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ON THE UNIFORM KADEC-KLEE PROPERTY WITH RESPECT TO CONVERGENCE IN MEASURE

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Abstract

Let $E(0, \infty)$ be a separable symmetric function space, let M be a semifinite von Neumann algebra with normal faithful semifinite trace μ, and let $E(M, \mu)$ be the symmetric operator space associated with $E(0, \infty)$. If $E(0, \infty)$ has the uniform Kadec-Klee property with respect to convergence in measure then $E(M, \mu)$ also has this property. In particular, if $L_\Phi(0, \infty)(\Lambda_\varphi(0, \infty))$ is a separable Orlicz (Lorentz) space then $L_\Phi(M, \mu)(\Lambda_\varphi(M, \mu))$ has the uniform Kadec-Klee property with respect to convergence in measure. It is established also that $E(0, \infty)$ has the uniform Kadec-Klee property with respect to convergence in measure on sets of finite measure if and only if the norm of $E(0, \infty)$ satisfies G. Birkhoff's condition of uniform monotonicity.

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0. Introduction

Let $(X, \|\cdot\|_x)$ be a Banach space, and let τ be a topological vector space topology on X that is weaker than the norm topology. The space $(X, \|\cdot\|_x)$ is said to have the uniform Kadec-Klee property with respect to τ (notation $X \in (\mathrm{UH}_\tau)$) if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that for every sequence $(x_n) \subset X$ with $\|x_n\|_x = 1$, $\|x_n - x_m\|_x > \epsilon$ $(m \neq n)$ and with limit x in the topology τ, we have $\|x\|_x < 1 - \delta(\epsilon)$.

We will consider the following cases:

(1) X is a Banach space with a Schauder basis (e_n), $\tau = \sigma(X, \Gamma)$ where $\Gamma = [e_n^*]$;

(2) X is a symmetric function space $E(0, \infty)$, $\tau =$ convergence locally in measure and

(3) $X = E(0, \infty)$ $(X = E(M, \mu))$, $\tau =$ convergence in Lebesgue measure m ($\tau =$ convergence in the measure topology on the set of all μ-measurable operators (see

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[5])). We will denote the ((UH_τ))-property by (UH_Γ) in the first case; by (UHlm) in the second and by (UHm)((UHμ)) in the third.

See [8,10,11] for general information concerning Banach spaces and symmetric function spaces. For relevant terminology from the theory of the von Neumann algebras we refer to [17], and for the theory of non-commutative integration we refer to [5].

1. Property (UH_Γ) in spaces with Schauder basis

Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis (e_n). Throughout this section (e_n^*) are the bi-orthogonal functionals associated with (e_n), and P_n are the projections onto $[e_k]_{k=1}^n$ with kernel $[e_k]_{k=n+1}^\infty$; that is for every $x = \sum_{k=1}^\infty e_k^*(x)e_k$, we have $P_n x = \sum_{k=1}^n e_k^*(x)e_k$.

The basis (e_n) is said to satisfy the condition (C) if for every $c > 0$ there exists $\delta = \delta(c) > 0$ such that for every $x \in X$ and for each integer n it follows from the conditions $\|P_n x\| = 1$ and $\|(I - P_n)x\| \geq c$ that

$$\|x\| \geq 1 + \delta.$$

The Theorem below is due (in implicit form) to D. van Dulst and V. de Valk [4] for the case when X is an Orlicz sequence space; however, it is also true more generally. We omit the proof and refer to [4, Proposition 3].

THEOREM 1. *If a basis (e_n) satisfies the condition (C) then $X \in (UH_\Gamma)$, where $\Gamma = [e_n^*]_{n=1}^\infty$.*

