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On Injective and Cofibrant Equivariant Minimal Models

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Abstract. Let X be a G-connected nilpotent simplicial set, where G is a finite Hamiltonian group. We construct a cofibrant equivariant minimal model of X with the strong homotopy type of the injective minimal model of X defined by Triantafillou.

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Rational homotopy theory provides a first approximation to integral homotopy theory that has attracted much research interest. Sullivan [11] introduced the rational de Rham theory for connected simplicial complexes and applied it to prove that the de Rham algebra A^*_X of $\mathbb{Q}$-differential forms on a simply connected complex X of finite type determines its rational homotopy type, where $\mathbb{Q}$ is the field of rationals.

Triantafillou [13–15] has generalized the central results of Sullivan theory to an equivariant context but under the assumption that a G-simplicial set X of finite type (with a finite group G action) is G-connected and nilpotent, i.e. the fixed point simplicial subsets X^H are nonempty, connected and nilpotent for all subgroups $H \subseteq G$. In this case not only A^*_X with the induced G-action is considered, but also the de Rham algebras $A^*_{X^H}$ of X^H for all subgroups $H \subseteq G$. Thus, a functor $\mathcal{A}^*_X$ on the category $\mathcal{O}(G)$ of canonical orbits is studied and its componentwise injectivity is the key property for the existence of an injective minimal model $\mathcal{M}^{inj}_X$. This is an equivariant analog of the Sullivan minimal model, for a G-connected simplicial set X. On the other hand, in [6, 7], Lambre showed that $\mathcal{M}^{inj}_X$ could be replaced by a cofibrant minimal model $\mathcal{M}_X$ provided G is the cyclic group $\mathbb{Z}_{p^k}$ with p a prime number and examples of cofibrant equivariant minimal models were presented as well.

This paper grew out of our attempt to study the results presented in [14] and generalize the construction of the cofibrant equivariant minimal model for G a finite Hamiltonian group.

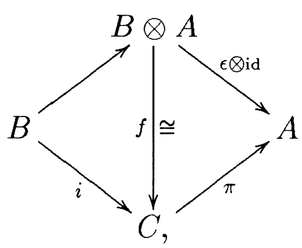
Let DGA_k be the category of (homologically) connected (i.e. $H^0(A) = k$ for $A \in DGA_k$) commutative differential graded k-algebras. It has been proved [1, 10] that this category forms a closed model category in the sense of Quillen [10] and the most important cofibrant algebras are the minimal ones. From [2], it follows that this category structure can be extended to the category $\mathcal{O}(G)$-DGA_k of $\mathcal{O}(G)$-algebras provided that G is a finite Hamiltonian group.

We explain how to construct cofibrant minimal models of $\mathcal{O}(G)$-algebras and list their basic properties. An idea of this construction (for a special case) has been presented in [6] and based on the notion of a Koszul–Sullivan extension considered in [5]. Then, some geometric applications are presented. For any G-connected nilpotent simplicial set X (of finite type) we can consider polynomial forms $A^*_{X^H}$ for all subgroups $H \subseteq G$ to obtain the $\mathcal{O}(G)$-algebra $\mathcal{A}^*(X)$ over the field $\mathbb{Q}$ of rationals. We show that the cofibrant minimal model $\mathcal{M}_X$ of $\mathcal{A}^*(X)$ has the (strong) homotopy type of its injective minimal model $\mathcal{M}_X^{inj}$ considered in [14, 15].

1. Preliminaries

Let k be a zero characteristic field and DGA_k the category of homologically connected commutative differential graded k-algebras. For a map $\gamma: B \to E$ in DGA_k, where B is augmented, Halperin [5] considers its 'minimal factorization'. Namely, he defines a *minimal KS-extension* as a sequence of augmented DGA_k's $\mathbb{E} : B \xrightarrow{i} C \xrightarrow{\pi} A$ such that

(i) A is a free graded algebra generated by a graded vector subspace $X \subset A$;
(ii) there is a commutative diagram of algebra homomorphisms



where $\varepsilon: A \to k$ is the augmentation map;

(iii) there is a homogeneous basis $\{x_\alpha\}_{\alpha \in I}$ of X indexed by an ordered set I with $d(x_\alpha) \in B \otimes A_{<\alpha}$ and $\deg x_\beta < \deg x_\alpha$ implies $\beta < \alpha$, where $A_{<\alpha}$ is the subalgebra of A generated by the x_β with $\beta < \alpha$.

