\xi_0$ *is a solution of* (1.5) *with* $(t,\, s) \in I^2_{\geqslant}$ *such that*

$$\begin{aligned}\hat{\Phi}(t,s)\hat{P}(s) = \Phi(t)\Phi^{-1}(s)P(s)\hat{P}(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\\ + \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\\ - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\\ - \int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau.\end{aligned} \tag{3.15}$$

Proof. Let $y(t) = \hat{P}(t)\hat{\Phi}(t,\, s)\xi_0$ with given $s \in I$, and denote $\xi = \hat{P}(s)\xi_0$ the initial condition at time s. Clearly, $y(t)$ is a solution of (1.5) with $y(s) = \hat{P}(s)\xi_0 = \hat{P}(s)\hat{P}(s)\xi_0 = \xi$. By lemma 3.6, $\hat{P}(t)\hat{\Phi}(t,\, s)$ is bounded in $(\mathscr{L}_c,\, \|\cdot\|_c)$. Since ξ_0 is arbitrary in $\mathbb{R}^n$, identity (3.15) follows now readily from lemma 3.4.

Remark 3.8. From lemma 3.7, we know that the explicit expressions (2.6) and (3.5) are the same under the condition of NMS-EC. In fact, as a special case of lemma 3.7, $\hat{\Phi}(t,\, s)$ is always bounded in $(\mathscr{L}_c,\, \|\cdot\|_c)$ with $I = [t_0,\, +\infty)$ since projections are the identity.

In the following lemma, we present the explicit expression of $\hat{\Phi}(t,\, s)\hat{Q}(s)$ with $(t,\, s) \in I^2_{\leqslant}$.

Lemma 3.9. *For any given initial value* $\xi_0 \in \mathbb{R}^n$*, the function* $\hat{Q}(t)\hat{\Phi}(t,\, s)\xi_0$ *is a solution of* (1.5) *with* $(t,\, s) \in I^2_{\leqslant}$ *such that*

$$\begin{aligned}\hat{\Phi}(t,s)\hat{Q}(s) = \Phi(t)\Phi^{-1}(s)Q(s)\hat{Q}(s) + \int_{t_0}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\omega(\tau)\\ + \int_{t_0}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\tau\\ - \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\omega(\tau)\\ - \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\tau.\end{aligned} \tag{3.16}$$

Proof. Following the same lines as given in the proof of lemma 2.2, one can prove that

$$\begin{aligned}\hat{\Phi}(t,s) = \Phi(t)\Phi^{-1}(s) + \int_s^t \Phi(t)\Phi^{-1}(\tau)H(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\omega(\tau)\\ + \int_s^t \Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\mathrm{d}\tau\end{aligned}$$

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for any $(t,\, s) \in I^2_{\leqslant}$. Write $K(t) = \hat{\Phi}(t,\, t_0)\hat{Q}(t_0)$. Therefore,

$$K(t) = \Phi(t)\Phi^{-1}(t_0)\hat{Q}(t_0) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau. \tag{3.17}$$

On the other hand, it follows from $\hat{P}(t) = \hat{\Phi}(t,\, t_0)U(t_0,\, t_0)\hat{\Phi}(t_0,\, t)$ and (3.5) with $t = s = t_0$ that

$$\hat{P}(t_0) = U(t_0, t_0) = P(t_0) - \int_{t_0}^{\infty} \Phi(t_0)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau, t_0)\mathrm{d}\omega(\tau) - \int_{t_0}^{\infty} \Phi(t_0)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau, t_0)\mathrm{d}\tau. \tag{3.18}$$

Since $P(t_0)$ and $Q(t_0)$ are complementary projections, multiplying (3.18) on the left with $P(t_0)$ gives

$$P(t_0)\hat{P}(t_0) = P(t_0). \tag{3.19}$$

In addition,

$$Q(t_0)\hat{Q}(t_0) = \big(Id - P(t_0)\big)\big(Id - \hat{P}(t_0)\big) = Id - \hat{P}(t_0) = \hat{Q}(t_0). \tag{3.20}$$

By (3.17), using (3.20), we have

$$\Phi(t)\Phi^{-1}(s)Q(s)K(s) = \Phi(t)\Phi^{-1}(t_0)\hat{Q}(t_0) + \int_{t_0}^{s} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau) + \int_{t_0}^{s} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau,$$

which can be rewritten as

$$\Phi(t)\Phi^{-1}(t_0)\hat{Q}(t_0) = \Phi(t)\Phi^{-1}(s)Q(s)K(s) - \int_{t_0}^{s} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau) - \int_{t_0}^{s} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau. \tag{3.21}$$

Substituting (3.21) into (3.17) leads to

$$K(t) = \Phi(t)\Phi^{-1}(s)Q(s)K(s) - \int_{t_0}^{s} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau) - \int_{t_0}^{s} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau)$$

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$$
\begin{aligned}
&+\int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau\\
&=\Phi(t)\Phi^{-1}(s)Q(s)K(s)-\int_{t}^{s}\Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau)\\
&\quad-\int_{t}^{s}\Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau+\int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)K(\tau)\mathrm{d}\omega(\tau)\\
&\quad+\int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)K(\tau)\mathrm{d}\tau. \qquad (3.22)
\end{aligned}
$$

Since (3.2) we have $K(t)=\hat{\Phi}(t,\,t_0)\hat{Q}(t_0)=\hat{Q}(t)\hat{\Phi}(t,\,t_0)$. Therefore, $K(t)\hat{\Phi}(t_0,\,s)=\hat{Q}(t)\hat{\Phi}(t,\,s)$ for every $(t,\,s)\in I^2_{\leqslant}$. Thus, multiplying (3.22) on the right with $\hat{\Phi}(t_0,\,s)$ yields the desired identity (3.16).

We proceed with the proof of theorem 3.1. Squaring both sides of (3.15), and taking expectations. Setting $z(t,\,s)=\mathbb{E}\|\hat{\Phi}(t,\,s)\hat{P}(s)\|^2$ with $(t,\,s)\in I^2_{\geqslant}$. It follows from (2.7) that

$$
\begin{aligned}
z(t,s)\leqslant\ &5\mathbb{E}\|\Phi(t)\Phi^{-1}(s)P(s)\hat{P}(s)\|^2\\
&+5\mathbb{E}\left\|\int_{s}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\right\|^2\\
&+5\mathbb{E}\left\|\int_{s}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\right\|^2\\
&+5\mathbb{E}\left\|\int_{t}^{\infty}\Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\right\|^2\\
&+5\mathbb{E}\left\|\int_{t}^{\infty}\Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\right\|^2. \qquad (3.23)
\end{aligned}
$$

By using the Itô isometry property and inequalities (1.3), the second term on the right-hand side of (3.23) can be deduced as follows:

$$
\begin{aligned}
&\mathbb{E}\left\|\int_{s}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\right\|^2\\
&\quad=\int_{s}^{t}\mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|\hat{\Phi}(\tau,s)\hat{P}(s)\|^2\mathrm{d}\tau\\
&\leqslant Mh^2\int_{s}^{t}e^{-\alpha(t-\tau)}\mathbb{E}\|\hat{\Phi}(\tau,s)\hat{P}(s)\|^2\mathrm{d}\tau.
\end{aligned}
$$

As to the third term in (3.23), it follows from $\mathbb{E}\|x\| \leqslant \sqrt{\mathbb{E}\|x\|^2}$, Cauchy–Schwarz inequality and inequalities (1.3) that

$$\begin{aligned}
&\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\right\|^2 \\
&\quad = \mathbb{E}\left\|\int_s^t \left(\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\right)^{\frac{1}{2}}\left(\left(\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\right)^{\frac{1}{2}}\hat{\Phi}(\tau,s)\hat{P}(s)\right)\mathrm{d}\tau\right\|^2 \\
&\quad \leqslant \left(\int_s^t \mathbb{E}\left\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\right\|\mathrm{d}\tau\right) \\
&\qquad \times \left(\int_s^t \mathbb{E}\left\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\right\|\mathbb{E}\left\|\hat{\Phi}(\tau,s)\hat{P}(s)\right\|^2\mathrm{d}\tau\right) \\
&\quad \leqslant 2M(b^2+g^2h^2)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathrm{d}\tau\right)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\hat{P}(s)\right\|^2\mathrm{d}\tau\right) \\
&\quad \leqslant \frac{4M(b^2+g^2h^2)}{\alpha}\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\left\|\hat{\Phi}(\tau,s)\hat{P}(s)\right\|^2\mathrm{d}\tau.
\end{aligned}$$