It is well known that every Banach space with an unconditional basis (e_n), whose unconditional constant is equal to 1, is a Banach lattice when the order is defined by $\sum_{n=1}^\infty a_n e_n \geq 0$ if and only if $a_n \geq 0$ for all n. A Banach lattice $(X, \|\cdot\|_x)$ is called a UMB-lattice (notation: $(X, \|\cdot\|_x) \in$ (UMB)) if its norm satisfies G. Birkhoff's condition of uniform monotonicity; that is for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that if $f, g \in X$, $f \geq 0$, $g \geq 0$, $\|f\|_x = 1$ and $\|f + g\|_x \leq 1 + \delta(\epsilon)$ then $\|g\|_x \leq \epsilon$ (see [2]). In addition, if we suppose that $f \wedge g = 0$, then X is said to have the property (UMBd) (notation: $(X, \|\cdot\|_x) \in$ (UMBd)). We remark that the property (UMBd) coincides with the property (C) which is considered in [3]. Evidently, if $(X, \|\cdot\|_x) \in$ (UMBd) and the order on X is defined by the unconditional basis (e_n), then (e_n) satisfies condition (C). So, we have the following

COROLLARY 1. (See also [3]). *Let X be a Banach lattice whose order is induced by the 1-unconditional basis (e_n). If $(X, \|\cdot\|_x) \in$ (UMBd) then $X \in (UH_\Gamma)$.*

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The next theorem shows that in the preceding corollary the condition $(X, \|\cdot\|_x) \in$ (UMBd) is also necessary if basis (e_n) is symmetric.

THEOREM 2. *Let X be a Banach lattice whose order is induced by the 1-symmetric basis (e_n). Then the following conditions are equivalent:*

(i) $(X, \|\cdot\|_x) \in$ (UMB);
(ii) $(X, \|\cdot\|_x) \in$ (UMBd);
(iii) $(X, \|\cdot\|_x) \in$ (UH_Γ).

PROOF. The implication (ii)→(iii) follows from Corollary 1. The implication (i)→(ii) is obvious.

Before proving the implication (iii)→(i) of the theorem, we recall the following facts. Let $a = (a_i)$, $b = (b_i) \in c_0$, let $a^* = (a_i^*)$, $b^* = (b_i^*)$ be the sequences $(|a_i|)$, $(|b_i|)$ arranged in non-increasing order. We say that a is weakly submajorized by b and write $a \prec_w b$ if $\sum_{i=1}^k a_i^* \leq \sum_{i=1}^k b_i^*$, for $k = 1, 2, 3, \dots$. It is well known that if $a \prec_w b$, $b = (b_i) \in X$ (we identify $b = \sum_i b_i e_i$ with (b_i)) then $a \in X$ and $\|a\|_x \leq \|b\|_x$.

Divide the set N into two disjoint subsets A and B with card A = card $B = \infty$. Let $\pi_1 : N \to A$, $\pi_2 : N \to B$ be arbitrary injections. Put $a' = \sum_k e_k^*(a) e_{\pi_1(k)}$, $b' = \sum_k e_k^*(b) e_{\pi_2(k)}$. It is clear that $a^* = (a')^*$, $b^* = (b')^*$.

The proof of the following lemma is straightforward. The details are therefore omitted. It should be pointed out that the lemma which follows is a special case of Lemma 3 below.

LEMMA 1. *Let $a, b \in X$, $a \geq 0$, $b \geq 0$. Then $a' + b' \prec_w a + b$ and therefore $\|a' + b'\|_x \leq \|a + b\|_x$.*

We remark that it follows immediately from Lemma 1 that (ii) implies (i).

Let us continue the proof of Theorem 2. Assume that $X \in (\mathrm{UH}_\Gamma)$ but that the condition of uniform monotonicity fails to hold for X. Then there exist $\epsilon > 0$; $(x_n), (y_n) \subset X$, $x_n, y_n \geq 0$, such that $\|x_n\|_x = 1$, $\|y_n\|_x \geq \epsilon$, and $\|x_n + y_n\|_x < 1 + n^{-1}$. Fix the integer n and divide the set N into an infinite family of disjoint subsets C_k with card $C_k = \infty$, $k = 0, 1, 2, \dots$ Let $\pi_k : N \to C_k$ be arbitrary bijections. Put $a = \sum_i e_i^*(x_n) e_{\pi_0(i)}$, $b_k = \sum_i e_i^*(y_n) e_{\pi_k(i)}$. It is clear that $(a + b_k)^* = (a + b_m)^*$ for $k, m = 1, 2, \dots$, and $\sigma(X, \Gamma) - \lim_k b_k = 0$. Using Lemma 1, we have

