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DILATIONS AND HAHN DECOMPOSITIONS FOR LINEAR MAPS

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Suppose $\mathscr{A}$ is a C^*-algebra and H is a Hilbert space. Let $\mathrm{CP}(\mathscr{A}, H)$ denote the set of completely positive maps from $\mathscr{A}$ into the set $B(H)$ of (bounded linear) operators on H. This paper studies the vector space $\mathscr{V}(\mathscr{A}, H)$ spanned by CP $(\mathscr{A}, H)$, i.e., the linear maps that are finite linear combinations of completely positive maps. From another viewpoint, a map φ is in $\mathscr{V}(\mathscr{A}, H)$ precisely when it has a decomposition $\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4)$ with $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ in CP $(\mathscr{A}, H)$; this decomposition is analogous to the Hahn decomposition for measures [**8**, III.4.10] (see also Theorem 20). The analogous class of maps with "completely positive" replaced by "positive" was studied by R. I. Loebl [**11**] and S.-K. Tsui [**17**], and when $\mathscr{A}$ is commutative, this latter class coincides with $\mathscr{V}(\mathscr{A}, H)$, since every positive linear map on a commutative C^*-algebra is completely positive [**16**].

Completely positive maps were introduced by W. F. Stinespring [**16**], who showed that CP $(\mathscr{A}, H)$ consists of precisely those maps that are compressions of (have dilations to) representations. We show that $\mathscr{V}(\mathscr{A}, H)$ consists precisely of those maps that are compressions of homomorphisms that are similar to representations. We show that a bounded homomorphism from $\mathscr{A}$ into $B(H)$ is in $\mathscr{V}(\mathscr{A}, H)$ if and only if it is similar to a representation.

In the case when $\mathscr{A}$ is the C^*-algebra $C(X)$ of continuous complex functions on a compact Hausdorff space X, R. I. Loebl [**11**, Theorem 4.4] has shown that the set of maps satisfying a certain "bounded variation" condition is included in $\mathscr{V}(\mathscr{A}, H)$. We provide an example that shows that the inclusion is usually proper. The set of bounded linear maps from $C(X)$ into $B(H)$ can be identified with certain $B(H)$-valued measures on X, and we show that a map is in $\mathscr{V}(\mathscr{A}, H)$ if and only if its associated measure is a linear combination of positive operator-valued measures. We also show that a bounded operator-valued measure is a linear combination of positive operator-valued measures if and only if it can be dilated to a bounded (non-self-adjoint) spectral measure.

The key new ideas are the following two relatively simple lemmas; the first is a factorization theorem for pairs of operators A, B for which $AB = 1$, and the second says that any two operators have dilations that are inverses of each other.

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LEMMA 1. *If $A, B \in B(H)$ and $AB = 1$, then there is an isometry V and a positive invertible operator S such that $A = V^*S^{-1}$ and $B = SV$.*

Proof. Let P be the orthogonal projection of H onto $(\text{ran } A^*)^\perp$, let $\epsilon = 1/(2\|A\|^2)$, let $t = \|BB^*\|^2/\epsilon$, and let $Q = BB^* + (t + \epsilon)P$. Clearly $Q \geqq 0$. We will show that Q is invertible by proving that $(Qf, f) \geqq \epsilon$ for every unit vector f in H. Suppose $f \in H$ and $\|f\| = 1$. Since $B^*A^* = 1$, we know that A^* is bounded from below; thus ran A^* is closed. Write $f = u + v$ with $u \in \text{ran } A^*$ and $v \in (\text{ran } A^*)^\perp$, and write $u = A^*h$. Note that $\|u\| \leqq \|A^*\|\|h\| = \|A\|\|h\|$ implies that $\|h\|^2 \geqq 2\epsilon\|u\|^2$. Thus

$$\begin{aligned}(Qf, f) &= (BB^*u, u) + (t + \epsilon)\|v\|^2 + 2 \operatorname{Re} (BB^*u, v) + (BB^*v, v) \\ &\geqq (BB^*u, u) + (t + \epsilon)\|v\|^2 - 2\|BB^*\| \, \|u\| \, \|v\|.\end{aligned}$$

Since

$$(BB^*u, u) = \|B^*u\|^2 = \|B^*A^*h\|^2 = \|h\|^2 \geqq 2\epsilon\|u\|^2,$$

we conclude that

$$\begin{aligned}(Qf, f) &\geqq 2\epsilon\|u\|^2 + (t + \epsilon)\|v\|^2 - 2\|BB^*\| \, \|u\| \, \|v\| \\ &= \epsilon(\|u\|^2 + \|v\|^2) + (\epsilon^{1/2}\|u\| - t^{1/2}\|v\|)^2 \geqq \epsilon.\end{aligned}$$

Thus Q is invertible. Let S be the positive square root of Q. Then S is positive and invertible and

$$S^2A^* = (BB^* + (t + \epsilon)P)A^* = B.$$

Let $V = SA^* = S^{-1}B$. Then $V^*V = (AS)(S^{-1}B) = 1$; whence V is an isometry, and $A = V^*S^{-1}$, and $B = SV$. This completes the proof.

LEMMA 2. *If $A, B \in B(H)$, then there is a Hilbert space $H_1 \supset H$ and an invertible operator S in $B(H_1)$ such that if P is the projection from H_1 outo H, then $PS^{-1}|H = A$ and $PS|H = B$.*

Proof. If

$$S = \begin{pmatrix} A & 1 \\ 1 - BA & -B \end{pmatrix} \text{ then } S^{-1} = \begin{pmatrix} B & 1 \\ 1 - AB & -A \end{pmatrix}.$$

The preceding 2×2 matrix argument, which replaces the author's original 8×8 version, is due to Man-Duen Choi.