Clearly, the proof above is also valid for proving the other terms on the right-hand side in (3.23). Thus, we can rewrite inequality (3.23) as

$$\begin{aligned}
z(t,s) &\leqslant 5Me^{-\alpha(t-s)+\varepsilon|s|}z(s,s) \\
&\qquad + 5Mh^2\left(\int_s^t e^{-\alpha(t-\tau)}z(\tau,s)\mathrm{d}\tau + \int_t^\infty e^{-\alpha(\tau-t)}z(\tau,s)\mathrm{d}\tau\right) \\
&\qquad + \frac{20M(b^2+g^2h^2)}{\alpha}\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}z(\tau,s)\mathrm{d}\tau + \int_t^\infty e^{-\frac{\alpha}{2}(\tau-t)}z(\tau,s)\mathrm{d}\tau\right) \\
&\leqslant 5Me^{-\frac{\alpha}{2}(t-s)+\varepsilon|s|}z(s,s) \\
&\qquad + \frac{5M\tilde{M}}{\alpha}\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}z(\tau,s)\mathrm{d}\tau + \int_t^\infty e^{-\frac{\alpha}{2}(\tau-t)}z(\tau,s)\mathrm{d}\tau\right)
\end{aligned} \tag{3.24}$$

with $\tilde{M} = 8b^2 + 8g^2h^2 + \alpha h^2$. Let

$$\begin{aligned}
Z(t,s) &= 5Me^{-\frac{\alpha}{2}(t-s)+\varepsilon|s|}z(s,s) \\
&\qquad + \frac{5M\tilde{M}}{\alpha}\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}z(\tau,s)\mathrm{d}\tau + \int_t^\infty e^{-\frac{\alpha}{2}(\tau-t)}z(\tau,s)\mathrm{d}\tau\right).
\end{aligned}$$

Clearly, inequality (3.24) can be rewritten as

$$z(t,s) \leqslant Z(t,s).$$

On the contrary,

$$\frac{d}{\mathrm{d}t}Z(t,s) = -\frac{\alpha}{2}Z(t,s) + \frac{10M\tilde{M}}{\alpha}z(t,s),$$

and therefore,

$$\frac{d}{\mathrm{d}t}Z(t,s) \leqslant \left(\frac{10M\tilde{M}}{\alpha} - \frac{\alpha}{2}\right) Z(t,s).$$

Integrating the above inequality from s to t and note that $Z(s,\, s) = 5Me^{\varepsilon|s|}z(s,\, s)$, we obtain

$$z(t,s) \leqslant Z(t,s) \leqslant 5Me^{\varepsilon|s|}e^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)}z(s,s), \quad \forall\ (t,s) \in I^2_{\geqslant}.$$

By $z(t,\, s) = \mathbb{E}\|\hat{\Phi}(t,\, s)\hat{P}(s)\|^2$, we have

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)\|^2 \leqslant 5Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)+\varepsilon|s|}\mathbb{E}\|\hat{P}(s)\|^2, \quad \forall\ (t,s) \in I^2_{\geqslant}. \tag{3.25}$$

Similarly, squaring both sides of (3.16), and taking expectations. Using the same way as above, we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s)\|^2 \leqslant 5Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(s-t)+\varepsilon|s|}\mathbb{E}\|\hat{Q}(s)\|^2, \quad \forall\ (t,s) \in I^2_{\leqslant}. \tag{3.26}$$

Now we try to find out the bounds in mean-square setting for the projections $\hat{P}(t)$, $\hat{Q}(t)$. Multiplying (3.15) with $Q(t)$ on the left side, and let $t = s$, we have

$$\begin{aligned} Q(t)\hat{P}(t) = &-\int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,t)\hat{P}(t)\mathrm{d}\omega(\tau) \\ &-\int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,t)\hat{P}(t)\mathrm{d}\tau. \end{aligned} \tag{3.27}$$

By (3.27), using (1.4), (3.1) and (3.25), we have

$$\begin{aligned} \mathbb{E}\|Q(t)\hat{P}(t)\|^2 \leqslant{}& 2\mathbb{E}\left\|\int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,t)\hat{P}(t)\mathrm{d}\omega(\tau)\right\|^2 \\ &+ 2\mathbb{E}\left\|\int_t^\infty \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,t)\hat{P}(t)\mathrm{d}\tau\right\|^2 \\ \leqslant{}& 2\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|\hat{\Phi}(\tau,t)\hat{P}(t)\|^2\mathrm{d}\tau \\ &+ 2\left(\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|\mathbb{E}\|\tilde{B}(\tau)\|^{\frac{1}{2}}\mathrm{d}\tau\right) \\ &\times\left(\int_t^\infty \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|\mathbb{E}\|\tilde{B}(\tau)\|^{\frac{3}{2}}\mathbb{E}\|\hat{\Phi}(\tau,t)\hat{P}(t)\|^2\mathrm{d}\tau\right) \\ \leqslant{}& \frac{10M^2\tilde{M}}{\alpha}\mathbb{E}\|\hat{P}(t)\|^2\int_t^\infty e^{-(\alpha+\tilde{\alpha}-\varepsilon)(\tau-t)}\mathrm{d}\tau \\ \leqslant{}& \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}\mathbb{E}\|\hat{P}(t)\|^2, \end{aligned} \tag{3.28}$$

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since $\alpha > \varepsilon$ and $\tilde{\alpha} = \frac{\alpha}{2} - 10M\tilde{M}/\alpha > 0$. In addition, it follows from (3.16) with $t = s$ that

$$P(t)\hat{Q}(t) = \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau, t)\hat{Q}(t)\mathrm{d}\omega(\tau) + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau, t)\hat{Q}(t)\mathrm{d}\tau. \tag{3.29}$$

Similarly, by (3.29), using (1.3), (3.1) and (3.26), we obtain

$$\mathbb{E}\|P(t)\hat{Q}(t)\|^2 \leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha + \tilde{\alpha} - \varepsilon)}\mathbb{E}\|\hat{Q}(t)\|^2. \tag{3.30}$$

Meanwhile, notice that

$$\begin{aligned}\hat{P}(t) - P(t) &= \hat{P}(t) - P(t)\hat{P}(t) - P(t) + P(t)\hat{P}(t)\\ &= (Id - P(t))\hat{P}(t) - (Id - \hat{P}(t))P(t)\\ &= Q(t)\hat{P}(t) - P(t)\hat{Q}(t).\end{aligned}$$

Thus it follows from (3.28) and (3.30) that

$$\mathbb{E}\|\hat{P}(t) - P(t)\|^2 \leqslant \frac{20M^2\tilde{M}}{\alpha(\alpha + \tilde{\alpha} - \varepsilon)}(\mathbb{E}\|\hat{P}(t)\|^2 + \mathbb{E}\|\hat{Q}(t)\|^2). \tag{3.31}$$

Furthermore, it follows from (1.3)–(1.4) with $t = s$ that

$$\mathbb{E}\|P(t)\|^2 \leqslant Me^{\varepsilon|t|}, \quad \text{and} \quad \mathbb{E}\|Q(t)\|^2 \leqslant Me^{\varepsilon|t|}.$$

Therefore,

$$\begin{aligned}\mathbb{E}\|\hat{P}(t)\|^2 &\leqslant 2\mathbb{E}\|\hat{P}(t) - P(t)\|^2 + 2\mathbb{E}\|P(t)\|^2\\ &\leqslant \frac{40M^2\tilde{M}}{\alpha(\alpha + \tilde{\alpha} - \varepsilon)}(\mathbb{E}\|\hat{P}(t)\|^2 + \mathbb{E}\|\hat{Q}(t)\|^2) + 2Me^{\varepsilon|t|}.\end{aligned}$$

Since $\hat{Q}(t) - Q(t) = (Id - \hat{P}(t)) - (Id - P(t)) = P(t) - \hat{P}(t)$, we also have

$$\begin{aligned}\mathbb{E}\|\hat{Q}(t)\|^2 &\leqslant 2\mathbb{E}\|\hat{P}(t) - P(t)\|^2 + 2\mathbb{E}\|Q(t)\|^2\\ &\leqslant \frac{40M^2\tilde{M}}{\alpha(\alpha + \tilde{\alpha} - \varepsilon)}(\mathbb{E}\|\hat{P}(t)\|^2 + \mathbb{E}\|\hat{Q}(t)\|^2) + 2Me^{\varepsilon|t|}.\end{aligned}$$

Then we know

$$(\mathbb{E}\|\hat{P}(t)\|^2 + \mathbb{E}\|\hat{Q}(t)\|^2) \leqslant \frac{80M^2\tilde{M}}{\alpha(\alpha + \tilde{\alpha} - \varepsilon)}(\mathbb{E}\|\hat{P}(t)\|^2 + \mathbb{E}\|\hat{Q}(t)\|^2) + 4Me^{\varepsilon|t|},$$

and hence,

$$\left(1 - \frac{80M^2\tilde{M}}{\alpha(\alpha + \tilde{\alpha} - \varepsilon)}\right)(\mathbb{E}\|\hat{P}(t)\|^2 + \mathbb{E}\|\hat{Q}(t)\|^2) \leqslant 4Me^{\varepsilon|t|}.$$

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Since $\tilde{M} := 8b^2 + 8g^2h^2 + \alpha h^2$, we can obtain

$$\frac{80M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)} \leqslant \frac{1}{2}$$

by letting b and h sufficiently small. This yields

$$\mathbb{E}\|\hat{P}(t)\|^2 \leqslant 8Me^{\varepsilon|t|} \quad \text{and} \quad \mathbb{E}\|\hat{Q}(t)\|^2 \leqslant 8Me^{\varepsilon|t|}. \tag{3.32}$$

By (3.25), (3.26), using (3.32) we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)\|^2 \leqslant 40M^2 e^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)+2\varepsilon|s|}, \quad \forall\ (t,s) \in I^2_{\geqslant},$$

and

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s)\|^2 \leqslant 40Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(s-t)+2\varepsilon|s|}, \quad \forall\ (t,s) \in I^2_{\leqslant}.$$

This completes the proof of the theorem.

