ON THE HIRSCH–PLOTKIN RADICAL OF A FACTORIZED GROUP

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1. Introduction. Let the group $G = AB$ be the product of two subgroups A and B. A normal subgroup K of G is said to be *factorized* if $K = (A \cap K)(B \cap K)$ and $A \cap B \leq K$, and this is well-known to be equivalent to the fact that $K = AK \cap BK$ (see [1]). Easy examples show that normal subgroups of a product of two groups need not, in general, be factorized. Therefore the determination of certain special factorized subgroups is of relevant interest in the investigation concerning the structure of a factorized group. In this direction E. Pennington [**5**] proved that the Fitting subgroup of a finite product of two nilpotent groups is factorized. This result was extended to infinite groups by B. Amberg and the authors, who proved in [**2**] that if the soluble group $G = AB$ with finite abelian section rank is the product of two locally nilpotent subgroups A and B, then the Hirsch–Plotkin radical (i.e. the maximum locally nilpotent normal subgroup) of G is factorized. If G is a soluble $\mathcal{S}_1$-group and the factors A and B are nilpotent, it was shown in [**3**] that also the Fitting subgroup of G is factorized. However, Pennington's theorem becomes false for finite soluble groups which are the product of two arbitrary subgroups. For instance, the symmetric group of degree 4 is the product of a subgroup isomorphic with the symmetric group of degree 3 and a cyclic subgroup of order 4, but its Fitting subgroup is not factorized.

The aim of this paper is to prove that even in the case of a group factorized by two arbitrary subgroups the Hirsch–Plotkin radical and the Fitting subgroup have some factorization properties.

THEOREM A. *Let the soluble-by-finite group $G = AB$ with finite abelian section rank be the product of two subgroups A and B, and let H be the Hirsch–Plotkin radical of G. Then $H = A_0H \cap B_0H$, where A_0 and B_0 are the Hirsch–Plotkin radicals of A and B, respectively.*

Here the requirement that G has finite abelian section rank cannot be removed, as Ya. P. Sysak [**10**] gave an example of a triply factorized group $G = AB = AK = BK$, where A, B and K are torsion-free abelian subgroups and K is normal in G, but G is not locally nilpotent.

In the hypotheses of Theorem A, if the subgroups A and B are locally nilpotent, one has in particular that the Hirsch–Plotkin radical of G is factorized. Similarly, the factorization of the Fitting subgroup of a soluble $\mathcal{S}_1$-group factorized by two nilpotent subgroups is a consequence of the following result.

THEOREM B. *Let the soluble-by-finite $\mathcal{S}_1$-group $G = AB$ be the product of two subgroups A and B, and let F be the Fitting subgroup of G. Then $F = A_0F \cap B_0F$, where A_0 and B_0 are the Fitting subgroups of A and B, respectively.*

Most of our notation is standard and can for instance be found in [**6**]. In particular:
If G is a group, $\bar{Z}(G)$ is the hypercentre of G.
If G is a group, $\pi(G)$ is the set of prime divisors of the orders of elements of G.

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A group G has *finite abelian section rank* if it has no infinite elementary abelian p-sections for every prime p.

A group G is an *$\mathcal{S}_1$-group* is it has finite abelian section rank and the set of primes $\pi(G)$ is finite.

If Q is a group and M is a Q-module, $H_n(Q,M)$ and $H^n(Q,M)$ are the n-th *homology group* and the n-th *cohomology group* of Q with coefficients in M, respectively.

If N is a normal subgroup of a factorized group $G = AB$, the *factorizer* of N in G is the subgroup $X(N) = AN \cap BN$.

2. Proof of the Theorems. Our first lemma shows that Theorems A and B hold in the finite case.

LEMMA 1. *Let the finite group $G = AB$ be the product of two subgroups A and B, and let F be the Fitting subgroup of G. Then $F = A_0F \cap B_0F$, where A_0 and B_0 are the Fitting subgroups of A and B, respectively.*

Proof. Assume that the lemma is false, and let $G = AB$ be a counterexample of minimal order. If N_1 and N_2 are distinct minimal normal subgroups of G, and F_i/N_i is the Fitting subgrup of $G/N_i (i = 1, 2)$, it follows that $A_0F_i \cap B_0F_i = F_i$, since the result holds for the factor group G/N_i. Then

$$A_0F \cap B_0F \leq F_1 \cap F_2 = F,$$

and $F = A_0F \cap B_0F$. This contradiction shows that G has a unique minimal normal subgroup N, and hence F is a p-group for some prime p. Put $F_0 = A_0F \cap B_0F$. Since $F \leq F_0 \leq A_0F$, the subgroup F_0 is subnormal in AF, and similarly it is subnormal in BF. Then it follows from Satz 1 of **[11]** that F_0 is subnormal also in the factorized group $G = (AF)(BF)$. Therefore F_0 is not nilpotent, and there exists a prime $q \neq p$ dividing the order of F_0. The Sylow q-subgroup Q_1 of A_0 is clearly also a Sylow q-subgroup of A_0F, and hence $Q = Q_1 \cap F_0$ is a Sylow q-subgroup of F_0. Moreover Q lies in A_0, and so is subnormal in A. Let Q_2 be the Sylow q-subgroup of B_0. Then Q_2 is a Sylow q-subgroup of B_0F, and thus there exists $x \in G$ such that

$$Q \leq Q_2^x \leq B_0^x.$$

As B_0^x is the Fitting subgroup of B^x, we obtain that Q is subnormal in B^x, and Satz 1 of **[11]** yields that Q is subnormal in $G = AB^x$. Since F is a p-group, it follows that $Q = 1$, and this contradiction proves the lemma.