In [5] the following result is proved.

THEOREM 1.1. *For any map $\gamma\colon B \to E$ of connected DGA_k's, where B is augmented, there is a unique (up to isomorphism) minimal KS-extension*

$$\mathbb{E}\colon B \xrightarrow{i} C \xrightarrow{\pi} A$$

and a homology isomorphism $\rho : C \to E$ such that $\rho \circ i = \gamma$.

The extension $\mathbb{E}$ together with the map $\rho\colon C \to E$ is called a *KS-minimal model* of the map γ. In particular, a minimal algebra M_A together with a homology isomorphism $\rho_A\colon M_A \to A$ is isomorphic to the *minimal model* for A.

Let now G be a finite group and G-DGA_k the category of differential graded algebras with an action of G. Then a notion of a minimal KS-extension may be considered in G-DGA_k as well and in [3] (see also [9, 13]) it has been shown that an equivariant version of Theorem 1.1 yields a G-KS-minimal model of a map $\gamma : B \to E$ in G-DGA_k.

Let $\mathcal{O}(G)$ be the category of canonical orbits; its objects are the left cosets G/H as H ranges over all subgroups of G and morphisms are the equivariant maps $G/H \longrightarrow G/K$ with respect to left translation. Following [8], we define a partial order (which is crucial for the sequel) on the set $\mathrm{Is}(\mathcal{O}(G))$ of isomorphism classes $\overline{G/H}$ of objects G/H in $\mathcal{O}(G)$ by

$$\overline{G/H} \leqslant \overline{G/K} \quad \text{if} \quad \mathcal{O}(G)(G/H, G/K) \neq \emptyset.$$

This induces a partial ordering on the set $\mathrm{Is}(\mathcal{O}(G))$. We say that $\overline{G/H} < \overline{G/K}$ if $\overline{G/H} \leqslant \overline{G/K}$ and $\overline{G/H} \neq \overline{G/K}$. The group G is finite, so $\mathcal{O}(G)$ is a *cofinite* category (i.e. each $\overline{G/H}$ isomorphism class of an element G/H has finitely many predecessors). Thus, for any $G/H \in \mathrm{Ob}\,\mathcal{O}(G)$ we can define its *height* as the maximum length of a chain $\overline{G/H_0} < \cdots < \overline{G/H_n} = \overline{G/H}$.

Denote by $\mathcal{O}(G)$-DGA_k the category of covariant functors from the category $\mathcal{O}(G)$ to the category DGA_k. Objects of $\mathcal{O}(G)$-DGA_k are called $\mathcal{O}(G)$-*algebras*.

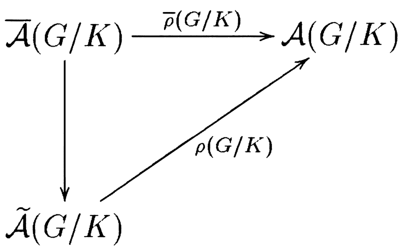
Throughout the rest of this section, G will be a finite Hamiltonian group (i.e. each subgroup of G is normal). In light of [4, Corollary 10.3.4] any finite Hamiltonian group is nilpotent and the key example of a non-abelian finite Hamiltonian group is the quaternion group. A full description of these groups has been presented by Hall in [4, Theorem 12.5.4]. Observe that for such a group G the automorphism group $\mathrm{Aut}(G/H)$ of any object G/H in $\mathcal{O}(G)$ can be identified with the quotient group G/H. Moreover, any map $\phi : G/H \to G/K$ in $\mathcal{O}(G)$ determines an element $g \in G$ with $g^{-1}Hg \subseteq K$ and an epimorphism $\hat{\phi} : G/H \to G/K$ given by $\hat{\phi}(xH) = g^{-1}xgK$ for any xH in G/H. Consequently, we may state that the category $\mathcal{O}(G)$ has the additional property (useful in the construction below).