Under the hypotheses of theorem 3.1, the following theorem tries to discuss the differences of projections $P(t)$ and $\hat{P}(t)$ in the mean-square sense. To illustrate it clearly, write

$$\Phi(t,s) = \Phi(t)\Phi^{-1}(s).$$

Obviously, $\Phi(t,\ s)$ is a fundamental matrix solution of (1.1) with $\Phi(s,\ s) = Id$.

Theorem 3.10. *Under the hypotheses of theorem 3.1, for any $t \in I$, we have*

$$P(t) = \Phi(t_0,t)P(t_0)\Phi(t,t_0), \quad \text{and} \quad \hat{P}(t) = \hat{\Phi}(t_0,t)\hat{P}(t_0)\hat{\Phi}(t,t_0), \tag{3.33}$$

and

$$\mathbb{E}\|P(t)-\hat{P}(t)\|^2 \leqslant \frac{320M^3\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}e^{\varepsilon|t|}. \tag{3.34}$$

In particular, for each fixed $t \in I$, we have $\mathbb{E}\|P(t)-\hat{P}(t)\|^2 \to 0$ as $b,\ h \to 0$.

Proof. The second equality of (3.33) is obvious from definition (3.14) of linear operators $\hat{P}(t)$. For the first term in (3.33), it follows from (1.2) that

$$P(t)\Phi(t,t_0)\Phi(t_0,s) = \Phi(t,t_0)\Phi(t_0,s)P(s), \quad \forall\ t,s \in I,$$

and then

$$\Phi(t_0,t)P(t)\Phi(t,t_0) = \Phi(t_0,s)P(s)\Phi(s,t_0), \quad \forall\ t,s \in I. \tag{3.35}$$

Taking $s = t_0$ in (3.35), we obtain

$$\Phi(t_0,t)P(t)\Phi(t,t_0) = P(t_0).$$

Thus,

$$P(t) = \Phi(t_0,t)P(t_0)\Phi(t,t_0).$$

In addition, (3.34) follows immediately from (3.31) and (3.32).

THEOREM 3.11. *Under the hypotheses of theorem 3.1, we have*

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)-\Phi(t,s)P(s)\|^2 \leqslant \frac{720M\tilde{M}}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}, \quad \forall\ (t,s)\in I^2_{\geqslant},$$

and

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s)-\Phi(t,s)Q(s)\|^2 \leqslant \frac{720M\tilde{M}}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(s-t)+\hat{\varepsilon}|s|}, \quad \forall\ (t,s)\in I^2_{\leqslant}.$$

Proof. By $\hat{P}(s)\hat{P}(s)=\hat{P}(s)$, it follows from (3.15) that

$$\begin{aligned}
&\mathbb{E}\left\|\hat{\Phi}(t,s)\hat{P}(s)\hat{P}(s)-\Phi(t,s)P(s)\hat{P}(s)\right\|^2\\
&\leqslant 4\mathbb{E}\left\|\int_s^t \Phi(t,\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\right\|^2\\
&\quad + 4\mathbb{E}\left\|\int_s^t \Phi(t,\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\right\|^2\\
&\quad + 4\mathbb{E}\left\|\int_t^\infty \Phi(t,\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\right\|^2\\
&\quad + 4\mathbb{E}\left\|\int_t^\infty \Phi(t,\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\right\|^2.
\end{aligned} \tag{3.36}$$

By (1.3) and (3.3), using $\alpha-\hat{\alpha}=\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha}>0$, the first term on the right-hand side in (3.36) can be deduced as follows:

$$\begin{aligned}
&\mathbb{E}\left\|\int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\right\|^2\\
&=\int_s^t \mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)P(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|\hat{\Phi}(\tau,s)\hat{P}(s)\|^2\mathrm{d}\tau\\
&\leqslant M\hat{M}h^2\int_s^t e^{-\alpha(t-\tau)}e^{-\hat{\alpha}(\tau-s)+\hat{\varepsilon}|s|}\mathrm{d}\tau\\
&= M\hat{M}h^2 e^{-\alpha(t-s)+\hat{\varepsilon}|s|}\int_s^t e^{(\alpha-\hat{\alpha})(\tau-s)}\mathrm{d}\tau\\
&\leqslant \frac{M\hat{M}h^2}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}.
\end{aligned}$$

As to the second term in (3.36), by $\frac{\alpha}{2}-\hat{\alpha}=\frac{10M\tilde{M}}{\alpha}>0$, we have $2\alpha^2-\alpha\hat{\alpha}>0$. It follows from $\mathbb{E}\|x\|\leqslant\sqrt{\mathbb{E}\|x\|^2}$, Cauchy–Schwarz inequality and inequalities (1.3), (3.3) that

$$\begin{aligned}
&\mathbb{E}\left\|\int_s^t \Phi(t,\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau\right\|^2\\
&=\mathbb{E}\left\|\int_s^t \left(\Phi(t,\tau)P(\tau)\tilde{B}(\tau)\right)^{\frac{1}{2}}\left(\left(\Phi(t,\tau)P(\tau)\tilde{B}(\tau)\right)^{\frac{1}{2}}\hat{\Phi}(\tau,s)\hat{P}(s)\right)\mathrm{d}\tau\right\|^2
\end{aligned}$$

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$$
\begin{aligned}
&\leqslant \left(\int_s^t \mathbb{E}\left\|\Phi(t,\tau)P(\tau)\tilde{B}(\tau)\right\| \mathrm{d}\tau\right) \\
&\quad \times \left(\int_s^t \mathbb{E}\left\|\Phi(t,\tau)P(\tau)\tilde{B}(\tau)\right\| \mathbb{E}\left\|\hat{\Phi}(\tau,s)\hat{P}(s)\right\|^2 \mathrm{d}\tau\right) \\
&\leqslant 2M\hat{M}(b^2+g^2h^2)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}\mathrm{d}\tau\right)\left(\int_s^t e^{-\frac{\alpha}{2}(t-\tau)}e^{-\hat{\alpha}(\tau-s)+\hat{\varepsilon}|s|}\mathrm{d}\tau\right) \\
&\leqslant \frac{8M\hat{M}(b^2+g^2h^2)}{2\alpha^2-\alpha\hat{\alpha}}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}.
\end{aligned}
$$

Clearly, the proof above is also valid for proving the other terms on the right-hand side in (3.36). Thus, we can rewrite inequality (3.36) as

$$
\begin{aligned}
&\mathbb{E}\left\|\hat{\Phi}(t,s)\hat{P}(s)\hat{P}(s)-\Phi(t,s)P(s)\hat{P}(s)\right\|^2 \\
&\quad \leqslant \left(\frac{8M\hat{M}h^2}{\alpha-\hat{\alpha}}+\frac{64M\hat{M}(b^2+g^2h^2)}{2\alpha^2-\alpha\hat{\alpha}}\right)e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|} \\
&\quad = \frac{320M\tilde{M}}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}.
\end{aligned} \tag{3.37}
$$

Additionally, as $\hat{P}(s)$ and $\hat{Q}(s)$ are complementary projections for each $s \in I$, it follows from(1.3), (3.4) and (3.29) that

$$
\begin{aligned}
&\mathbb{E}\left\|\hat{\Phi}(t,s)\hat{P}(s)\hat{Q}(s)-\Phi(t,s)P(s)\hat{Q}(s)\right\|^2 = \mathbb{E}\left\|\Phi(t,s)P(s)\hat{Q}(s)\right\|^2 \\
&\quad \leqslant 2\mathbb{E}\left\|\int_{t_0}^s \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,t)\hat{Q}(t)\mathrm{d}\omega(\tau)\right\|^2 \\
&\qquad + 2\mathbb{E}\left\|\int_{t_0}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,t)\hat{Q}(t)\mathrm{d}\tau\right\|^2 \\
&\quad \leqslant \frac{40M\tilde{M}}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}.
\end{aligned} \tag{3.38}
$$

Combining (3.37) and (3.38) yields

$$
\begin{aligned}
&\mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)-\Phi(t,s)P(s)\|^2 \\
&\quad = \mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)(\hat{P}(s)+\hat{Q}(s))-\Phi(t,s)P(s)(\hat{P}(s)+\hat{Q}(s))\|^2 \\
&\quad \leqslant \frac{720M\tilde{M}}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(t-s)+\hat{\varepsilon}|s|}.
\end{aligned}
$$

Similarly, by (3.16) we obtain

$$
\mathbb{E}\left\|\hat{\Phi}(t,s)\hat{Q}(s)\hat{Q}(s)-\Phi(t,s)Q(s)\hat{Q}(s)\right\|^2 \leqslant \frac{320M\tilde{M}}{\alpha-\hat{\alpha}}e^{-\hat{\alpha}(s-t)+\hat{\varepsilon}|s|}. \tag{3.39}
$$

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Also, as $\hat{P}(s)$ and $\hat{Q}(s)$ are complementary projections for each $s \in I$, by (3.28) we obtain

$$\mathbb{E}\left\|\hat{\Phi}(t,s)\hat{Q}(s)\hat{P}(s) - \Phi(t,s)Q(s)\hat{P}(s)\right\|^2 \leqslant \frac{40M\tilde{M}}{\alpha - \hat{\alpha}} e^{-\hat{\alpha}(s-t)+\hat{\varepsilon}|s|}. \tag{3.40}$$

Combining (3.39) and (3.40) yields

$$\begin{aligned}
&\mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s) - \Phi(t,s)Q(s)\|^2 \\
&\quad = \mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s)(\hat{P}(s)+\hat{Q}(s)) - \Phi(t,s)Q(s)(\hat{P}(s)+\hat{Q}(s))\|^2 \\
&\quad \leqslant \frac{720M\tilde{M}}{\alpha - \hat{\alpha}} e^{-\hat{\alpha}(s-t)+\hat{\varepsilon}|s|}.
\end{aligned}$$

This completes the proof of the theorem.