Suppose $\varphi : A \to B(H)$ is a linear map. For each positive integer n let $\mathfrak{M}_n$ denote the $n \times n$ complex matrices and let

$$\varphi^{(n)} : A \otimes \mathfrak{M}_n \to B(H) \otimes \mathfrak{M}_n$$

be the linear map defined by $\varphi^{(n)}(a \otimes b) = \varphi(a) \otimes b$. The map φ is *positive* if $\varphi(a) \geqq 0$ whenever $a \in \mathscr{A}$ and $a \geqq 0$. The map φ is *completely positive* if $\varphi^{(n)}$ is positive for $n = 1, 2, \ldots$, and φ is *completely bounded* if

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$\sup_n \|\varphi^{(n)}\| < \infty$. We also define $\varphi^* : \mathscr{A} \to B(H)$ by $\varphi^*(a) = \varphi(a^*)^*$, and we define

$$\operatorname{Re} \varphi = (\varphi + \varphi^*)/2 \quad \text{and} \quad \operatorname{Im} \varphi = (\varphi - \varphi^*)/2i.$$

We are now ready for the main results.

THEOREM 3. *If $\varphi : \mathscr{A} \to B(H)$, then the following are equivalent:*

(1) $\varphi \in \mathscr{V}(\mathscr{A}, H)$;

(2) *there is a positive integer n, Hilbert spaces H_i, completely positive maps $\psi_i : \mathscr{A} \to B(H_i)$, and operators $A_i : H_i \to H$, $B_i : H \to H_i$, for $i = 1, 2, \ldots, n$ such that $\varphi(a) = \sum_i A_i \psi_i(a) B_i$ for every a in $\mathscr{A}$;*

(3) *there is a representation $\pi : \mathscr{A} \to B(H_\pi)$, a positive invertible operator S in $B(H_\pi)$, and an isometry $V : H \to H_\pi$ such that*

$$\varphi(a) = V^*[S^{-1}\pi(a)S]V$$

for every a in $\mathscr{A}$.

Proof. Clearly (3) $\Rightarrow$ (2). To prove (2) $\Rightarrow$ (1) assume that (2) holds. There is no harm in assuming $n = 1$, since $\mathscr{V}(\mathscr{A}, H)$ is closed under addition. Write $\varphi(\cdot) = A\psi(\cdot)B^*$ with ψ completely positive. Since $C\psi(\cdot)C^*$ is completely positive for every operator C, we can conclude $\varphi \in \mathscr{V}(\mathscr{A}, H)$ from the polarization identity.

$$\varphi(a) = \tfrac{1}{4}[(A + B)\psi(a)(A + B)^* - (A - B)\psi(a)(A - B)^* + i(A + iB)\psi(a)(A + iB)^* - i(A - iB)\psi(a)(A - iB)^*)$$

for every a in $\mathscr{A}$.

We next prove (1) $\Rightarrow$ (3). Suppose $\varphi = \alpha_1\psi_1 + \ldots + \alpha_n\psi_n$ where $\alpha_1, \ldots, \alpha_n$ are scalars and $\psi_1, \ldots, \psi_n \in \mathrm{CP}(\mathscr{A}, H)$. It follows from Stinespring's theorem [**16**] that there are Hilbert spaces H_i, representations $\pi_i : \mathscr{A} \to B(H_i)$, and operators $W_i : H \to H_i$ for $i = 1, 2, \ldots, n$ such that

$$\varphi(a) = \alpha_1 W_1^* \pi_1(a) W_1 + \ldots + \alpha_n W_n^* \pi_n(a) W_n$$

for every a in $\mathscr{A}$. Define $A : H \to H_1 \oplus \ldots \oplus H_n$ by

$$Ah = \alpha_1 W_1 h \oplus \ldots \oplus \alpha_n W_n h,$$

and define $B : H_1 \oplus \ldots \oplus H_n \to H$ by

$$B(h_1 \oplus \ldots \oplus h_n) = W_1^* h_1 + W_2^* h_2 + \ldots + W_n^* h_n.$$

Let $\rho = \pi_1 \oplus \ldots \oplus \pi_n$. Then ρ is a representation of $\mathscr{A}$ and $\varphi(a) = A\rho(a)B$ for every a in $\mathscr{A}$. Since we are trying to prove that φ can be dilated to a map that is similar to a representation, there is no harm in replacing φ by some dilation of φ. In particular, we can replace φ by a direct sum of arbitrarily many copies of φ. Thus we can assume that H and $H_1 \oplus \ldots \oplus H_\pi$ have the same dimension. Hence there is no harm in

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assuming that $H = H_1 \oplus \ldots \oplus H_n$. Thus $A, B \in B(H)$, and, by Lemma 2, there is a Hilbert space $H_\pi \supset H$ and an invertible operator T in $B(H_\pi)$ such that $PT^{-1}|H = A$ and $PT|H = B$, where P is the projection from H_π onto H. Let $T = US$ be the polar decomposition of T with U unitary and S positive (and invertible). Define $\pi : \mathscr{A} \rightarrow B(H_\pi)$ by

$$\pi(a) = U^*(\rho(a) \oplus 0)U,$$

and define $V : H \rightarrow H_\pi$ by $Vh = h$. Then π is a representation, V is an isometry, and

$$\varphi(a) = V^*S^{-1}\pi(a)SV$$

for every a in $\mathscr{A}$. This completes the proof.

Note that the representation π in the preceding theorem is not necessarily nondegenerate. In fact, if $1 \in \mathscr{A}$, and π is nondegenerate, then $\pi(1) = 1$, which would imply that $\varphi(1) = 1$ for every map φ that is a compression of a map that is similar to π. It turns out that if $\varphi(1) = 1$, then the representation π in the preceding theorem can be chosen so that $\pi(1) = 1$.

THEOREM 4. *If $\varphi \in \mathscr{V}(\mathscr{A}, H)$ and $\varphi(1) = 1$, then there is a representation $\pi : \mathscr{A} \rightarrow B(H_\pi)$, a positive invertible operator S in $B(H_\pi)$, and an isometry $V : H \rightarrow H_\pi$ such that*

$$\varphi(a) = V^*[S^1\pi(a)S]V$$

for every a in $\mathscr{A}$ and such that $\pi(1) = 1$.