For any map $\phi : G/H \to G/K$ in $\mathcal{O}(G)$, there is an epimorphism $\hat{\phi} : G/H \to G/K$ with $\phi \circ \gamma = \hat{\phi}(\gamma) \circ \phi$ for all γ in $\mathrm{Aut}(G/H) = G/H$ and $\widehat{\psi\phi} = \hat{\psi}\hat{\phi}$ for a map $\psi : G/K \to G/L$ in $\mathcal{O}(G)$.

For a given $\mathcal{A}$ in $\mathcal{O}(G)$-DGA_k and a map $\phi : G/H \to G/K$ there is an action of the quotient group G/H on $\mathcal{A}(G/K)$ and $\mathcal{A}(\phi) : \mathcal{A}(G/H) \to \mathcal{A}(G/K)$ is an G/H-map. Denote by $I_\phi(G/H)(\mathcal{A})$ the ideal in $\mathcal{A}(G/H)$ generated by elements $a - ga$, for $a \in \mathcal{A}(G/H)$ and $g \in \ker \hat{\phi}$. Then $\mathcal{A}_\phi(G/H) = \mathcal{A}(G/H)/I_\phi(G/H)(\mathcal{A})$ is an object in G/K-DGA_k and the induced map $\mathcal{A}_\phi(G/H) \to \mathcal{A}(G/K)$ preserves the G/K-action.

For a fixed G/K in $\mathrm{Ob}\,(\mathcal{O}(G))$, let $\mathcal{O}(G)_{G/K}$ be the category, where objects are pairs $(G/H, \phi)$, where $\phi : G/H \to G/K$ is a non-isomorphism and maps from $(G/H_1, \phi_1)$ to $(G/H_2, \phi_2)$ are determined by maps $\psi : G/H_1 \to G/H_2$ such that $\phi_2\psi = \phi_1$.

We also have a functor $\overline{\mathcal{A}}_{G/K} : \mathcal{O}(G)_{G/K} \to DGA_k$ such that $\overline{\mathcal{A}}_{G/K}(G/H, \phi) = \mathcal{A}_\phi(G/H)$. Hence $\overline{\mathcal{A}}(G/K) = \varinjlim_{\mathcal{O}(G)_{G/K}} \overline{\mathcal{A}}_{G/K}$ is an object in G/K-DGA_k and there is the induced G/K-map $\overline{\rho}(G/K) : \overline{\mathcal{A}}(G/K) \to \mathcal{A}(G/K)$. The algebra $\overline{\mathcal{A}}(G/K)$ is augmented and we may take the G/K-KS-minimal model

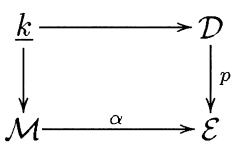


of the map $\overline{\rho}(G/K)$.

We say that an object $\mathcal{M}$ in $\mathcal{O}(G)$-DGA_k is *KS-minimal* if $\mathcal{M}(G/H) \cong \widetilde{\mathcal{M}}(G/H)$ for any object $G/H \in \mathrm{Ob}\,(\mathcal{O}(G))$. We recall from [2] that the category $\mathcal{O}(G)$-DGA_k is a closed model category in the sense of Quillen [10] with respect to the following three classes of maps. A map $f : \mathcal{A} \to \mathcal{B}$ in $\mathcal{O}(G)$-DGA_k is called a *weak equivalence* (resp. *fibration*) if for all $G/H \in \mathrm{Ob}\,(\mathcal{O}(G))$ the maps $f(G/H) : \mathcal{A}(G/H) \to \mathcal{B}(G/H)$ are homology isomorphisms (resp. surjections) in the category DGA_k. A map $f : \mathcal{A} \to \mathcal{B}$ is a *cofibration* if it has the left-lifting property with respect to all maps which are both fibrations and weak equivalences, i.e. trivial fibrations.