REMARK 3.12. Since $I = [t_0, +\infty)$, the second-moment Lyapunov exponent is bounded by $-\hat{\alpha}$ for any fixed b, $h > 0$, i.e.

$$\lim_{t\to+\infty} \frac{1}{t} \log \mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s) - \Phi(t,s)P(s)\|^2 = -\hat{\alpha} < 0.$$

This shows that in the stable direction, any two solutions $\hat{\Phi}(t, s)\hat{P}(s)\xi$ and $\Phi(t, s)P(s)\xi$ with the same initial condition are forward asymptotic in the mean-square sense. Furthermore, since $M = 8b^2 + 8g^2h^2 + \alpha h^2$, for each fixed $T_1 \in (s, +\infty)$ and $T_2 \in (t_0, s)$, we have

$$\lim_{b,h\to 0} \sup_{t\in[s,T_1]} \mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s) - \Phi(t,s)P(s)\|^2 = 0,$$

and

$$\lim_{b,h\to 0} \sup_{t\in[T_2,s]} \mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s) - \Phi(t,s)Q(s)\|^2 = 0.$$

This means that the solution $\hat{\Phi}(t, s)\hat{P}(s)$ (or $\hat{\Phi}(t, s)\hat{Q}(s)$) of the perturbed system (1.5) approaches uniformly the solution $\Phi(t, s)P(s)$ (or $\Phi(t, s)Q(s)$) of system (1.1) in the mean-square sense on any compact interval.

4. Robustness of NMS-ED on the half line $(-\infty, t_0]$

In this section, we deal with the robustness of NMS-ED on $I = (-\infty, t_0]$, which is analogous to the case $[t_0, +\infty)$. So in what follows, we highlight the main steps of the proof which only indicate the major differences.

THEOREM 4.1. *The assertion in theorem 3.1 remains true for* $I = (-\infty, t_0]$.

Proof of theorem 4.1. Consider the Banach space

$$\mathscr{L}_d := \{\hat{\Phi} : I^2_{\leqslant} \to \mathfrak{B}(L^2(\Omega, \mathbb{R}^n)) : \ \hat{\Phi} \text{ is continuous and } \|\hat{\Phi}\|_d < \infty\} \tag{4.1}$$

with the norm

$$\|\hat{\Phi}\|_d = \sup\left\{(\mathbb{E}\|\hat{\Phi}(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon}{2}|s|} : (t,s) \in I^2_{\leqslant}\right\}. \tag{4.2}$$

Following the same steps as in the proof of theorem 1, we establish the following statements.

Lemma 4.2. *For any given initial value $\xi_0 \in \mathbb{R}^n$, (1.5) has a unique solution $V(t, s)\xi_0$ with $V \in (\mathscr{L}_d, \|\cdot\|_d)$ such that*

$$\begin{aligned} V(t,s) = \Phi(t)\Phi^{-1}(s)Q(s) &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)V(\tau,s)\mathrm{d}\omega(\tau) \\ &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)V(\tau,s)\mathrm{d}\tau \\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)V(\tau,s)\mathrm{d}\omega(\tau) \\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)V(\tau,s)\mathrm{d}\tau. \end{aligned} \tag{4.3}$$

Lemma 4.3. *For any $u \in (t, s)$ in I, we have*

$$V(s,t) = V(s,u)V(u,t)$$

in the sense of $(\mathscr{L}_d, \|\cdot\|_d)$.

Lemma 4.4. *Given $s \in I$, if $y(t) := \tilde{\Lambda}(t, s)\xi : (-\infty, s] \to L^2(\Omega, \mathbb{R}^n)$ is a solution of (1.5) with $y(s) = \tilde{\Lambda}(s, s)\xi = \xi$ such that $\tilde{\Lambda}$ is bounded in $(\mathscr{L}_d, \|\cdot\|_d)$. Then*

$$\begin{aligned} y(t) = \Phi(t)\Phi^{-1}(s)Q(s)\xi &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau + \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau) \\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau. \end{aligned}$$

For each $t \in I$, define linear operators as

$$\hat{Q}(t) = \hat{\Phi}(t,t_0)V(t_0,t_0)\hat{\Phi}(t_0,t) \quad \text{and} \quad \hat{P}(t) = Id - \hat{Q}(t), \tag{4.4}$$

where t_0 is the right boundary point of the interval I.

Lemma 4.5. *The operator $\hat{P}(t)$ is a linear projection for $t \in I$, and (3.2) holds.*

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Lemma 4.6. *For any given initial value $\xi_0 \in \mathbb{R}^n$, the function $\hat{Q}(t)\hat{\Phi}(t, s)\xi_0$ is a solution of (1.5) with $\hat{Q}(t)\hat{\Phi}(t, s)$ bounded in $(\mathscr{L}_d, \|\cdot\|_d)$.*

Lemma 4.7. *For any given initial value $\xi_0 \in \mathbb{R}^n$, the function $\hat{Q}(t)\hat{\Phi}(t, s)\xi_0$ is a solution of (1.5) with $(t, s) \in I^2_{\leqslant}$ such that*

$$\begin{aligned}\hat{\Phi}(t,s)\hat{Q}(s) = \Phi(t)\Phi^{-1}(s)Q(s)\hat{Q}(s) &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\omega(\tau)\\ &- \int_t^s \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\tau\\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\omega(\tau)\\ &+ \int_{-\infty}^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\tau. \end{aligned} \tag{4.5}$$

Lemma 4.8. *For any given initial value $\xi_0 \in \mathbb{R}^n$, the function $\hat{P}(t)\hat{\Phi}(t, s)\xi_0$ is a solution of (1.5) with $(t, s) \in I^2_{\geqslant}$ such that*

$$\begin{aligned}\hat{\Phi}(t,s)\hat{P}(s) = \Phi(t)\Phi^{-1}(s)P(s)\hat{P}(s) &- \int_t^{t_0} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\omega(\tau)\\ &- \int_t^{t_0} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{Q}(s)\mathrm{d}\tau\\ &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\omega(\tau)\\ &+ \int_s^t \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\hat{P}(s)\mathrm{d}\tau. \end{aligned} \tag{4.6}$$

Proceeding as in the proof of theorem 3.1. Squaring both sides of (4.5), and taking expectations, we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s)\|^2 \leqslant 5Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(s-t)+\varepsilon|s|}\mathbb{E}\|\hat{Q}(s)\|^2, \quad \forall\ (t,s) \in I^2_{\leqslant}. \tag{4.7}$$

Similarly, squaring both sides of (4.6), and taking expectations, we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)\|^2 \leqslant 5Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)+\varepsilon|s|}\mathbb{E}\|\hat{P}(s)\|^2, \quad \forall\ (t,s) \in I^2_{\geqslant}. \tag{4.8}$$

Meanwhile, multiplying (4.5) with $P(t)$ and (4.6) with $Q(t)$ on the left side, respectively, and let $t = s$, we obtain

$$\mathbb{E}\|P(t)\hat{Q}(t)\|^2 \leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}\mathbb{E}\|\hat{Q}(t)\|^2,$$

and

$$\mathbb{E}\|Q(t)\hat{P}(t)\|^2 \leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}\mathbb{E}\|\hat{P}(t)\|^2.$$

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Since

$$\mathbb{E}\|P(t)\|^2 \leqslant Me^{\varepsilon|t|}, \quad \mathbb{E}\|Q(t)\|^2 \leqslant Me^{\varepsilon|t|},$$

and $\hat{P}(t) - P(t) = Q(t)\hat{P}(t) - P(t)\hat{Q}(t)$, for sufficiently small b and h, we obtain the bounds for the projections $\hat{P}(t)$ and $\hat{Q}(t)$ as follows:

$$\mathbb{E}\|\hat{P}(t)\|^2 \leqslant 8Me^{\varepsilon|t|} \quad \text{and} \quad \mathbb{E}\|\hat{Q}(t)\|^2 \leqslant 8Me^{\varepsilon|t|}. \tag{4.9}$$

By (4.7), (4.8), using (4.9) we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{P}(s)\|^2 \leqslant 40M^2 e^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)+2\varepsilon|s|}, \quad \forall\ (t,s) \in I^2_{\geqslant},$$

and

$$\mathbb{E}\|\hat{\Phi}(t,s)\hat{Q}(s)\|^2 \leqslant 40Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(s-t)+2\varepsilon|s|}, \quad \forall\ (t,s) \in I^2_{\leqslant}.$$

This completes the proof of the theorem.