Proof. If we follow the proof of (1) $\Rightarrow$ (3) in the preceding theorem, we can reduce the present proof to the case when there is a representation $\pi : \mathscr{A} \rightarrow B(H)$ such that $\pi(1) = 1$ (this follows from [**16**]), and operators A, B in $B(H)$ such that $\varphi(a) = A\pi(\mathrm{a})B$ for every a in $\mathscr{A}$. Since $\varphi(1) = 1 = \pi(1)$, we conclude $AB = 1$. Thus, by Lemma 1, there is a positive invertible operator S and an isometry V such that $A = V^*S^{-1}$ and $B = SV$. This completes the proof.

In [**1**, Theorem 1.2.3] W. Arveson proved that if $\mathscr{S}$ is a norm closed self-adjoint linear subspace of $\mathscr{A}$ with $1 \in \mathscr{S}$, and if $\varphi \in \mathrm{CP}(\mathscr{S}, H)$, then $\varphi = \psi|\mathscr{S}$ for some ψ in CP $(\mathscr{A}, H)$. Clearly Arveson's extension theorem implies its analogue for $\mathscr{V}(\mathscr{A}, H)$.

LEMMA 5. *If $\mathscr{S}$ is a normed closed self-adjoint linear subspace of $\mathscr{A}$ with $1 \in \mathscr{S}$, and if $\varphi \in V(\mathscr{S}, H)$, then $\varphi = \psi|S$ for some ψ in $\mathscr{V}(\mathscr{A}, H)$.*

The next lemma shows that the class of maps that are finite linear combinations of completely positive maps is closed under composition.

LEMMA 6. *If $\varphi \in \mathscr{V}(\mathscr{A}, H)$, H_1 is a Hilbert space, and*

$$\psi \in \mathscr{V}(C^*(\varphi(\mathscr{A})), H_1),$$

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then

$$\psi \circ \varphi \in \mathscr{V}(\mathscr{A}, H_1).$$

Proof. It follows from the preceding lemma that we can assume that $\psi \in \mathscr{V}(B(H), H_1)$. Write

$$\varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4) \quad \text{and} \quad \psi = (\psi_1 - \psi_2) + i(\psi_3 - \psi_4)$$

where $\varphi_j \in \text{CP}(\mathscr{A}, H)$ and $\psi_j \in \text{CP}(B(H), H_1)$ for $j = 1, 2, 3, 4$. Clearly $\psi \circ \varphi$ is a linear combination of the completely positive maps $\psi_j \circ \varphi_k$ $(1 \leqq j, k \leqq 4)$.

COROLLARY 7. *Suppose* $\varphi : \mathscr{A} \to B(H)$ *is a bounded linear map and*

$$\pi : C^*(\varphi(\mathscr{A})) \to B(H_\pi)$$

is a one to one $*$*-homomorphism. Then* $\varphi \in \mathscr{V}(\mathscr{A}, H)$ *if and only if* $\pi \circ \varphi \in \mathscr{V}(\mathscr{A}, H_\pi)$.

Proof. This follows from Lemma 6 and the fact that $\varphi = \pi^{-1} \circ (\pi \circ \varphi)$.

The following example shows that $\mathscr{V}(\mathscr{A}, H)$ is generally not closed under norm limits (and a posteriori limits in the standard notions of pointwise convergence). This example relies on the observation of Loebl [**11**] that every map in $\mathscr{V}(\mathscr{A}, H)$ is completely bounded.

Example 8. Let

$$\mathscr{A} = \sum_n^{\oplus} \mathfrak{M}_n = \{\{A_n\} : \|A_n\| \to 0, A_n \in \mathfrak{M}_n \text{ for } n \geqq 1\}.$$

Let A^t denote the transpose of a complex matrix A. Define $\varphi : \mathscr{A} \to \mathscr{A}$ by

$$\varphi(\{A_n\}) = \{A_n{}^t / n^{1/2}\}.$$

For each positive integer k define $\varphi_k : \mathscr{A} \to \mathscr{A}$ by

$$\varphi_k(\{A_n\}) = \{B_n\}$$

where

$$B_n = \begin{cases} A_n{}^t / n^{1/2} & \text{if } 1 \leqq n \leqq k \\ A_n / n^{1/2} & \text{if } n > k. \end{cases}$$

If $\mathscr{A} \subseteq B(H)$, then clearly $\varphi_k \in \mathscr{V}(\mathscr{A}, H)$ for $k = 1, 2, \ldots$. Since $\|\varphi_k - \varphi\| \to 0$ (since $\|\varphi_k - \varphi\| \leqq 2/k^{1/2}$ for $k = 1, 2, \ldots$), we know that $\varphi_k \to \varphi$ in all of the familiar (point-norm, point-strong, point-weak) topologies. However, $\varphi \notin \mathscr{V}(\mathscr{A}, H)$ because φ is not completely bounded. To see this, note that Arveson [**1**, p. 144] shows that if k is a positive integer and $\psi : \mathfrak{M}_n \to \mathfrak{M}_n$ is defined by $\psi(A) = A^t$, then $\|\psi^{(k)}\| \geqq k$. Thus (by looking at kth coordinates)

$$\|\varphi^{(k)}\| \geqq \|\psi^{(k)}/k^{1/2}\| \geqq k^{1/2} \quad \text{for} \quad k = 1, 2, \ldots.$$

In spite of the preceding example we define a norm $\| \|_{\mathscr{V}}$ on $\mathscr{V}(\mathscr{A}, H)$ that makes $\mathscr{V}(\mathscr{A}, H)$ into a Banach space. First note that if $\psi \in \mathrm{CP}(\mathscr{A}, H)$ and r is a non-negative number, then $r\psi \in \mathrm{CP}(\mathscr{A}, H)$. Thus if $z_1, \ldots, z_n$ are scalars and $\psi_1, \ldots, \psi_n \in \mathrm{CP}(\mathscr{A}, H)$, then we can write

$$z_1\psi_1 + \ldots + z_n\psi_n = \lambda_1|z_1|\psi_1 + \ldots + \lambda_n|z_n|\psi_n$$

where $|\lambda_1| = \ldots = |\lambda_n| = 1$.