PROPOSITION 1.2. *If G is a finite Hamiltonian group, then any KS-minimal object $\mathcal{M}$ in $\mathcal{O}(G)$-DGA_k is cofibrant.*

Proof. Consider a commutative diagram



in $\mathcal{O}(G)$-DGA_k, where $\underline{k}$ is the constant $\mathcal{O}(G)$-algebra determined by the field k and p is a trivial fibration. We proceed inductively with respect to height of objects in $\mathcal{O}(G)$. For an object $G/H \in \mathrm{Ob}\,(\mathcal{O}(G))$ of height 0, there is a map $\beta(G/H) : \mathcal{M}(G/H) \to \mathcal{D}(G/H)$ such that $p(G/H) \circ \beta(G/H) = \alpha(G/H)$, since

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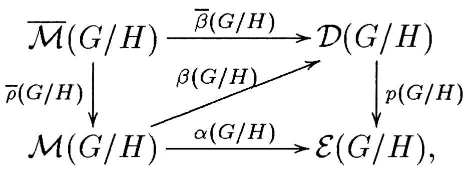
$\mathcal{M}(G/H)$ is G/H-KS-minimal. Suppose now that for all $G/K \in \mathrm{Ob}(\mathcal{O}(G))$ of height smaller than height of G/H there are maps $\beta(G/K)\colon \mathcal{M}(G/K) \to \mathcal{A}(G/K)$ such that $p(G/K) \circ \beta(G/K) = \alpha(G/K)$. Hence, we get a map $\overline{\beta}(G/H)\colon \overline{\mathcal{M}}(G/H) = \lim_{\to \mathcal{O}(G)_{G/H}} \overline{\mathcal{M}}_{G/H} \longrightarrow \mathcal{D}(G/H)$. Then, in the commutative diagram

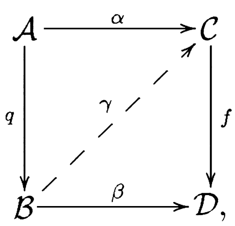


there is a filler $\beta(G/H)$ since the map $\overline{\mathcal{M}}(G/H) \to \mathcal{M}(G/H)$ is a cofibration in the category G/H-DGA_k. □

2. The Main Result

Let $\mathcal{A}$ be in $\mathcal{O}(G)$-DGA_k and let $\rho : \mathcal{M} \to \mathcal{A}$ be a weak equivalence, where $\mathcal{M}$ is KS-minimal. Then $\mathcal{M}$ is called the KS- (or *cofibrant*) *minimal model* of $\mathcal{A}$. Proposition 2.2 below (cf. [1, 6]) implies that this definition is meaningful. First, however, we establish a lemma that plays an important role in the proof of this proposition.

LEMMA 2.1. *If G is a finite Hamiltonian group, then for a diagram in* $\mathcal{O}(G)$-DGA_k *commutative up to homotopy*



where q is a cofibration and f a weak equivalence, there exists an arrow γ *making this diagram commutative up to homotopy.*

Proof. Using [2, Theorem 1.3], we may factor the map f as $\mathcal{C} \xrightarrow{q'} \mathcal{C}' \xrightarrow{p} \mathcal{D}$ with q' a trivial cofibration and p a trivial fibration. Every object in $\mathcal{O}(G)$-DGA_k is fibrant, hence by [10] the map $q' : \mathcal{C} \to \mathcal{C}'$ has a homotopy inverse $q'' : \mathcal{C}' \to \mathcal{C}$. But the map $q : \mathcal{A} \to \mathcal{B}$ is a cofibration, so there are maps $\beta' : \mathcal{B} \to \mathcal{D}$ and $\gamma' : \mathcal{B} \to \mathcal{C}'$ such

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that $\beta \simeq \beta'$ and the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{q' \circ \alpha} & \mathcal{C}' \\ {\scriptstyle q}\downarrow & \overset{\gamma'}{\nearrow} & \downarrow{\scriptstyle p} \\ \mathcal{B} & \xrightarrow{\beta'} & \mathcal{D}, \end{array}$$

strictly commutes. Then $\gamma = q'' \circ \gamma'$ is the required map. □

PROPOSITION 2.2. *Let G be a finite Hamiltonian group and let $\mathcal{M}$ and $\mathcal{M}'$ be KS-minimal $\mathcal{O}(G)$-algebras, and $\rho : \mathcal{M} \to \mathcal{A}$, $\rho' : \mathcal{M}' \to \mathcal{A}$ be weak equivalences. Then:*