5. Robustness of NMS-ED on the whole $\mathbb{R}$

In this section, we consider the robustness of NMS-ED on the whole $I = \mathbb{R}$. From the last two sections we know that if (3.1) holds, the perturbed equation (1.5) remains NMS-ED on $[t_0, +\infty)$ with the operators:

$$\hat{P}_+(t) = \hat{\Phi}(t,t_0)U(t_0,t_0)\hat{\Phi}(t_0,t), \quad \hat{Q}_+(t) = Id - \hat{P}_+(t),$$

and on $(-\infty, t_0]$ with the operators:

$$\hat{Q}_-(t) = \hat{\Phi}(t,t_0)V(t_0,t_0)\hat{\Phi}(t_0,t), \quad \hat{P}_-(t) = Id - \hat{Q}_-(t).$$

The most important part in this section is to show that (1.5) has an NMS-ED on both half lines with the same projections. For this purpose, we introduce modified projections, which combine the advantages of projections $\hat{P}_+(t)$ and $\hat{Q}_-(t)$. Actually, this technique has been used in several papers to deal with this problem, see e.g. [**5**, **7**, **39**, **40**, **44**, **45**] for details.

In the following, for convenience and brevity, let us denote by $G(t, s)$ the Green function of (1.1):

$$G(t,s) := \begin{cases} P(t)\Phi(t,s), & \forall\ (t,s) \in \mathbb{R}^2_{\geqslant}, \\ -Q(t)\Phi(t,s), & \forall\ (t,s) \in \mathbb{R}^2_{\leqslant}. \end{cases}$$

Green function is a classical concept in the study of exponential dichotomy as for example [**8**, **15**]. Now we deal with the robustness of NMS-ED for (1.1) on the whole $\mathbb{R}$.

Theorem 5.1. *The assertion in theorem 3.1 remains true for $I = \mathbb{R}$.*

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Proof of theorem 5.1. Consider the Banach spaces

$$\mathscr{L}_c = \{\hat{\Phi} : \mathbb{R}^2_{\geqslant} \to \mathfrak{B}(L^2(\Omega, \mathbb{R}^n)) : \ \hat{\Phi} \text{ is continuous and } \|\hat{\Phi}\|_c < \infty\},$$

and

$$\mathscr{L}_d = \{\hat{\Phi} : \mathbb{R}^2_{\leqslant} \to \mathfrak{B}(L^2(\Omega, \mathbb{R}^n)) : \ \hat{\Phi} \text{ is continuous and } \|\hat{\Phi}\|_d < \infty\}$$

with the norm

$$\|\hat{\Phi}\|_c = \sup_{(t,s)\in\mathbb{R}^2_{\geqslant}} \left\{ (\mathbb{E}\|\hat{\Phi}(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon}{2}|s|} \right\},$$

and

$$\|\hat{\Phi}\|_d = \sup_{(t,s)\in\mathbb{R}^2_{\leqslant}} \left\{ (\mathbb{E}\|\hat{\Phi}(t,s)\|^2)^{\frac{1}{2}} e^{-\frac{\varepsilon}{2}|s|} \right\}$$

respectively. Define operator $\Gamma_1 : \mathscr{L}_c \to \mathscr{L}_c$ by

$$(\Gamma_1 U)(t,s) = \Phi(t)\Phi^{-1}(s)P(s) + \int_s^\infty G(t,\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau) + \int_s^\infty G(t,\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau,$$

and operator $\Gamma_2 : \mathscr{L}_d \to \mathscr{L}_d$,

$$(\Gamma_2 V)(t,s) = \Phi(t)\Phi^{-1}(s)Q(s) + \int_{-\infty}^s G(t,\tau)H(\tau)V(\tau,s)\mathrm{d}\omega(\tau) + \int_{-\infty}^s G(t,\tau)\tilde{B}(\tau)V(\tau,s)\mathrm{d}\tau.$$

Similar arguments to those in the proofs of lemma 3.2 and lemma 4.2 can be used to deduce that

$$\|\Gamma_1 U_1 - \Gamma_1 U_2\|_c \leqslant \theta \|U_1 - U_2\|_c,$$
$$\|\Gamma_2 V_1 - \Gamma_2 V_2\|_d \leqslant \theta \|V_1 - V_2\|_d.$$

with $\theta = \sqrt{\frac{10M\tilde{M}}{\alpha^2}} < 1$. Thus, we have the following lemma.

LEMMA 5.2. *Operators* Γ_1, Γ_2 *have unique fixed points* $U \in (\mathscr{L}_c, \|\cdot\|_c)$, *respectively* $V \in (\mathscr{L}_d, \|\cdot\|_d)$ *such that*

$$U(t,s) = \Phi(t)\Phi^{-1}(s)P(s) + \int_s^\infty G(t,\tau)H(\tau)U(\tau,s)\mathrm{d}\omega(\tau) + \int_s^\infty G(t,\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau,$$

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and

$$V(t,s)=\Phi(t)\Phi^{-1}(s)Q(s)+\int_{-\infty}^{s}G(t,\tau)H(\tau)V(\tau,s)\mathrm{d}\omega(\tau)$$
$$+\int_{-\infty}^{s}G(t,\tau)\tilde{B}(\tau)V(\tau,s)\mathrm{d}\tau.$$

Repeating arguments in the proofs of theorems 3.1 and 4.1, we obtain the following statements.

LEMMA 5.3. *For any $u\in(s,\,t)$ in I, we have*

$$U(t,s)=U(t,u)U(u,s)$$

in the sense of $(\mathscr{L}_c,\,\|\cdot\|_c)$, respectively,

$$V(t,s)=V(t,u)V(u,s)$$

in the sense of $(\mathscr{L}_d,\,\|\cdot\|_d)$.

LEMMA 5.4. *Given $s\in I$, if $x(t)=\Lambda(t,\,s)\xi:[s,\,+\infty)\to L^2(\Omega,\,\mathbb{R}^n)$ (respectively, $y(t):=\tilde{\Lambda}(t,\,s)\xi:(-\infty,\,s]\to L^2(\Omega,\,\mathbb{R}^n)$) is a solution of (1.5) with $x(s)=\Lambda(s,\,s)\xi=\xi$ (respectively, $y(s)=\tilde{\Lambda}(s,\,s)\xi=\xi$) such that Λ (respectively, $\tilde{\Lambda}$) is bounded in $(\mathscr{L}_c,\,\|\cdot\|_c)$ (respectively, $(\mathscr{L}_d,\,\|\cdot\|_d)$), then*

$$x(t)=\Phi(t)\Phi^{-1}(s)P(s)\xi+\int_{s}^{\infty}G(t,\tau)H(\tau)x(\tau)\mathrm{d}\omega(\tau)$$
$$+\int_{s}^{\infty}G(t,\tau)\tilde{B}(\tau)x(\tau)\mathrm{d}\tau, \tag{5.1}$$

and

$$y(t)=\Phi(t)\Phi^{-1}(s)Q(s)\xi+\int_{-\infty}^{s}G(t,\tau)H(\tau)y(\tau)\mathrm{d}\omega(\tau)$$
$$+\int_{-\infty}^{s}G(t,\tau)\tilde{B}(\tau)y(\tau)\mathrm{d}\tau. \tag{5.2}$$

Now we present that projection $S=\hat{P}_+(t_0)+\hat{Q}_-(t_0)$ is invertible for some $t_0\in\mathbb{R}$ with b and h sufficiently small. Using this result, we are able to define modified operators.

LEMMA 5.5. *If b and h are sufficiently small, then the operator $S=\hat{P}_+(t_0)+\hat{Q}_-(t_0)$ is invertible.*

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Proof. We first derive $\hat{P}_+(t_0)P(t_0) = \hat{P}_+(t_0)$. In fact, following the same procedure as we did for lemma 3.3, we find that

$$U(t, s) = U(t, s)P(s). \tag{5.3}$$

Since $\hat{P}_+(t_0) = U(t_0, t_0)$, by (5.3) with $t = s = t_0$ we have

$$\hat{P}_+(t_0)P(t_0) = \hat{P}_+(t_0). \tag{5.4}$$

In addition, we have (see (3.19))

$$P(t_0)\hat{P}_+(t_0) = P(t_0). \tag{5.5}$$

Since $\hat{Q}_-(t_0) = V(t_0, t_0)$, a similar argument using lemma 4.3 with $t = s = t_0$ yields

$$\hat{Q}_-(t_0)Q(t_0) = \hat{Q}_-(t_0). \tag{5.6}$$

Furthermore, it follows from $\hat{Q}_-(t) = \hat{\Phi}(t, t_0)V(t_0, t_0)\hat{\Phi}(t_0, t)$ and (4.3) with $t = s = t_0$ that

$$\begin{aligned}\hat{Q}_-(t_0) = V(t_0, t_0) = Q(t_0) &+ \int_{-\infty}^{t_0} \Phi(t_0)\Phi^{-1}(\tau)P(\tau)H(\tau)V(\tau, t_0)\mathrm{d}\omega(\tau)\\ &+ \int_{-\infty}^{t_0} \Phi(t_0)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)V(\tau, t_0)\mathrm{d}\tau.\end{aligned} \tag{5.7}$$