We define the norm $\| \|_{\mathscr{V}}$ on $\mathscr{V}(\mathscr{A}, H)$ by

$$\|\varphi\|_{\mathscr{V}} = \inf\left\{ \sum_{j=1}^{n} \|\psi_j\| : \varphi = \sum_{j=1}^{n} \lambda_j\psi_j;\ \psi_1, \ldots, \psi_n \in \mathrm{CP}(\mathscr{A}, H); \right. \\ \left. |\lambda_1| = \ldots = |\lambda_n| = 1 \right\}.$$

It is easily seen that $\| \|_{\mathscr{V}}$ is indeed a norm on $\mathscr{V}(\mathscr{A}, H)$ that dominates $\| \|$. Furthermore, it is clear that $\|\varphi\| = \|\varphi\|_{\mathscr{V}}$ for every φ in $\mathrm{CP}(\mathscr{A}, H)$.

It is useful to compare $\| \|_{\mathscr{V}}$ with some other numerical quantities that arise naturally from the preceding characterizations of the elements of $\mathscr{V}(\mathscr{A}, H)$. Define functions $\alpha, \beta, \gamma : \mathscr{V}(\mathscr{A}, H) \to [0, \infty)$ by

$$\alpha(\varphi) = \inf\left\{ \sum_{j=1}^{4} \|\varphi_j\| : \varphi = (\varphi_1 - \varphi_2) + i(\varphi_3 - \varphi_4); \right. \\ \left. \varphi_1, \ldots, \varphi_4 \in \mathrm{CP}(\mathscr{A}, H) \right\},$$

$$\beta(\varphi) = \inf\{\|A\|\,\|B\| : \varphi(\cdot) = A\pi(\cdot)B \text{ for some representation } \pi \text{ of } \mathscr{A}\},$$

$$\gamma(\varphi) = \inf\{\|S\|\,\|S^{-1}\| : \varphi(\cdot) = V^*S^{-1}\pi(\cdot)SV \text{ for some isometry } V \\ \text{and some representation } \pi \text{ of } \mathscr{A}, \text{ and } S \text{ invertible}\}.$$

The following lemma is based on very crude estimates involving the proofs of Lemma 2 and Theorem 3. The main significance of these estimates is contained in the corollary and in Theorem 11.

LEMMA 9. *For each φ in $\mathscr{V}(\mathscr{A}, H)$ we have*
(1) $\|\varphi\|_{\mathscr{V}} \leqq \alpha(\varphi) \leqq 2\|\varphi\|_{\mathscr{V}}$,
(2) $\beta(\varphi) \leqq \|\varphi\|_{\mathscr{V}} \leqq 4\beta(\varphi)$,
(3) $\beta(\varphi) \leqq \gamma(\varphi) \leqq 4(1 + \beta(\varphi))^2$.

Proof. (1) is obvious.

A key idea in the proofs of (2) and (3) is that the added restriction $\|A\| = \|B\|$ in the definition of $\beta(\varphi)$ does not alter $\beta(\varphi)$.

(2). The inequality $\beta(\varphi) \leqq \|\varphi\|_{\mathscr{V}}$ follows from the proof of (1) $\Rightarrow$ (3) in Theorem 3. The inequality $\|\varphi\|_{\mathscr{V}} \leqq 4\beta(\varphi)$ follows from the proof of (2) $\Rightarrow$ (1) in Theorem 3.

(3). The inequality $\beta(\varphi) \leqq \gamma(\varphi)$ is obvious. The inequality $\gamma(\varphi) \leqq 4(1 + \beta(\varphi))^2$ follows from the proof of Lemma 2.

COROLLARY 10. *Suppose* $\{\varphi_n\}$ *is a net in* $\mathscr{V}(\mathscr{A}, H)$. *If one of the nets* $\{\|\varphi_n\|_{\mathscr{V}}\}$, $\{\alpha(\varphi_n)\}$, $\{\beta(\varphi_n)\}$, $\{\gamma(\varphi_n)\}$ *is bounded, then so are the others.*

In view of Example 8, the following theorem seems a little surprising.

THEOREM 11. *If* $\{\varphi_n\}$ *is a* $\| \ \|_{\mathscr{V}}$*-bounded net in* $\mathscr{V}(\mathscr{A}, H)$ *and* $\varphi_n(a) \rightarrow \varphi(a)$ *in the weak operator topology for every* a *in* $\mathscr{A}$, *then* $\varphi \in \mathscr{V}(\mathscr{A}, H)$.

Proof. Since $\{\alpha(\varphi_n)\}$ is bounded, we can find $\| \ \|$-bounded nets $\{\varphi_{n1}\}$, $\{\varphi_{n2}\}$, $\{\varphi_{n3}\}$, $\{\varphi_{n4}\}$ in CP $(\mathscr{A}, H)$ such that, for each n, we have

$$\varphi_n = (\varphi_{n1} - \varphi_{n2}) + i(\varphi_{n3} - \varphi_{n4}).$$

By replacing $\{\varphi_n\}$ by an appropriate subnet if necessary, we can assume that there are maps $\psi_1, \psi_2, \psi_3, \psi_4$ in CP $(\mathscr{A}, H)$ such that $\varphi_{nk}(a) \rightarrow \psi_k(a)$ in the weak operator topology for $k = 1, 2, 3, 4$ and every a in $\mathscr{A}$. Hence $\varphi \in \mathscr{V}(\mathscr{A}, H)$ since $\varphi = (\psi_1 - \psi_2) + i(\psi_3 - \psi_4)$.

COROLLARY 12. *Suppose* $\varphi : A \rightarrow B(H)$, $\mathscr{D}$ *is a dense subset of* $\mathscr{A}$, *and* D *is a dense subset of* H. *Then* $\varphi \in \mathscr{V}(\mathscr{A}, H)$ *if and only if there is a positive number* M *such that for each* $\epsilon > 0$, *each finite subset* $\mathscr{F}$ *of* $\mathscr{A}$, *and each finite subset* F *of* H *there is a representation* $\pi : \mathscr{A} \rightarrow B(H_\pi)$ *and operators* $A, B : H \rightarrow H_\pi$ *such that*

$$\|A\| \, \|B\| \leqq M \quad \textit{and} \quad |(A^*\pi(a)Bf, g) - (\varphi(a)f, g)| < \epsilon$$

for each a *in* $\mathscr{F}$ *and each* f, g *in* F.