$$(1) \qquad 1 \leq \|a + b_k\|_x \leq \|x_n + y_n\|_x < 1 + n^{-1}.$$

Therefore $\|\alpha^{-1}(a + b_k)\|_x = 1$, $\|\alpha^{-1}(b_k - b_m)\|_x \geq \alpha^{-1}\|b_k\|_x = \alpha^{-1}\|y_n\|_x \geq \alpha^{-1}\epsilon \geq 2^{-1}\epsilon$ and $\sigma(X, \Gamma) - \lim_k(\alpha^{-1}(a + b_k)) = \alpha^{-1}a$, where $\alpha = \|a + b_k\|_x$. Since $X \in (\mathrm{UH}_\Gamma)$ we have $\|\alpha^{-1}a\|_x < 1 - \delta(2^{-1}\epsilon)$. Using (1), it now follows that

$$1 = \|x_n\|_x = \|a\| < (1 + n^{-1})(1 - \delta(2^{-1}\epsilon))$$

for all $n = 1, 2, \ldots$ This contradicts the fact that $\delta(2^{-1}\epsilon) > 0$ and thus completes the proof of Theorem 2.

REMARK. The equivalence of (ii) and (iii) has been established also in [6].

2. (UHlm)-property for the symmetric function spaces $E(0, \infty)$

Throughout this section $E(0, \infty)$ is a separable symmetric function space. It is well known (see [11]), that every space X with a symmetric basis (e_n) is a symmetric function space on an infinite discrete measure space Ω in which the mass of every point is one. In this context $\sigma(X, \Gamma)$-convergence coincides with convergence in measure on sets of finite measure on the unit sphere of X. The following question therefore naturally arises from Theorem 2: are the properties (UHlm)and (UMB) equivalent in $E(0, \infty)$? The theorem below gives a positive answer to this question.

THEOREM 3. *For a separable symmetric space* $(E(0, \infty), \|\cdot\|)$ *the following conditions are equivalent:*

(i) $(E(0, \infty), \|\cdot\| \in$ (UMB)*;*
(ii) $(E(0, \infty), \|\cdot\| \in$ (UMBd)*;*
(iii) $(E(0, \infty), \|\cdot\| \in$ (UHlm).

PROOF. Assertion (ii) is a consequence of assertion (i). It follows from [3], Theorem 3.3, that assertion (iii) is a consequence of (ii). The implication (iii)$\to$(i) is proved exactly in the same way as in Theorem 2. Instead of relation $a \prec_w b$ between sequences $a = (a_n)$ and $b = (b_n)$ we consider the relation $f \prec g$ between functions $f, g \in L_1(0, \infty) + L_\infty(0, \infty)$, where $f \prec g$ means that for every $0 < s < \infty$

$$\int_0^s f^*(t)dt \le \int_0^s g^*(t)dt.$$

Here $f^*(t)$ is the decreasing rearrangement of $|f(t)|$. Lemma 1 is reformulated for this situation in the obvious way (see also Lemma 3 and Remark 2 below). Instead of a partition of N into disjoint subsets C_k and bijections π_k we use a partition of $(0, \infty)$ into an infinite family of disjoint subsets of infinite measure and measure preserving transformations.

COROLLARY 2. *Let* Φ *be an Orlicz function, and let* $L_\Phi(0, \infty)$ *be the corresponding Orlicz space equipped with the Luxemburg norm. Then the following conditions are equivalent:*

(1) Φ *satisfies the* Δ_2*-condition;*

(2) $L_\Phi(0,\infty) \in$ (UHlm).

PROOF. By [1] the Δ_2 condition for Φ is equivalent to (UMB) property for $L_\Phi(0,\infty)$.