(1) *there is an isomorphism $\theta : \mathcal{M} \to \mathcal{M}'$ in $\mathcal{O}(G)$-DGA_k such that $\rho'(G/H) \circ \theta(G/H) \simeq \rho(G/H)$ in the category G/H-DGA_k for all $G/H \in \mathrm{Ob}(\mathcal{O}(G))$;*
(2) *if $\hat{\theta} : \mathcal{M} \to \mathcal{M}'$ is a map in $\mathcal{O}(G)$-DGA_k such that $\rho'(G/H) \circ \hat{\theta}(G/H) \simeq \rho(G/H)$ in the category G/H-DGA_k then $\hat{\theta}$ is an isomorphism and $\hat{\theta}(G/H) \simeq \theta(G/H)$ in the category G/H-DGA_k, for all $G/H \in \mathrm{Ob}(\mathcal{O}(G))$.*

Proof. (1) We again proceed inductively with respect to height of objects in $\mathrm{Ob}(\mathcal{O}(G))$. If $G/H \in \mathrm{Ob}(\mathcal{O}(G))$ has height 0, then $\mathcal{M}(G/H)$ and $\mathcal{M}'(G/H)$ are G/H-minimal and by [5, Proposition 4.3] there is an G/H-isomorphism $\theta(G/H) : \mathcal{M}(G/H) \to \mathcal{M}'(G/H)$ such that $\rho'(G/H) \circ \theta(G/H) \simeq \rho(G/H)$ in the category G/H-DGA_k.

Suppose that for all $G/K \in \mathrm{Ob}(\mathcal{O}(G))$ of height smaller than that of G/H, there exists $\theta(G/K) : \mathcal{M}(G/K) \to \mathcal{M}'(G/K)$ such that $\rho'(G/K) \circ \theta(G/K) \simeq \rho(G/K)$ in the category G/K-DGA_k and the diagrams commute

$$\begin{array}{ccc} \mathcal{M}(G/L) & \xrightarrow{\theta(G/L)} & \mathcal{M}'(G/L) \\ \downarrow & & \downarrow \\ \mathcal{M}(G/K) & \xrightarrow{\theta(G/K)} & \mathcal{M}'(G/K) \end{array}$$

for $\overline{G/L} < \overline{G/K}$. Then, we obtain the induced isomorphism $\bar{\theta}(G/H) : \overline{\mathcal{M}}(G/H) \to \overline{\mathcal{M}}'(G/H)$. But the map $\alpha(G/H) : \overline{\mathcal{M}}(G/H) \to \mathcal{M}(G/H)$ is a cofibration in the category G/H-DGA_k and $\rho(G/H) : \mathcal{M}(G/H) \to \mathcal{A}(G/H)$ is a weak equivalence, hence by Lemma 2.1 there is an G/H-map

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$\theta'(G/H) : \mathcal{M}(G/H) \to \mathcal{M}'(G/H)$ such that the diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}(G/H) & \xrightarrow{\alpha(G/H)} & \mathcal{M}(G/H) & & \\ & & & \searrow^{\rho(G/H)} & \\ \Big\downarrow{\scriptstyle \overline{\theta}(G/H)} & & \Big\downarrow{\scriptstyle \theta'(G/H)} & & \mathcal{A}(G/H) \\ & & & \nearrow_{\rho(G/H)} & \\ \overline{\mathcal{M}}'(G/H) & \xrightarrow{\alpha'(G/H)} & \mathcal{M}'(G/H) & & \end{array}$$

commutes up to homotopy. In particular, $\theta'(G/H) \circ \alpha(G/H) \simeq \alpha'(G/H) \circ \overline{\theta}(G/H)$. But the map $\alpha(G/H)$ is a cofibration, hence there is a map $\theta(G/H)$ such that $\theta'(G/H) \simeq \theta(G/H)$ and $\theta(G/H) \circ \alpha(G/H) = \alpha'(G/H) \circ \overline{\theta}(G/H)$. The maps $\alpha(G/H)$ and $\alpha'(G/H)$ are G/H-KS-minimal extensions and $\overline{\theta}(G/H) : \overline{\mathcal{M}}(G/H) \to \overline{\mathcal{M}}'(G/H)$ is an isomorphism, hence by [5, Proposition 4.6] the map $\theta(G/H)$ is an isomorphism.