THEOREM 13. *With the norm* $\| \ \|_{\mathscr{V}}$ *the space* $\mathscr{V}(\mathscr{A}, H)$ *is a Banach space. Furthermore,* $\mathscr{V}(B(H), H)$ *is a Banach algebra with composition as multiplication.*

Proof. The only part of the proof that is not completely elementary involves completeness. To this end suppose that $\{\varphi_\pi\}$ is a sequence in $\mathscr{V}(\mathscr{A}, H)$ such that $\sum_n \|\varphi_n\|_{\mathscr{V}} < \infty$. Since $\sum_n \alpha(\varphi_n) < \infty$, we can find sequences $\{\varphi_{n1}\}$, $\{\varphi_{n2}\}$, $\{\varphi_{n3}\}$, $\{\varphi_{n4}\}$ in CP $(\mathscr{A}, H)$ such that

$$\varphi_\pi = (\varphi_{n1} - \varphi_{n2}) + i(\varphi_{n3} - \varphi_{n4}) \quad \text{for} \quad n = 1, 2, \ldots \quad \text{and}$$
$$\sum_n \|\varphi_{nk}\| < \infty \quad \text{for } k = 1, 2, 3, 4.$$

Since $\| \ \|$ and $\| \ \|_{\mathscr{V}}$ agree on CP $(\mathscr{A}, H)$, it follows that $\sum_n \varphi_{nk}$ is $\| \ \|_{\mathscr{V}}$-convergent for $k = 1, 2, 3, 4$. Thus $\sum_n \varphi_n$ is $\| \ \|_{\mathscr{V}}$-convergent. Hence $\mathscr{V}(\mathscr{A}, H)$ is complete.

The question of R. V. Kadison [**10**] that asks whether every bounded unital homomorphism from a C^*-algebra $\mathscr{A}$ into $B(H)$ is similar to a $*$-homomorphism is still unanswered, although significant progress has been made [**5**], [**6**], [**2**], [**7**], [**18**]. (Theorem 16 shows that the answer is

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affirmative whenever the homomorphism is in $\mathscr{V}(\mathscr{A}, H)$). We first need two lemmas; the first is due to Sarason [**15**], and the second is probably well known.

LEMMA 14. *Suppose $\mathscr{S}$ is a unital subalgebra of $B(H)$ and P is a projection in $B(H)$ such that the mapping $S \to PSP$ is a homomorphism on $\mathscr{S}$. If Q is the smallest $\mathscr{S}$-invariant projection (i.e., $(1 - Q)\mathscr{S}Q = 0$) whose range contains* ran P, *then $Q - P$ is an $\mathscr{S}$-invariant projection.*

LEMMA 15. *Suppose $\pi:\mathscr{A} \to B(H)$ is a unital representation of the C^*-algebra $\mathscr{A}$ and $\pi = \pi_1 \oplus \pi_2$ relative to $H = H_1 \oplus H_2$. Suppose M is a (closed) subspace of H such that $M \cap H_1 = 0$ and $M + H_1 = H$. Let P be the non-orthogonal projection of H onto M along H_1, and define $\rho:\mathscr{A} \to B(M)$ by $\rho(a) = P\pi(a)|M$. Then there is an invertible operator $S:H_2 \to M$ such that $\rho(a) = S\pi_2(a)S^{-1}$ for every a in $\mathscr{A}$.*

Proof. Actually, the required operator S is $P|H_2$. Since $\ker P = H_1$, we can write $P = \begin{pmatrix} 0 & A \\ 0 & 1 \end{pmatrix}$ relative to $H = H_1 \oplus H_2$. If $T = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 1 & -A \\ 0 & 1 \end{pmatrix}$. Clearly, $T|H_2 = P|H_2$. Let $S = T|H_2$. Since $T(H_2) = P(H_2) = M$, it follows that $S^{-1} = T^{-1}|M$. A simple matrix calculation shows that

$$T\pi(a)T^{-1}P = P\pi(a)P \quad \text{for every } a \text{ in } \mathscr{A}.$$

Thus

$$\begin{aligned} S\pi_2(a)S^{-1} &= S(\pi(a)|H_2)S^{-1} = T\pi(a)T^{-1}|M \\ &= P\pi(a)|M = \rho(a) \quad \text{for every } a \text{ in } \mathscr{A}. \end{aligned}$$

THEOREM 16. *Suppose $\rho:A \to B(H)$ is a bounded unital homomorphism. Then ρ is similar to a $*$-homomorphism if and only if $\rho \in \mathscr{V}(\mathscr{A}, H)$.*

Proof. First suppose that $\rho \in \mathscr{V}(\mathscr{A}, H)$. It follows from Theorem 4 that there is a Hilbert space $H_\pi \supset H$, an invertible operator S in $B(H_\pi)$, and a $*$-homomorphism $\pi:\mathscr{A} \to B(H_\pi)$ such that if P is the projection of H_π onto H, then

$$\rho(a) = PS^{-1}\pi(a)S|H \quad \text{for every } a \text{ in } \mathscr{A}.$$

It follows from Lemma 14 that there is a subspace M of H_π that contains H such that M and $M \ominus H$ are invariant for $S^{-1}\pi(\mathscr{A})S$. There is no harm in assuming that $M = H_\pi$, since the mapping $a \to S^{-1}\pi(a)S|M$ is similar to the mapping $a \to \pi(a)|S(M)$, which is a $*$-homomorphism (because $S(M)$ is $\pi(\mathscr{A})$-invariant and thus $\pi(\mathscr{A})$-reducing). Thus we can assume that $H^\perp = H_\pi \ominus H$ is invariant for $S^{-1}\pi(\mathscr{A})S$. Let $H_1 = S(H^\perp)$ and $H_2 = H_1^\perp$, and let $Q = SPS^{-1}$. Since $H^\perp$ is invariant for $S^{-1}\pi(\mathscr{A})S$, we know that H_1 is invariant for (and thus reduces) $\pi(\mathscr{A})$.