The Banach lattice E is said to satisfy a *lower q-estimate* if there exists a constant $C > 0$ such that for all finite sequence (x_n) of mutually disjoint elements in E

$$\left(\sum_n \|x_n\|_E^q\right)^{1/q} \le C \left\|\sum_n x_n\right\|_E .$$

Combining Theorem 3 and Corollary 2.11 [3] we obtain the following:

COROLLARY 3. *If E is a symmetric function space on $(0,\infty)$, then E satisfies a lower q-estimate for some $1 < q < \infty$ if and only if there is an equivalent symmetric norm $\|\cdot\|_0$ on E such that $(E, \|\cdot\|_0) \in$ (UMB).*

3. (UHμ)-property for the symmetric operator spaces $E(M,\mu)$

In this section (M,μ) will be a semifinite von Neumann algebra M with a faithful semifinite normal trace μ on M. Let $K(M,\mu)$ denote the space of all μ-measurable operators affiliated with M (see [5]). $K(M,\mu)$ is the closure of M with respect to the measure topology generated by the trace μ with fundamental system of neighbourhoods around 0 given by $V(\epsilon,\delta) = \{T \in K(M,\mu)$; there exists a projection P in M such that $\|TP\|_\infty \le \epsilon$ and $\mu(1-P) \le \delta\}$ for $\epsilon, \delta > 0$. Here 1 is the unit of M and $\|\cdot\|_\infty$ is the C^*-norm on M. We shall denote by $x_n \xrightarrow{\mu} x$ the convergence of the sequence (x_n) to x in the measure topology generated by the trace μ. Let $A \in K(M,\mu)$. The *t-th singular number of A* $\mu_t(A)$ is

$$\mu_t(A) = \inf\{\|AP\|_\infty : P \text{ is a projection in } M \text{ with } \mu(1-P) \le t\}, t > 0$$

(see, for example [5, Definition 2.1]). It is known [5] that $\mu_t(A) = \mu_t(A^*) = \mu_t(|A|)$ where $|A| = (A^*A)^{1/2}$.

Let $E(0,\mu(1))$ be a separable symmetric function space. The symmetric operator space $E(M,\mu)$ is the space of operators $A \in K(M,\mu)$ such that $\mu_t(A)$ belongs to $E(0,\mu(1))$ and

$$\|A\|_{E(M,\mu)} = \|\mu(A)\|_{E(0,\mu(1))}.$$

Before formulating the main result of this section which concerns the (UHμ) property, we note that if $E(M,\mu) \in$ (UHμ) then $E(M,\mu)$ possesses the Kadec-Klee property with respect to measure convergence (notation: $E(M,\mu) \in$ (Hμ)). In the setting of symmetric function spaces, the property (Hm) has been investigated in [13, 14]. It has been shown ([5, 7]) that the non-commutative L^p-spaces have property (Hμ); subsequently, it was proved in [16] that $E(0, \mu(1)) \in$ (Hm) implies $E(M, \mu) \in$ (Hm). Further, it has been established in [16] that if $E(0, \mu(1))$ is an arbitrary separable symmetric space then $E(M, \mu)$ can be renormed equivalently so that $E(M, \mu)$ endowed with the new norm $\|\cdot\|'$ is a symmetric operator space and $(E(M, \mu), \|\cdot\|') \in$ (Hμ).

The main result of this section shows the uniform Kadec-Klee property with respect to convergence in measure extends from the symmetric function space $E(0, \mu(1))$ to $E(M, \mu)$.

THEOREM 4. *If $E(0, \mu(1)) \in$ (UHm), then $E(M, \mu) \in$ (UHμ).*

The proof of Theorem 4 is based mainly on the following result (cf. [9, Theorem 2.1]).

LEMMA 2. *Let $M, \mu, E(0, \mu(1)), E(M, \mu)$ be as above, and let $(x_n) \subseteq E(M, \mu)$, satisfy $x_n \xrightarrow{\mu} 0$. There exist two sequences of pairwise orthogonal projections $(p_k), (q_k) \subseteq M$ and subsequence (x_{n_k}) such that*

$$\|x_{n_k} - q_k x_{n_k} p_k\|_{E(m,\mu)} \to 0.$$

We also need the following non-commutative analog of Lemma 1 (see also proof of the implication (iii)→(i) of Theorem 3).