(2) If $G/H \in \mathrm{Ob}(\mathcal{O}(G))$ has height 0, then $\mathcal{M}(G/H)$ and $\mathcal{M}'(G/H)$ are G/H-minimal and by [5, Proposition 4.3] the map $\hat{\theta}(G/H)$ is an G/H-isomorphism and $\theta(G/H) \simeq \hat{\theta}(G/H)$ in the category G/H-DGA_k.

Suppose that, for all $G/K \in \mathrm{Ob}(\mathcal{O}(G))$ of a height smaller than that of G/H, the maps $\hat{\theta}(G/K)$ are G/K-isomorphisms and there exists an G/K-homotopy $\theta(G/K) \simeq \hat{\theta}(G/K)$. Then the diagram

$$\begin{array}{ccc} \overline{\mathcal{M}}(G/H) & \xrightarrow{\alpha(G/H)} & \mathcal{M}(G/H) \\ {\scriptstyle \overline{\theta}(G/H)}\Big\downarrow & & \Big\downarrow{\scriptstyle \hat{\theta}(G/H)} \\ \overline{\mathcal{M}}'(G/H) & \xrightarrow{\alpha'(G/H)} & \mathcal{M}'(G/H) \end{array}$$

satisfies the hypothesis of Theorem 10.4 in [5]; hence $\theta(G/H) \simeq \hat{\theta}(G/H)$ in the category G/H-DGA_k and the map $\hat{\theta}(G/H)$ is an isomorphism. □

We now show that the cofibrant minimal models exist.

THEOREM 2.3. *Let G be a finite Hamiltonian group. Then for any $\mathcal{A}$ in $\mathcal{O}(G)$-DGA_k there exist a cofibrant equivariant minimal model $\mathcal{M}_{\mathcal{A}}$ and a weak equivalence $\rho : \mathcal{M}_{\mathcal{A}} \to \mathcal{A}$.*

Proof. For any $\mathcal{A}$ in $\mathcal{O}(G)$-DGA_k, we construct its cofibrant equivariant minimal model $\mathcal{M}_{\mathcal{A}}$ as follows.

(1) If $G/H \in \mathrm{Ob}(\mathcal{O}(G))$ has height 0, then for $\mathcal{M}_{\mathcal{A}}(G/H)$ take the G/H-minimal model of $\mathcal{A}(G/H)$. Let $\rho(G/H) : \mathcal{M}_{\mathcal{A}}(G/H) \to \mathcal{A}(G/H)$ be a fixed G/H-weak equivalence.

(2) Suppose that for all $G/K \in \mathrm{Ob}(\mathcal{O}(G))$ of a height smaller than the height of G/H there are G/K-weak equivalences $\rho(G/K) : \mathcal{M}_{\mathcal{A}}(G/K) \to \mathcal{A}(G/K)$ such that

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for $\overline{G/K_1}, \overline{G/K_2} < \overline{G/K}$ with $\overline{G/K_1} < \overline{G/K_2}$ all diagrams

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{A}}(G/K_1) & \xrightarrow{\rho(G/K_1)} & \mathcal{A}(G/K_1) \\ \downarrow & & \downarrow \\ \mathcal{M}_{\mathcal{A}}(G/K_2) & \xrightarrow{\rho(G/K_2)} & \mathcal{A}(G/K_2) \end{array}$$

commute. To get $\mathcal{M}_{\mathcal{A}}(G/K)$ and a G/H-weak equivalence $\rho(G/H): \mathcal{M}_{\mathcal{A}}(G/H) \to \mathcal{A}(G/H)$, consider the induced G/H-map $\overline{\rho}(G/H): \overline{\mathcal{M}_{\mathcal{A}}}(G/H) \to \mathcal{A}(G/H)$ and its G/H-KS-minimal model

$$\begin{array}{ccc} \overline{\mathcal{M}_{\mathcal{A}}}(G/H) & \xrightarrow{\overline{\rho}(G/H)} & \mathcal{A}(G/H) \\ \downarrow & \nearrow_{\rho(G/H)} & \\ \mathcal{M}_{\mathcal{A}}(G/H). & & \end{array}$$