Write $\pi = \pi_1 \oplus \pi_2$ relative to $H_\pi = H_1 \oplus H_2$. Clearly Q is the non-orthogonal projection of H_π onto $S(H)$ along H_1. It follows from Lemma 15 that π_2 is similar to the map $a \to Q\pi(a)|S(H)$, which is clearly similar to ρ. Thus ρ is similar to a $*$-homomorphism. The other half of the theorem follows from Theorem 3. This completes the proof.

COROLLARY 17. *Suppose* $\rho:\mathscr{A} \to B(H)$ *and* $\tau: C^*(\rho(\mathscr{A})) \to B(H_1)$. *If* ρ *and* τ *are both similar to* $*$*-homomorphisms, then so is* $\tau \circ \rho$. *If* τ *is a one to one* $*$*-homomorphism and* $\tau \circ \rho$ *is similar to a* $*$*-homomorphism, then so is* ρ.

COROLLARY 18. *Suppose* $\rho:\mathscr{A} \to B(H)$ *is a bounded homomorphism,* $\{\pi_n\}$ *is a net of representations of* $\mathscr{A}$, *and* $\{A_n\}$, $\{B_n\}$ *are bounded nets of operators such that* $A_n\pi_n(a)B_n \to \rho(a)$ *in the weak operator topology for each* a *in* $\mathscr{A}$. *Then* ρ *is similar to a* $*$*-homomorphism.*

Note that the preceding corollary implies Theorem 7 in [**10**].

Stinespring's theorem [**16**] can be viewed as an extension of a theorem of Naimark [**13**] about dilating certain positive operator-valued measures to self-adjoint spectral measures. Accordingly, our results show that certain operator-valued measures can be dilated to non-self-adjoint spectral measures.

Suppose that X is a compact Hausdorff space. A $B(H)$*-valued measure* on X is a map E from the Borel sets of X into $B(H)$ that is countably additive with respect to the weak operator topology on $B(H)$. A $B(H)$-valued measure E is

(a) *bounded* if $\|E\| = \sup\{\|E(M)\|: M \text{ a Borel set}\} < \infty$,

(b) *regular* if the complex measure $E_{f,g}$ defined by $E_{f,g}(M) = (E(M)f, g)$ is regular for every f, g in H,

(c) *self-adjoint* if $E(M)^* = E(M)$ for every Borel set M,

(d) *positive* if $E(M) \geqq 0$ for every Borel set M,

(e) *spectral* if $E(M \cap N) = E(M)E(N)$ for all Borel sets M and N.

Let meas $(X, B(H))$ denote the set of all bounded regular $B(H)$-valued measures on X. If E is a $B(H)$-valued measure on X, define the measure E^* by $E^*(M) = E(M)^*$. Each measure E in meas $(X, B(H))$ uniquely determines a bounded linear mapping $\Phi_E: C(X) \to B(H)$ defined by

$$(\Phi_E(\varphi)f, g) = \int_X \varphi dE_{f,g}$$

for φ in $C(X)$ and f, g in H.

The author was unable to find the following proposition in the literature. The main ideas of the proof (which is omitted) appear in [**8**, VI.7., XVII.2.5] and [**3**, Theorem 19].

PROPOSITION 19. *Suppose* X *is a compact Hausdorff space. The mapping*

$E \to \Phi_E$ *from* meas $(X, B(H))$ *to the set of linear operators from* $C(X)$ *to* $B(H)$ *is a Banach space isomorphism. In addition*

(1) $\|E\| \leqq \|\Phi_E\| \leqq 4\|E\|$ *for every* E *in* meas $(X, B(H))$,

(2) $\Phi_{E^*} = \Phi_E{}^*$ *for every* E *in* meas $(X, B(H))$,

(3) Φ_E *is (completely) positive if and only if* E *is positive.*

Thus bounded linear mappings from $C(X)$ to $B(H)$ correspond to measures in meas $(X, B(H))$, and self-adjoint (completely positive) mappings correspond to self-adjoint (positive) measures. Furthermore, bounded homomorphisms from $C(X)$ to $B(H)$ correspond to spectral measures in meas $(X, B(H))$ [**8**, XV.6.2]; if the homomorphism is a $*$-homomorphism, then the measure is self-adjoint. The theorem of Naimark [**13**] mentioned earlier says that a positive measure E in meas $(X, B(H))$ with $\|E\| \leqq 1$ can be dilated to a self-adjoint spectral measure. (Note that we do not require that $E(X) = 1$.) Naimark's theorem is a special case of Stinespring's theorem [**16**], which says that completely positive maps can be dilated to $*$-homomorphisms. In the same circle of ideas, the following theorem is a reformulation of Theorem 3 in the case when $\mathscr{A} = C(X)$.

THEOREM 20. *Suppose* X *is a compact Hausdorff space and* $E \in$ meas $(X, B(H))$. *The following are equivalent.*

(1) E *has a Hahn decomposition* $E = (E_1 - E_2) + i(E_3 - E_4)$ *where* E_1, E_2, E_3, E_4 *are positive measures in* meas $(X, B(H))$,

(2) $\Phi_E \in \mathscr{V}(C(X), H)$,

(3) *there is a Hilbert space* H_1 *containing* H *and a spectral measure* F *in* meas $(X, B(H))$ *such that if* P *is the orthogonal projection of* H_1 *onto* H, *then* $PF(M)|H = E(M)$ *for every Borel subset* M *of* X.

Note that it follows from Theorem 4 that if $E(X) = 1$ in the preceding theorem, then the measure F can be chosen so that $F(X) = 1$. Note also that the measure F in the preceding theorem is (see Theorem 3) similar to a self-adjoint spectral measure. In fact, every spectral measure in meas $(X, B(H))$ is similar to a self-adjoint spectral measure [**8**, XV.6.2]; perhaps Theorem 16 can be used to reprove this result by showing that every spectral measure in meas $(X, B(H))$ has a Hahn decomposition.