LEMMA 3. *Let $a, b, c, d \in E(M, \mu)$ be positive operators such that $\mu_t(b) = \mu_t(c)$, $\mu_t(a) \leq \mu_t(d)$ for all $t \geq 0$ and $ac = 0$. Then*

$$\|a + c\|_{E(M,\mu)} \leq \|d + b\|_{E(M,\mu)}.$$

PROOF OF LEMMA 3. Let $x, y \in K(M, \mu)$. The notation $x \prec y$ means

$$\int_0^t \mu_\tau(x) d\tau \leq \int_0^t \mu_\tau(y) d\tau$$

for all $t > 0$. Since $E(0, \mu(1))$ is a separable symmetric space the relation $x \prec y$, $y \in E(M, \mu)$ implies $x \in E(M, \mu)$ and $\|x\|_{E(M,\mu)} \leq \|y\|_{E(M,\mu)}$ (see, for example [16]). So, it suffices to prove that $a + c \prec d + b$. Fix $t > 0$. Without loss of generality we can assume that M has no minimal projections and therefore there are projections $P_1, P_2 \in M$ such that $P_1 P_2 = 0$, $\mu(P_1 + P_2) = t$ and $\int_0^t \mu_\tau(a + c) d\tau = \mu(aP_1 + cP_2)$ (see [5, Lemma 4.1 and subsequent remarks]). Since $\mu_t(c) = \mu_t(b)$, $\mu_t(a) \leq \mu_t(d)$ we can find the projections $P_3, P_4 \in M$ such that $\mu(P_3) = \mu(P_2)$, $\mu(P_4) = \mu(P_1)$

and $\mu(bP_3) = \mu(cP_2)$, $\mu(aP_1) \leq \mu(dP_4)$. Put $P_5 = P_4 \vee P_3$. Then $\mu(P_5) \leq t$ and it follows that

$$\int_0^t \mu_\tau(d+b)d\tau \geq \mu((d+b)P_5) \geq \mu(dP_4) + \mu(bP_3))$$
$$\geq \mu(aP_1 + cP_2) = \int_0^t \mu_\tau(a+c)d\tau.$$

REMARK. Lemma 1, and its continuous analog in the implication (iii)$\rightarrow$(i) of Theorem 3, follow from Lemma 3 as particular cases.

PROOF OF THEOREM 4. Suppose that $x_n, x \in E(M,\mu)$, $\|x_n\|_{E(M,\mu)} = 1$, $\|x_n - x_m\|_{E(M,\mu)} \geq \epsilon$ $(m \neq n)$, and $x_n \xrightarrow{\mu} x$. We can assume that $x \neq 0$. Put $x_n = x + y_n$. It is clear that $y_n \xrightarrow{\mu} 0$ and $\|y_n - y_m\|_{E(M,\mu)} \geq \epsilon$. By Lemma 2 we may assume by passing to a subsequence, if necessary, that there exist (p_n), $(q_n) \subset M$ such that $p_n = p_n^* = p_n^2$, $q_n = q_n^* = q_n^2$, for all $n = 1, 2, \ldots$, $p_n p_m = q_n q_m = 0$ $(n \neq m)$ and $\|y_n - p_n y_n q_n\|_{E(M,\mu)} \to 0$. Put $P_n = \bigvee_{i=n}^\infty p_i$, $Q_n = \bigvee_{i=n}^\infty q_i$. It is evident that $P_n \downarrow 0$, $Q_n \downarrow 0$ and hence $P_n^\perp \uparrow 1$, $Q_n^\perp \uparrow 1$. Our first objective is to show that

$$\left\|x - P_n^\perp x Q_n^\perp\right\|_{E(M,\mu)} \to 0 \tag{2}$$

for all $x \in E(M,\mu)$. Indeed, without loss of generality we can assume that $x \geq 0$. Since $x = P_n^\perp x Q_n^\perp + P_n x Q_n^\perp + P_n^\perp x Q_n + P_n x Q_n$ it is sufficient to prove that $\|P_n^\perp x Q_n\|_{E(M,\mu)}$, $\|P_n^\perp x Q_n\|_{E(M,\mu)}$, $\|P_n x Q_n\|_{E(M,\mu)} \to 0$.