Since $P(t_0)$ and $Q(t_0)$ are complementary projections, multiplying (5.7) on the left with $Q(t_0)$ gives

$$Q(t_0)\hat{Q}_-(t_0) = Q(t_0). \tag{5.8}$$

We now consider the linear operators

$$S_1 := Id - P(t_0) + \hat{P}_+(t_0) \quad \text{and} \quad T_1 := Id + P(t_0) - \hat{P}_+(t_0). \tag{5.9}$$

It follows easily from (5.4) and (5.5) that $S_1T_1 = T_1S_1 = Id$. Therefore, S_1 is invertible and $S_1^{-1} = T_1$. In addition, using again (5.5) we obtain

$$\begin{aligned}S_1 - Id &= \hat{P}_+(t_0) - P(t_0)\\ &= \hat{P}_+(t_0) - P(t_0)\hat{P}_+(t_0)\\ &= Q(t_0)\hat{P}_+(t_0).\end{aligned} \tag{5.10}$$

By (3.18), we have

$$\begin{aligned}Q(t_0)\hat{P}_+(t_0) = &- \int_{t_0}^{\infty} \Phi(t_0)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau, t_0)\mathrm{d}\omega(\tau)\\ &- \int_{t_0}^{\infty} \Phi(t_0)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau, t_0)\mathrm{d}\tau.\end{aligned} \tag{5.11}$$

To estimate the bounds of the integral in the mean-square sense, we need to find out the bounds for $U(t, t_0)$ with $t \geqslant t_0$. Squaring both sides of (3.5), taking expectations

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and proceeding as in the proof of theorem 3.1, for any $t \geqslant t_0$, we have

$$\begin{aligned}
&\mathbb{E}\|U(t,t_0)\|^2 \\
&\quad \leqslant 5\mathbb{E}\|\Phi(t)\Phi^{-1}(t_0)P(t_0)\|^2 \\
&\qquad + 5\mathbb{E}\left\|\int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)U(\tau,t_0)\mathrm{d}\omega(\tau)\right\|^2 \\
&\qquad + 5\mathbb{E}\left\|\int_{t_0}^{t}\Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)U(\tau,t_0)\mathrm{d}\tau\right\|^2 \\
&\qquad + 5\mathbb{E}\left\|\int_{t}^{\infty}\Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,t_0)\mathrm{d}\omega(\tau)\right\|^2 \\
&\qquad + 5\mathbb{E}\left\|\int_{t}^{\infty}\Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,t_0)\mathrm{d}\tau\right\|^2 \\
&\quad \leqslant 5Me^{-\frac{\alpha}{2}(t-t_0)+\varepsilon|t_0|} \\
&\qquad + \frac{5M\tilde{M}}{\alpha}\left(\int_{t_0}^{t}e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\|U(\tau,t_0)\|^2\mathrm{d}\tau + \int_{t}^{\infty}e^{-\frac{\alpha}{2}(\tau-t)}\mathbb{E}\|U(\tau,t_0)\|^2\mathrm{d}\tau\right) \\
&\quad \leqslant 5Me^{-\hat{\alpha}(t-t_0)+\varepsilon|t_0|}.
\end{aligned} \tag{5.12}$$

By (5.10), using (5.11) and (5.12), we obtain

$$\begin{aligned}
\mathbb{E}\|S_1 - Id\|^2 &= \mathbb{E}\|Q(t_0)\hat{P}_+(t_0)\|^2 \\
&\leqslant 2\mathbb{E}\left\|\int_{t_0}^{\infty}\Phi(t_0)\Phi^{-1}(\tau)Q(\tau)H(\tau)U(\tau,t_0)\mathrm{d}\omega(\tau)\right\|^2 \\
&\quad + 2\mathbb{E}\left\|\int_{t_0}^{\infty}\Phi(t_0)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,t_0)\mathrm{d}\tau\right\|^2 \\
&\leqslant 2\int_{t}^{\infty}\mathbb{E}\|\Phi(t)\Phi^{-1}(\tau)Q(\tau)\|^2\mathbb{E}\|H(\tau)\|^2\mathbb{E}\|U(\tau,t_0)\|^2\mathrm{d}\tau \\
&\quad + 2\left(\int_{t_0}^{\infty}\mathbb{E}\|\Phi(t_0)\Phi^{-1}(\tau)Q(\tau)\|\mathbb{E}\|\tilde{B}(\tau)\|^{\frac{1}{2}}\mathrm{d}\tau\right) \\
&\quad \times\left(\int_{t_0}^{\infty}\mathbb{E}\|\Phi(t_0)\Phi^{-1}(\tau)Q(\tau)\|\mathbb{E}\|\tilde{B}(\tau)\|^{\frac{3}{2}}\mathbb{E}\|U(\tau,t_0)\|^2\mathrm{d}\tau\right) \\
&\leqslant \frac{10M^2\tilde{M}}{\alpha}\int_{t_0}^{\infty}e^{-(\alpha+\tilde{\alpha}-\varepsilon)(\tau-t_0)}\mathrm{d}\tau \\
&\leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}.
\end{aligned} \tag{5.13}$$

Meanwhile, we consider the linear operators

$$S_2 := Id - Q(t_0) + \hat{Q}_-(t_0) \quad \text{and} \quad T_2 := Id + Q(t_0) - \hat{Q}_-(t_0). \tag{5.14}$$

It follows easily from (5.6) and (5.8) that $S_2T_2 = T_2S_2 = Id$. Therefore, S_2 is invertible and $S_2^{-1} = T_2$. In addition, using again (5.8) we obtain

$$\begin{aligned} S_2 - Id &= \hat{Q}_-(t_0) - Q(t_0) \\ &= \hat{Q}_-(t_0) - Q(t_0)\hat{Q}_-(t_0) \\ &= P(t_0)\hat{Q}_-(t_0). \end{aligned} \tag{5.15}$$

By (5.7),

$$\begin{aligned} P(t_0)\hat{Q}_-(t_0) = &\int_{-\infty}^{t_0} \Phi(t_0)\Phi^{-1}(\tau)P(\tau)H(\tau)V(\tau,t_0)\mathrm{d}\omega(\tau) \\ &+ \int_{-\infty}^{t_0} \Phi(t_0)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)V(\tau,t_0)\mathrm{d}\tau. \end{aligned} \tag{5.16}$$

Similarly, for any $t \leqslant t_0$, one can deduce from (4.3) that

$$\begin{aligned} &\mathbb{E}\|V(t,t_0)\|^2 \\ &\quad\leqslant 5\mathbb{E}\|\Phi(t)\Phi^{-1}(t_0)Q(t_0)\|^2 \\ &\qquad + 5\mathbb{E}\left\|\int_t^{t_0} \Phi(t)\Phi^{-1}(\tau)Q(\tau)H(\tau)V(\tau,s)\mathrm{d}\omega(\tau)\right\|^2 \\ &\qquad + 5\mathbb{E}\left\|\int_t^{t_0} \Phi(t)\Phi^{-1}(\tau)Q(\tau)\tilde{B}(\tau)U(\tau,s)\mathrm{d}\tau\right\|^2 \\ &\qquad + 5\mathbb{E}\left\|\int_{-\infty}^{t} \Phi(t)\Phi^{-1}(\tau)P(\tau)H(\tau)V(\tau,t_0)\mathrm{d}\omega(\tau)\right\|^2 \\ &\qquad + 5\mathbb{E}\left\|\int_{-\infty}^{t} \Phi(t)\Phi^{-1}(\tau)P(\tau)\tilde{B}(\tau)V(\tau,t_0)\mathrm{d}\tau\right\|^2 \\ &\quad\leqslant 5Me^{-\frac{\alpha}{2}(t_0-t)+\varepsilon|t_0|} \\ &\qquad + \frac{5M\tilde{M}}{\alpha}\left(\int_t^{t_0} e^{-\frac{\alpha}{2}(\tau-t)}\mathbb{E}\|V(\tau,t_0)\|^2\mathrm{d}\tau + \int_{-\infty}^{t} e^{-\frac{\alpha}{2}(t-\tau)}\mathbb{E}\|V(\tau,t_0)\|^2\mathrm{d}\tau\right) \\ &\quad\leqslant 5Me^{-\hat{\alpha}(t_0-t)+\varepsilon|t_0|}. \end{aligned} \tag{5.17}$$

Therefore, by (5.15), using (5.16) and (5.17) we obtain

$$\mathbb{E}\|S_2 - Id\|^2 = \mathbb{E}\|P(t_0)\hat{Q}_-(t_0)\|^2 \leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}. \tag{5.18}$$

Besides, it follows easily from (5.8) that $P(t_0)\hat{P}_-(t_0) = \hat{P}_-(t_0)$. Using also (5.5) yields

$$\begin{aligned} \hat{P}_+(t_0) + \hat{Q}_-(t_0) - Id &= \hat{P}_+(t_0) - P(t_0) + P(t_0) - \hat{P}_-(t_0) \\ &= \hat{P}_+(t_0) - P(t_0)\hat{P}_+(t_0) + P(t_0) - P(t_0)\hat{P}_-(t_0) \\ &= Q(t_0)\hat{P}_+(t_0) + P(t_0)\hat{Q}_-(t_0). \end{aligned}$$

By (5.13) and (5.18) we obtain

$$\begin{aligned}\mathbb{E}\|\hat{P}_+(t_0)+\hat{Q}_-(t_0)-Id\|^2 &= \mathbb{E}\|Q(t_0)\hat{P}_+(t_0)+P(t_0)\hat{Q}_-(t_0)\|^2\\ &\leqslant 2\mathbb{E}\|Q(t_0)\hat{P}_+(t_0)\|^2+2\mathbb{E}\|P(t_0)\hat{Q}_-(t_0)\|^2\\ &\leqslant \frac{20M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}. \end{aligned}\tag{5.19}$$

Moreover,

$$\begin{aligned} S &= \hat{P}_+(t_0)+\hat{Q}_-(t_0)\\ &= (\hat{P}_+(t_0)+Q(t_0))+(P(t_0)+\hat{Q}_-(t_0))-Id\\ &= S_1+S_2-Id. \end{aligned}\tag{5.20}$$

Since $\tilde{M}:=8b^2+8g^2h^2+\alpha h^2$, by (5.13), respectively, (5.18), we can make invertible operators S_1 and S_2 such that $\mathbb{E}\|S_1-Id\|^2$ and $\mathbb{E}\|S_2-Id\|^2$ as small as desired with b and h sufficiently small. So if taking b and h sufficiently small, it follows from (5.19) and (5.20) that $S=\hat{P}_+(t_0)+\hat{Q}_-(t_0)$ is invertible.