Each measure E in meas $(X, B(H))$ can be uniquely written as a sum $\operatorname{Re} E + i \operatorname{Im} E$ where $\operatorname{Re} E$ and $\operatorname{Im} E$ are Hermitian-valued measures. The condition that a measure E in meas $(X, B(H))$ have a Hahn decomposition is a sort of bounded variation condition in that it is clearly equivalent to the condition that there is a positive measure F in meas $(X, B(H))$ such that $\pm\operatorname{Re} E$, $\pm\operatorname{Im} E \leqq F$. Unfortunately, the existence of such an F does not seem to lead to the existence of some canonically defined total variation measure for E. In [**11**] R. I. Loebl defines a notion of bounded variation for linear maps; i.e., a linear map $\varphi: \mathscr{A} \to$

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$B(H)$ has *finite total variation* if

$$\sup\left\{\left\|\sum_{i=1}^{n} |\varphi(a_i)|\right\| : n \geqq 1, 0 \leqq a_1, \ldots, a_n \in \mathscr{A}, \sum_i a_i \leqq 1\right\} < \infty.$$

(Here $|A| = (A^*A)^{1/2}$.)

Loebl [**11**, Theorem 4.4] proved that if $E \in$ meas $(X, B(H))$, $E = E^*$, and Φ_E has finite total variation, then $\Phi_E \in \mathscr{V}(C(X), H)$. The following example shows that the converse of this result is false.

Example 21. Let $X = \{n/n! : n = 0, 1, \ldots\}$, and let $\{e_1, \ldots\}$ be an orthonormal basis for H. Let $A_0 = B_0 = 0$, and for $n = 1, 2, \ldots,$ define operators A_n and B_n on H by

$$A_n f = (1/n)[(f, e_1)e_n + (f, e_n)e_1] \quad \text{and}$$
$$B_n f = (1/n^2)(f, e_1)e_1 + (f, e_n)e_n \quad \text{for } f \text{ in } H.$$

A matrix calculation shows that $E(\{n/n!\}) = A_n$ defines a self-adjoint measure E in meas $(X, B(H))$. Since $\sum_n |A_n|$ does not converge in the weak operator topology, it is clear that Φ_E does not have finite total variation in the sense of Loebl. However, $F(\{n/n!\}) = B_n$ defines a positive operator-valued measure F in meas $(X, B(H))$ such that $\pm\text{Re}\, E$, $\pm\text{Im}\, E \leqq F$. To see this, note that for $n \geqq 0$, we have $B_n \pm A_n$ has rank 1 and positive trace; whence $B_\pi \pm A_\pi \geqq 0$ for $n \geqq 0$. Thus $\Phi_E \in \mathscr{V}(C(X), H)$ (because E has a Hahn decomposition), although Φ_E does not have finite total variation.

Complete boundedness is also related to this discussion. In [**11**] Loebl proved that every map in $\mathscr{V}(\mathscr{A}, H)$ is completely bounded, and this author knows of no example of a completely bounded linear map from a C^*-algebra $\mathscr{A}$ into $B(H)$ that is not in $\mathscr{V}(\mathscr{A}, H)$. The following example gives an idea of some of the relationships involved.

Example 22. Let X be as in Example 21. Suppose $E(\{n/n!\}) = D_n$ defines a measure in meas $(X, B(H))$ and that Φ_E is completely bounded. Claim: $\sum_n D_n^* D_n$ converges weakly to a bounded operator. To prove this, let a_n be the (continuous) characteristic function of $\{n/n!\}$ for $n = 1, 2, \ldots$. Let $\varphi = \Phi_E$ and let

$$s = \sup\{\|\varphi^{(n)}\| : n = 1, 2, \ldots\}.$$

For each positive integer n, define T_n in $C(X) \otimes \mathfrak{M}_n$ by

$$T_n = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ a_n & 0 & & 0 \end{pmatrix}.$$

Then $\|T_n\| = 1$ implies that

$$\left\|\sum_{k=1}^{n} D_k^* D_k\right\|^{1/2} = \|\varphi^{(n)}(T_n)\| \leqq s \quad \text{for } n \geqq 1.$$

Thus $\sum_k D_k^*D_k \leqq s^2$. This proves the claim. Note that the claim does not imply that $\Phi_E \in \mathscr{V}(C(X), H)$.

Let CB $(\mathscr{A}, H)$ denote the set of completely bounded linear maps from $\mathscr{A}$ into $B(H)$. The space CB $(\mathscr{A}, H)$ is a Banach space with the norm $|||\ \ |||$ defined by

$$|||\varphi||| = \sup_n \|\varphi^{(n)}\|.$$

Conjecture I. $\mathrm{CB}(\mathscr{A}, H) = \mathscr{V}(\mathscr{A}, H)$ and the norms $|||\ \ |||$ and $\|\ \ \|_{\mathscr{V}}$ are equivalent.

To prove the preceding conjecture one needs to consider only certain finite-dimensional cases.

Conjecture II. If $\varphi: \mathscr{A} \rightarrow \mathfrak{M}_n$ is a bounded linear map, then $\beta(\varphi) \leqq \|\varphi^{(n)}\|$.

To see that Conjecture II implies Conjecture I, assume that Conjecture II is true and suppose that $\varphi \in$ CB $(\mathscr{A}, H)$. Let $\{P_n\}$ be a net of finite rank projections in $B(H)$ that converges strongly to 1. For each n define $\varphi_n: \mathscr{A} \rightarrow B(H)$ by

$$\varphi_n(a) = P_n \varphi(a) P_n.$$

It follows from Conjecture II that $\sup_n \beta(\varphi_n) \leqq |||\varphi|||$. Since $\varphi_n(a) \rightarrow \varphi(a)$ in the weak operator topology for every a in $\mathscr{A}$, it follows from Theorem 11 that $\varphi \in \mathscr{V}(\mathscr{A}, H)$. It follows from the proof of Theorem 11 that

$$\alpha(\varphi) \leqq 4 \limsup_n \alpha(\varphi_n);$$

thus, by Lemma 9, $\|\varphi\|_{\mathscr{V}} \leqq 8|||\varphi|||$ since $\|\varphi^{(k)}\| \leqq \beta(\varphi^{(k)}) \leqq \beta(\varphi)$ for $k = 1, 2, \ldots$. Thus the norms $\|\ \ \|_{\mathscr{V}}$ and $|||\ \ |||$ are equivalent.