□

At the end we present briefly some geometric applications of cofibrant minimal models. The de Rham functor A^* of polynomial forms determines the $\mathcal{O}(G)$-algebra $\mathbb{Q}$-algebra $\mathcal{A}^*_X$ such that $\mathcal{A}^*(X)(G/H) = A^*(X^H)$ for a G-simplicial set X and $H \subseteq G$.

We say that a G-simplicial set X is *G-connected* and *nilpotent* (resp. *pointed*) if all fixed point simplicial subsets X^H are connected and nilpotent, for all subgroups $H \subseteq G$ (resp. $X^G \neq \emptyset$). By [14, 15], for any G-connected pointed simplicial set X of finite type there is the (componentwise) injective minimal model $\mathcal{M}^{inj}_X$ and a weak equivalence $\mathcal{M}^{inj}_X \to \mathcal{A}^*(X)$.

The cofibrant minimal model $\mathcal{M}_X$ of the $\mathcal{O}(G)$-algebra $\mathcal{A}^*(X)$ is called the *cofibrant minimal model* of a G-simplicial set X. A slight variation of the construction in [7, 4.9. Exemples] (cf. also [6, IV.3 - Exemples]) allows to present

EXAMPLE 2.4. (1) Let $X = \mathbb{S}^3 \times \cdots \times \mathbb{S}^3$ be the product of six copies of the three-dimensional sphere $\mathbb{S}^3$ with an action of the cyclic group $\mathbb{Z}_6$ like in the case 4 of [7, Exemples 4.9]. Then the injective and cofibrant minimal models of X coincide.

(2) Let $X = \mathbb{S}^4_a \vee \mathbb{S}^4_b \vee \mathbb{S}^4_c \vee \mathbb{S}^4_d \vee \mathbb{S}^7_e \cup_{\omega_1} e^8_1 \cup_{\omega_2} e^8_2$ be a space constructed like in case 6 of [7, Exemples 4.9] and with a similar action of the cyclic group $\mathbb{Z}_2$. Then the injective and cofibrant minimal models of X do not coincide.

From Proposition 2.2 one sees that the cofibrant minimal models of $\mathcal{M}^{inj}_X$ and $\mathcal{A}^*(X)$ are isomorphic. Hence there is a weak equivalence $\rho: \mathcal{M}_X \to \mathcal{M}^{inj}_X$. By [14, Proposition 5.5] there is a map $\rho': \mathcal{M}^{inj}_X \to \mathcal{M}_X$ such that $\rho \circ \rho' \simeq \mathrm{id}_{\mathcal{M}^i_X}$. Thus the map ρ' is a weak equivalence and by [2, Corollary 1.5] and Proposition 2.2 there is a map $\rho'': \mathcal{M}_X \to \mathcal{M}^{inj}_X$ such that $\rho' \circ \rho'' \simeq \mathrm{id}_{\mathcal{M}_X}$. In the light of [14, 15], for nilpotent G-connected pointed simplicial sets X, Y of finite type, there is a bijection $[X, Y]_G \approx [\mathcal{M}_Y^{inj}, \mathcal{M}_X^{inj}]$ provided Y is rational, where $[X, Y]_G$ is the set of pointed G-homotopy classes of G-maps from X to Y. We have thus proved the following proposition.

PROPOSITION 2.5. *Let G be a finite Hamiltonian group. If X and Y are nilpotent G-connected pointed simplicial sets of finite type, then there is a bijection* $[X, Y]_G \approx [\mathcal{M}_Y, \mathcal{M}_X]$, *provided Y is rational.*

A construction of the cofibrant equivariant minimal model of any nilpotent G-disconnected simplicial set requires more subtle methods and will be published elsewhere.

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