By ([16, Lemma 3]), we have

$$\left\|P_n^\perp x Q_n\right\|_{E(M,\mu)} = \left\|P_n^\perp \sqrt{x}\sqrt{x} Q_n\right\|_{E(M,\mu)} \leq \left\|\sqrt{x} P_n^\perp \sqrt{x}\right\|_{E(M,\mu)}^{1/2} \|Q_n x Q_n\|_{E(M,\mu)}^{1/2}.$$

Since

$$\left\|\sqrt{x} P_n^\perp \sqrt{x}\right\|_{E(M,\mu)} = \left\|P_n^\perp x P_n^\perp\right\|_{E(M,\mu)} \leq \|x\|_{E(M,\mu)} \quad \text{and}$$
$$\|Q_n x Q_n\|_{E(M,\mu)} \to 0$$

([16, Proposition 4]) we have $\|P_n^\perp x Q_n\|_{E(M,\mu)} \to 0$. Similarly, $\|P_n x Q_n^\perp\|_{E(M,\mu)}$, $\|P_n x Q_n\|_{E(M,\mu)} \to 0$.

Now, using (2) and the fact that $\|y_n - p_n y_n q_n\|_{E(M,\mu)} \to 0$, we may assume by passing to a subsequence and relabelling if necessary, that

$$x_n = P_n^\perp x Q_n^\perp + p_n y_n q_n + z_n \tag{3}$$

where $\|z_n\|_{E(M,\mu)} \leq \epsilon 2^{-n}$.

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Fix $\epsilon > 0$. Now let $\delta = \delta(2^{-1}\epsilon)$ be chosen as in the definition of (UHm) for $E(0, \mu(1))$, and let the integer N simultaneously satisfy the inequalities

$$\left\|x - P_N^\perp x Q_N^\perp\right\|_{E(M,\mu)} < \delta 2^{-N}, \tag{4}$$

$$\left\|P_N^\perp x Q_N^\perp\right\|_{E(M,\mu)} \geq 2^{-1}\|x\|_{E(M,\mu)}. \tag{5}$$

Divide $(0, \mu(1))$ into an infinite family of disjoint subsets $(A_i)_{i=N}^\infty$ such that $m(A_N) = \mu(Q_N^\perp)$, $m(A_i) = \mu(q_i)$, $i = N+1, \ldots$ and choose sequence $(f_i(t))_{i=N}^\infty \subset E(0, \mu(1))$ such that $f_i(t)\chi_{A_i}(t) = f_i(t)$ for all $i \geq N$ and $f_N^*(t) = \mu_t(P_N^\perp x Q_N^\perp)$, $f_i^*(t) = \mu_t(p_i y_i q_i)$, $i \geq N$. Notice that for $n \geq N$

$$\begin{aligned}\left|P_N^\perp x Q_N^\perp + p_n y_n q_n\right|^2 &= \left(Q_N^\perp x^* P_N^\perp + q_n y_n^* p_n\right)\left(P_N^\perp x Q_N^\perp + p_n y_n q_n\right)\\ &= \left|P_N^\perp x Q_N^\perp\right|^2 + |p_n y_n q_n|^2 .\end{aligned}$$

Hence $|P_N^\perp x Q_N^\perp + p_n y_n q_n| = |P_N^\perp x Q_N^\perp| + |p_n y_n q_n|$, and therefore

$$\mu_t\left(P_N^\perp x Q_N^\perp + p_n y_n q_n\right) = (f_N + f_n)^*(t) \tag{6}$$

for all $t > 0$ and for all $n \geq N$. Using (6) it then follows that for $n \geq N$,

$$\|f_N + f_n\|_{E(0,\mu(1))} = \left\|P_N^\perp x Q_N^\perp + p_n y_n q_n\right\|_{E(M,\mu)}. \tag{7}$$