For each $t\in I$, define linear operators as

$$\tilde{P}(t)=\hat{\Phi}(t,t_0)SP(t_0)S^{-1}\hat{\Phi}(t_0,t)\quad\text{and}\quad \tilde{Q}(t)=Id-\tilde{P}(t).\tag{5.21}$$

LEMMA 5.6. *The operator $\tilde{P}(t)$ is a linear projection for $t\in I$, and (3.2) holds for any $t,\ s\in\mathbb{R}$.*

Proof. Obviously,

$$\tilde{P}(t)\tilde{P}(t)=\hat{\Phi}(t,t_0)SP^2(t_0)S^{-1}\hat{\Phi}(t_0,t)=\tilde{P}(t).$$

Moreover, for any $t,\ s\in\mathbb{R}$, we obtain

$$\begin{aligned}\tilde{P}(t)\hat{\Phi}(t,s) &= \hat{\Phi}(t,t_0)SP(t_0)S^{-1}\hat{\Phi}(t_0,t)\hat{\Phi}(t,s)\\ &= \hat{\Phi}(t,s)\hat{\Phi}(s,t_0)SP(t_0)S^{-1}\hat{\Phi}(t_0,s)\\ &= \hat{\Phi}(t,s)\tilde{P}(s),\end{aligned}$$

and this completes the proof of the lemma.

LEMMA 5.7. *For any given initial value $\xi_0\in\mathbb{R}^n$, the function $\tilde{P}(t)\hat{\Phi}(t,\ s)\xi_0$ is a solution of (1.5) with $\tilde{P}(t)\hat{\Phi}(t,\ s)$ bounded in $(\mathscr{L}_c,\ \|\cdot\|_c)$, respectively, the function $\tilde{Q}(t)\hat{\Phi}(t,\ s)\xi_0$ is a solution of (1.5) with $\tilde{Q}(t)\hat{\Phi}(t,\ s)$ bounded in $(\mathscr{L}_d,\ \|\cdot\|_d)$.*

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Proof. In view of (5.4) and (5.6), we have

$$SP(t_0) = \hat{P}_+(t_0)P(t_0) + \hat{Q}_-(t_0)P(t_0) = \hat{P}_+(t_0),$$
$$SQ(t_0) = \hat{P}_+(t_0)Q(t_0) + \hat{Q}_-(t_0)Q(t_0) = \hat{Q}_-(t_0).$$

Thus,

$$\tilde{P}(t)\hat{\Phi}(t,s) = \hat{\Phi}(t,t_0)SP(t_0)S^{-1}\hat{\Phi}(t_0,t)\hat{\Phi}(t,s) = \hat{\Phi}(t,t_0)\hat{P}_+(t_0)S^{-1}\hat{\Phi}(t_0,s),$$
$$\tilde{Q}(t)\hat{\Phi}(t,s) = \hat{\Phi}(t,t_0)SQ(t_0)S^{-1}\hat{\Phi}(t_0,t)\hat{\Phi}(t,s) = \hat{\Phi}(t,t_0)\hat{Q}_-(t_0)S^{-1}\hat{\Phi}(t_0,s).$$

Therefore, it follows from lemma 3.6 that $\tilde{P}(t)\hat{\Phi}(t,\, s)\xi_0 = \hat{\Phi}(t,\, t_0)\hat{P}_+(t_0)S^{-1}\hat{\Phi}(t_0,\, s)\xi_0$ is a solution of (1.5) with initial value $S^{-1}\hat{\Phi}(t_0,\, s)\xi_0 \in \mathbb{R}^n$ with $\tilde{P}(t)\hat{\Phi}(t,\, s)$ bounded in $(\mathscr{L}_c,\, \|\cdot\|_c)$. Similarly, by lemma 4.6, we have $\tilde{Q}(t)\hat{\Phi}(t,\, s)\xi_0$ as a solution of (1.5) with initial value $S^{-1}\hat{\Phi}(t_0,\, s)\xi_0 \in \mathbb{R}^n$ with $\tilde{Q}(t)\hat{\Phi}(t,\, s)$ bounded in $(\mathscr{L}_d,\, \|\cdot\|_d)$.

LEMMA 5.8. *For any given initial value $\xi_0 \in \mathbb{R}^n$, the function $\tilde{P}(t)\hat{\Phi}(t,\, s)\xi_0$ is a solution of (1.5) with $(t,\, s) \in \mathbb{R}^2_{\geqslant}$ such that*

$$\hat{\Phi}(t,s)\tilde{P}(s) = \Phi(t)\Phi^{-1}(s)P(s)\tilde{P}(s) + \int_s^{\infty} G(t,\tau)H(\tau)\hat{\Phi}(\tau,s)\tilde{P}(s)\mathrm{d}\omega(\tau) + \int_s^{\infty} G(t,\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\tilde{P}(s)\mathrm{d}\tau, \tag{5.22}$$

and the function $\tilde{Q}(t)\hat{\Phi}(t,\, s)\xi_0$ is a solution of (1.5) with $(t,\, s) \in \mathbb{R}^2_{\leqslant}$ such that

$$\hat{\Phi}(t,s)\tilde{Q}(s) = \Phi(t)\Phi^{-1}(s)Q(s)\tilde{Q}(s) + \int_{-\infty}^{s} G(t,\tau)H(\tau)\hat{\Phi}(\tau,s)\tilde{Q}(s)\mathrm{d}\omega(\tau) + \int_{-\infty}^{s} G(t,\tau)\tilde{B}(\tau)\hat{\Phi}(\tau,s)\tilde{Q}(s)\mathrm{d}\tau. \tag{5.23}$$

Proof. Let $x(t) = \tilde{P}(t)\hat{\Phi}(t,\, s)\xi_0$ (respectively, $y(t) = \tilde{Q}(t)\hat{\Phi}(t,\, s)\xi_0$) with given $s \in \mathbb{R}$, and denote $\xi = \tilde{P}(s)\xi_0$ the initial condition at time s. Clearly, $x(t)$ (respectively, $y(t)$) is a solution of (1.5) with $x(s) = \tilde{P}(s)\xi = \tilde{P}(s)\tilde{P}(s)\xi_0 = \xi$ (respectively, $y(s) = \tilde{Q}(s)\xi = \tilde{Q}(s)\tilde{Q}(s)\xi_0 = \xi$). By lemma 5.7, $\tilde{P}(t)\hat{\Phi}(t,\, s)$ (respectively, $\tilde{Q}(t)\hat{\Phi}(t,\, s)$) is bounded in $(\mathscr{L}_c,\, \|\cdot\|_c)$ (respectively, $(\mathscr{L}_d,\, \|\cdot\|_d)$). Since ξ_0 is arbitrary in $\mathbb{R}^n$, identity (5.22) (respectively, (5.23)) follows now readily from (5.1) (respectively, (5.2)).