It is clear from the preceding argument that Conjecture I is equivalent to the statement:

$$\sup \{\beta(\varphi): \varphi: A \rightarrow \mathfrak{M}_n \text{ linear}, |||\varphi||| = 1\} < \infty.$$

We next prove Conjecture II in the case when $n = 1$.

LEMMA 23. *If φ is a continuous linear functional on $\mathscr{A}$, then there is a representation $\pi: \mathscr{A} \rightarrow B(H_\pi)$ and vectors $f, g \in H_\pi$ such that $\|f\|\,\|g\| = \|\varphi\|$ and $\varphi(a) = (\pi(a)f, g)$ for every a in $\mathscr{A}$.*

Proof. It follows from Proposition 1.17.1 in [**14**] that $\varphi \in \mathscr{V}(\mathscr{A}, H)$. It follows from Theorem 3 that there is a representation $\rho: \mathscr{A} \rightarrow B(H_\rho)$ and vectors $u, v \in H_\rho$ such that $\varphi(a) = (\rho(a)u, v)$ for every a in $\mathscr{A}$. Clearly we can assume that ρ is unitarily equivalent to $\rho \oplus \rho \oplus \ldots$. Thus $\rho(\mathscr{A})$ has "property C" as defined in [**9**]. Thus $\rho(\mathscr{A})$ has "property

$D(1)$" as defined in [**9**]. Hence there are sequences $\{u_n\}$, $\{v_n\}$ in H_ρ such that

$$\|u_n\| = \|v_n\| \leqq (\|\varphi\| + 1/n)^{1/2} \quad \text{for } n = 1, 2, \ldots$$

and such that $\varphi(a) = (\rho(a)u_n, v_n)$ for all a in $\mathscr{A}$ and for $n = 1, 2, \ldots$. We define H_π to be a "Berberian space" [**4**] obtained from H_ρ by first defining a sesquilinear functional $\langle\ ,\ \rangle$ on the space X of all bounded sequences in H_ρ by

$$\langle f_n, g_n \rangle = \text{glim } (f_n, g_n)$$

where "glim" denotes a Banach limit. If

$$M = \{h \in X : \langle h, h \rangle = 0\},$$

then M is a linear subspace of X and $\langle\ ,\ \rangle$ induces an inner product $(\ ,\ )$ on X/M. We define H_π to be the completion of X/M with respect to this induced inner product. For each a in $\mathscr{A}$, the mapping on X that sends a sequence $\{h_n\}$ to the sequence $\{\rho(a)h_n\}$ induces an operator $\pi(a)$ on H_π. Clearly $\pi : \mathscr{A} \to B(H_\pi)$ is a representation. Let f, g be the respective images of $\{u_n\}$, $\{v_n\}$ in H_π. Then

$$\|f\| = \|g\| = \|\varphi^{1/2}\| \quad \text{and}$$

$$(\pi(a)f, g) = \text{glim } (\rho(a)u_n, v_n) = \varphi(a) \text{ for every } a \text{ in } \mathscr{A}.$$

COROLLARY 24. *If $\varphi : \mathscr{A} \to B(H)$ is a bounded linear map and $\varphi(\mathscr{A})$ is finite-dimensional, then $\varphi \in \mathscr{V}(\mathscr{A}, H)$.*

Proof. If $\{S_1, \ldots, S_n\}$ is a linear basis for $\varphi(\mathscr{A})$, then there are continuous linear functionals $\varphi_1, \ldots, \varphi_n$ on $\mathscr{A}$ such that

$$\varphi(a) = \varphi_1(a)S_1 + \ldots + \varphi_n(a)S_n$$

for each a in $\mathscr{A}$. The preceding lemma implies that $\varphi_1, \ldots, \varphi_n \in \mathscr{V}(\mathscr{A}, H)$. Thus, by Theorem 3, $\varphi \in \mathscr{V}(\mathscr{A}, H)$.

Another consequence of Conjecture II is a Hahn-Banach theorem type theorem whose validity (or lack of it) could be used to test the validity of Conjecture II.

LEMMA 25. *Suppose Conjecture II is true. If $\mathscr{A}_1$ is a unital C^*-subalgebra of $\mathscr{A}$, $\epsilon > 0$, and $\varphi : \mathscr{A}_1 \to \mathfrak{M}_n$ is a bounded linear map, then φ can be extended to a linear map $\psi : \mathscr{A} \to \mathfrak{M}_n$ such that $\|\psi\| \leqq \|\varphi^{(n)}\| + \epsilon$.*

Proof. It follows from Conjecture II that there is a representation π of $\mathscr{A}_1$ and operators A, B such that $\|A\|\,\|B\| \leqq \|\varphi^{(n)}\| + \epsilon$ and $\varphi(a) = A\pi(a)B$ for every a in $\mathscr{A}_1$. We can extend π to a completely positive map ρ on $\mathscr{A}$ with $\|\rho\| = 1$. Define $\psi : \mathscr{A} \to \mathfrak{M}_n$ by $\psi(a) = A\rho(a)B$.

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Another consequence of Conjecture II is that $|||\varphi||| = \|\varphi^{(n)}\|$ for every bounded linear map $\varphi:\mathscr{A} \to \mathfrak{M}_n$ and $n = 1, 2, \ldots$, which was conjectured by Loebl [**12**].

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Addendum. The author has recently learned the Uffe Haagerup has proved that every completely bounded homomorphism from $\mathscr{A}$ into $B(H)$ is similar to a $*$-homomorphism. His results therefore subsume most of our results on bounded homomorphisms. Also Haagerup has proved that every bounded homomorphism from $\mathscr{A}$ into $B(H)$ that has a cyclic vector is similar to a $*$-homomorphism.

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