Since $P_n^\perp \geq P_N^\perp$, $Q_n^\perp \geq Q_N^\perp$ for $n \geq N$ we have $\mu_t(P_N^\perp x Q_N^\perp) = \mu_t(P_N^\perp P_n^\perp x Q_n^\perp Q_N^\perp) \leq \mu_t(P_n^\perp x Q_n^\perp)$ for all $t \geq 0$ (see [5]) and hence by Lemma 3

$$\begin{aligned}\left|P_N^\perp x Q_N^\perp + p_n y_n q_n\right| &= \left|P_N^\perp x Q_N^\perp\right| + |p_n y_n q_n| \prec \left|P_n^\perp x Q_n^\perp\right| + |p_n y_n q_n|\\ &= \left|P_n^\perp x Q_n^\perp + p_n y_n q_n\right| .\end{aligned}$$

It follows that $|P_N^\perp x Q_N^\perp + p_n y_n q_n|_{E(M,\mu)} \leq \|P_n^\perp x Q_n^\perp + p_n y_n q_n\|_{E(M,\mu)}$ and so, by (7) and (3)

$$\|f_N + f_n\|_{E(0,\mu(1))} \leq 1 + \epsilon 2^{-n} \text{ for } n > N.$$

Using (5) and passing to a subsequence if necessary we may assume that $\|f_N + f_n\|_{E(0,\mu(1))} \to \alpha$, where $2^{-1}\|x\|_{E(M,\mu)} \leq \alpha \leq 1$. It follows that $\|\beta_n^{-1}(f_N + f_n)\|_{E(0,\mu(1))} = 1$ and

$$\left\|\beta_n^{-1}(f_N + f_n) - \beta_m^{-1}(f_N + f_m)\right\|_{E(0,\mu(1))} \geq 2^{-1}\epsilon$$

for sufficiently large m, n, where $\beta_n = \|f_N + f_n\|_{E(0,\mu(1))}$ (the last inequality is the simple consequence of the following correlations

$$\begin{aligned}&\|p_n y_n q_n - y_n\|_{E(M,\mu)} \to 0, \quad \|p_n y_n q_n - p_m y_m q_m\|_{E(M,\mu)} = \|f_n - f_m\|_{E(0,\mu(1))},\\ &\beta_n \to \alpha \quad \text{and} \quad \|y_n - y_m\|_{E(M,\mu)} \geq \epsilon).\end{aligned}$$

Observe that $p_n y_n q_n \xrightarrow{\mu} 0$ implies $f_n \xrightarrow{m} 0$ and therefore $\beta_n^{-1}(f_N + f_n) \xrightarrow{m} \alpha^{-1} f_N$. So, $E(0, \mu(1)) \in$ (UHm) implies $\|\alpha^{-1} f_N\|_{E(0,\mu(1))} < 1 - \delta$. It follows that $\|P_N^{\perp} x Q_N^{\perp}\|_{E(M,\mu)} = \|f_N\|_{E(0,\mu(1))} \leq 1 - \delta$. Then, by (4), $\|x\| \leq 1 - \delta + \delta 2^{-N}$. This completes the proof of Theorem 4.

The following corollary extends results of [5, 7] which assert $L_p(M, \mu) \in$ (Hμ) for all $p \geq 1$.

COROLLARY 4. *If* $\Phi \in \Delta_2$, *then* $L_\Phi(M, \mu) \in$ (UHμ).

The proof immediately follows from Corollary 2 and Theorem 4.

COROLLARY 5. *If* $\phi(t), \phi(0) = 0$, *is a concave, increasing function on* $(0, \mu(1))$ *such that* $\phi(\infty) = \infty$ *if* $\mu(1) = \infty$, *then the Lorentz space* $\Lambda_\phi(M, \mu)$ *has the uniform Kadec-Klee property with respect to convergence in measure.*

PROOF. By Theorem 4 it is sufficient to prove that $\Lambda_\phi(0, \mu(1)) \in$ (UHm). But this is easy follows from the results [14, 15] which assert that if $y \xrightarrow{m} 0$, $y_n \in \Lambda_\phi(0, \mu(1))$ then

$$\|x_n + y_n\|_{\Lambda_\phi(0,\mu(1))} = \|x\|_{\Lambda_\phi(0,\mu(1))} + \|y_n\|_{\Lambda_\phi(0,\mu(1))} + o(1).$$

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