Proceed as in the proof of theorem 3.1. Squaring both sides of (5.22), and taking expectations, we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\tilde{P}(s)\|^2 \leqslant 5Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(s-t)+\varepsilon|s|}\mathbb{E}\|\tilde{P}(s)\|^2, \quad \forall\ (t,s) \in \mathbb{R}^2_{\geqslant}. \tag{5.24}$$

Similarly, squaring both sides of (5.23), and taking expectations, we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\tilde{Q}(s)\|^2 \leqslant 5Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)+\varepsilon|s|}\mathbb{E}\|\tilde{Q}(s)\|^2, \quad \forall\ (t,s) \in \mathbb{R}^2_{\leqslant}. \tag{5.25}$$

Meanwhile, multiplying (5.22) with $Q(t)$ and (5.23) with $P(t)$ on the left side, respectively, and let $t = s$, we obtain

$$\mathbb{E}\|Q(t)\tilde{P}(t)\|^2 \leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}\mathbb{E}\|\tilde{P}(t)\|^2,$$

and

$$\mathbb{E}\|P(t)\tilde{Q}(t)\|^2 \leqslant \frac{10M^2\tilde{M}}{\alpha(\alpha+\tilde{\alpha}-\varepsilon)}\mathbb{E}\|\tilde{Q}(t)\|^2.$$

Since

$$\mathbb{E}\|P(t)\|^2 \leqslant Me^{\varepsilon|t|}, \quad \mathbb{E}\|Q(t)\|^2 \leqslant Me^{\varepsilon|t|},$$

and $\tilde{P}(t) - P(t) = Q(t)\tilde{P}(t) - P(t)\tilde{Q}(t)$, for sufficiently small b and h, we obtain the bounds for the projections $\tilde{P}(t)$ and $\tilde{Q}(t)$ as follows:

$$\mathbb{E}\|\tilde{P}(t)\|^2 \leqslant 8Me^{\varepsilon|t|} \quad \text{and} \quad \mathbb{E}\|\tilde{Q}(t)\|^2 \leqslant 8Me^{\varepsilon|t|}. \tag{5.26}$$

By (5.24), (5.25), using (5.26) we obtain

$$\mathbb{E}\|\hat{\Phi}(t,s)\tilde{P}(s)\|^2 \leqslant 40M^2e^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(t-s)+2\varepsilon|s|}, \quad \forall\ (t,s)\in\mathbb{R}^2_{\geqslant},$$

and

$$\mathbb{E}\|\hat{\Phi}(t,s)\tilde{Q}(s)\|^2 \leqslant 40Me^{(-\frac{\alpha}{2}+\frac{10M\tilde{M}}{\alpha})(s-t)+2\varepsilon|s|}, \quad \forall\ (t,s)\in\mathbb{R}^2_{\leqslant}.$$

This completes the proof of the theorem.

Remark 5.9. By (5.9), using (5.4) and (5.5), we obtain

$$\begin{aligned} S_1P(t_0)S_1^{-1} &= (Id - P(t_0) + \hat{P}_+(t_0))P(t_0)(Id + P(t_0) - \hat{P}_+(t_0)) \\ &= \hat{P}_+(t_0) = U(t_0,t_0). \end{aligned}$$

Thus, it follows from (3.14) that

$$\hat{P}_+(t) = \hat{\Phi}(t,t_0)U(t_0,t_0)\hat{\Phi}(t_0,t) = \hat{\Phi}(t,t_0)S_1P(t_0)S_1^{-1}\hat{\Phi}(t_0,t). \tag{5.27}$$

Meanwhile, by (5.14), using (5.6) and (5.8), we obtain

$$\begin{aligned} S_2Q(t_0)S_2^{-1} &= (Id - Q(t_0) + \hat{Q}_-(t_0))Q(t_0)(Id + Q(t_0) - \hat{Q}_-(t_0)) \\ &= \hat{Q}_-(t_0) = V(t_0,t_0). \end{aligned}$$

Thus it follows from (4.4) that

$$\hat{Q}_-(t) = \hat{\Phi}(t,t_0)V(t_0,t_0)\hat{\Phi}(t_0,t) = \hat{\Phi}(t,t_0)S_2Q(t_0)S_2^{-1}\hat{\Phi}(t_0,t),$$

and consequently,

$$\hat{P}_-(t) = \hat{\Phi}(t,t_0)V(t_0,t_0)\hat{\Phi}(t_0,t) = \hat{\Phi}(t,t_0)S_2P(t_0)S_2^{-1}\hat{\Phi}(t_0,t). \tag{5.28}$$

By (5.21), (5.27) and (5.28), we know that linear operators $\hat{P}_+(t)$, $\hat{P}_-(t)$ and $\tilde{P}(t)$, defined on $[t_0, +\infty)$, $(-\infty, t_0]$ and $\mathbb{R}$ respectively, are actually obtained under the same rules.

REMARK 5.10. Throughout this paper, we choose any fixed $t_0 \in \mathbb{R}$ instead of $0 \in \mathbb{R}$, which is a little different from the one given in uniform exponential dichotomy (see e.g. [**44**]), where the initial point 0 is used for simplicity, and there is no substantial difference in inequalities thus obtained. However, here we have to choose general term t_0 instead of 0 since the nonuniform item will vanish at time 0, and hence there is a significant difference in some calculations.

6. Example

In what follows we use an example to demonstrate our results. The following example shows that there exists a linear SDE which admits an NMS-ED but not uniform.

EXAMPLE 6.1. Let $a > b > 0$ be real parameters. Then the following linear SDE:

$$\begin{cases} \mathrm{d}u &= (-a - bt\sin t)u(t)\mathrm{d}t + \sqrt{2b\cos t}\exp(-at + bt\cos t)\mathrm{d}\omega(t) \\ \mathrm{d}v &= (a + bt\sin t)v(t)\mathrm{d}t - \sqrt{2b\cos t}\exp(at - bt\cos t)\mathrm{d}\omega(t) \end{cases} \tag{6.1}$$

with the initial condition $u(0) = v(0) = 1$ admits an NMS-ED that is not a uniform MS-ED.

Proof. Let

$$\Phi(t) = \begin{pmatrix} U(t) & 0 \\ 0 & V(t) \end{pmatrix}$$

be a fundamental matrix solution of (6.1). Thus we have $u(t) = U(t)U^{-1}(s)u(s)$ and $v(t) = V(t)V^{-1}(s)v(s)$. In addition, it is easy to verify that

$$\begin{pmatrix} \exp(-at + bt\cos t - b\sin t) & 0 \\ 0 & \exp(at - bt\cos t + b\sin t) \end{pmatrix}$$

is a fundamental matrix solution of

$$\begin{cases} \mathrm{d}u &= (-a - bt\sin t)u(t)\mathrm{d}t, \\ \mathrm{d}v &= (a + bt\sin t)v(t)\mathrm{d}t. \end{cases}$$

Hence, by [**16**, p. 97], the solution of (6.1) is given by

$$\begin{cases} u(t) &= \exp(-at + bt\cos t - b\sin t)\left(1 + \sqrt{2b}\int_0^t e^{b\sin s}\sqrt{\cos s}\mathrm{d}\omega(s)\right), \\ v(t) &= \exp(at - bt\cos t + b\sin t)\left(1 - \sqrt{2b}\int_0^t e^{-b\sin s}\sqrt{\cos s}\mathrm{d}\omega(s)\right), \end{cases}$$

since $u(0) = v(0) = 1$. Therefore,

$$\begin{aligned} \mathbb{E}\|u(t)\|^2 &= \exp(-2at + 2bt\cos t - 2b\sin t)\left(1 + 2b\int_0^t e^{2b\sin s}\cos s\mathrm{d}s\right) \\ &= \exp(-2at + 2bt\cos t). \end{aligned}$$

Thus, one can obtain

$$\mathbb{E}\|U(t)U^{-1}(s)\|^2 = \frac{\mathbb{E}\|u(t)\|^2}{\mathbb{E}\|u(s)\|^2} = e^{-2a(t-s)+2b(t\cos t - s\cos s)}$$

since $\mathbb{E}\|u(s)\|^2 > 0$. It is easy to see that

$$\mathbb{E}\|U(t)U^{-1}(s)\|^2 = e^{(-2a+2b)(t-s)+2bt(\cos t-1)-2bs(\cos s-1)},$$

and thus

$$\mathbb{E}\|U(t)U^{-1}(s)\|^2 \leqslant e^{(-2a+2b)(t-s)+2bs}, \quad \forall\ (t,s) \in I^2_{\geqslant}. \tag{6.2}$$

Furthermore, if $t = 4k\pi$ and $s = 3k\pi$ with $k \in \mathbb{N}$, then

$$\mathbb{E}\|U(t)U^{-1}(s)\|^2 = e^{(-2a+2b)(t-s)+2bs}, \quad \forall\ (t,s) \in I^2_{\geqslant}. \tag{6.3}$$

Similarly, one can prove that

$$\mathbb{E}\|V(t)V^{-1}(s)\|^2 \leqslant e^{(-2a+2b)(s-t)+2bs}, \quad \forall\ (t,s) \in I^2_{\leqslant}, \tag{6.4}$$

and

$$\mathbb{E}\|V(t)V^{-1}(s)\|^2 = e^{(-2a+2b)(s-t)+2bs}, \quad \forall\ (t,s) \in I^2_{\leqslant} \tag{6.5}$$

if $t = 4k\pi$ and $s = 3k\pi$ with $k \in \mathbb{N}$. Thus, (6.1) admits an NMS-ED. By (6.3) and/or (6.5), the exponential e^{2bs} in (6.2) and/or (6.4) cannot be removed. This shows that the NMS-ED is not uniform.

REMARK 6.2. The SDE (6.1) in example 6.1 admitting an NMS-ED is linear in the narrow sense. Following the same idea and method in [**60**], one can establish a general linear SDE, which admits an NMS-ED. For example, let $a > b > 0$ be real parameters, one can prove the following linear SDE

$$\begin{cases} \mathrm{d}u &= (-a - bt\sin t)u(t)\mathrm{d}t + u(t)\mathrm{d}\omega(t) \\ \mathrm{d}v &= (a + bt\sin t)v(t)\mathrm{d}t + v(t)\mathrm{d}\omega(t) \end{cases}$$

with the initial condition $u(0) = v(0) = 1$ admitting an NMS-ED that is not a uniform MS-ED.

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