LEMMA 2. *Let the group $G = AB = AK = BK$ be the product of two subgroups A and B and a radicable abelian normal p-subgroup K satisfying the minimal condition. If A_0 and B_0 are nilpotent normal subgroups of A and B, respectively, then the subgroup $A_0K \cap B_0K$ is nilpotent.*

Proof. Assume that the lemma is false, and choose a counterexample

$$G = AB = AK = BK$$

such that K has minimal Prüfer rank. Clearly the subgroups A_0K and B_0K are normal in G, and hence also $K_0 = A_0K \cap B_0K$ is a normal subgroup of G. Moreover $K_0/K \leq$

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A_0K/K is obviously nilpotent. Suppose that K_0 is finite-by-nilpotent. Then there exists a positive integer r such that the index $|K_0:Z_r(K_0)|$ is finite (see [**6**] Part 1, Theorem 4.25), so that $K \leq Z_r(K_0)$ and K_0 is nilpotent. This contradiction shows that K_0 is not finite-by-nilpotent. Let L be an infinite G-invariant subgroup of K with minimal Prüfer rank. Then L is radicable and all its proper G-invariant subgroups are finite. By the minimality of the rank of K the result holds for the factor group G/L, and hence K_0/L is nilpotent. It follows that $[L, K_0] \neq 1$, and so $[L, K_0] = L$, since $[L, K_0]$ is radicable and L has no infinite proper G-invariant subgroups. This means that $H_0(K_0/L, L) = 0$, and Theorem C of [**8**] yields that $H^2(G/L, L)$ has finite exponent. Therefore there exists a subgroup J of G such that $G = LJ$ and $L \cap J$ is finite. As $L \cap J$ is normal in G and K_0 is not finite-by-nilpotent, also the factor group $G/(L \cap J)$ is a counterexample, and hence we may suppose that $L \cap J = 1$. Thus $K = L \times (J \cap K)$ and $J \cap K \simeq K/L$ is a radicable normal subgroup of G. If $J \cap K \neq 1$, the result holds for the factor group $G/(J \cap K)$, and so $K_0/(J \cap K)$ is nilpotent. It follows that K_0 is nilpotent, and this contradiction proves that $J \cap K = 1$. Therefore $K = L$, and K has no infinite proper G-invariant subgroups. Assume that $A \cap K$ is infinite. As $A \cap K$ is normal in $G = AK$, we obtain that $A \cap K = K$ and $K \leq A$. Then A_0K is nilpotent, so that also K_0 is nilpotent. This contradiction shows that $A \cap K$ is finite, and similarly $B \cap K$ is finite. Thus the normal subgroup $N = (A \cap K)(B \cap K)$ of G is also finite, and as above the factor group G/N is a counterexample. Hence we may suppose that $A \cap K = B \cap K = 1$. If A_1 and B_1 are the Fitting subgroups of A and B, respectively, it follows that $A_1K = B_1K$ is a normal subgroup of G containing K_0. Since $H_0(K_0/K, K) = 0$, application of Theorem C of [**8**] yields that $H^1(A_1K/K, K)$ has finite exponent. But K is a radicable abelian p-group of finite rank, and hence there exists a finite characteristic subgroup E of K such that the complements of K/E in A_1K/E are conjugate (see [**7**]). The factor group G/E is also a counterexample, so that we may suppose that the complements of K in A_1K are conjugate. As A_1 and B_1 are both complements of K in A_1K, there exists $x \in G$ such that $A_1^x = B_1$. Write $x = ab$, where $a \in A$ and $b \in B$. Then

$$A_1 = A_1^a = B_1^{b^{-1}} = B_1,$$

so that $A_1 = B_1$ is normal in G, and A_1K is nilpotent. This last contradiction completes the proof of the lemma.

Lemma 3. *Let G be a group, and let K be a periodic abelian normal subgroup of infinite exponent of G whose proper G-invariant subgroups are finite. Then K is contained in the centre of the Fitting subgroup of G. In particular, if $C_G(K) = K$, then K is the Fitting subgroup of G.*

Proof. Let N be a nilpotent normal subgroup of G. Then KN is also nilpotent, and hence $K \cap Z(KN)$ is infinite, since K has infinite exponent (see for instance [**6**], Theorem 2.23). But $K \cap Z(KN)$ is normal in G, and K has no infinite proper G-invariant subgroups, so that $K \cap Z(KN) = K$. Therefore $K \leq Z(KN)$ and $N \leq C_G(K)$. This proves that K lies in the centre of the Fitting subgroup of G.

Proof of Theorem A. Assume that the result is false, and among all the counterexamples for which the soluble radical S of G has minimal index choose one $G = AB$ such that S has minimal derived length. As the theorem is true for finite groups by Lemma 1, the group G is infinite, and hence its soluble radical is not trivial. It follows that G contains an abelian normal subgroup K such that the theorem holds for the factor group G/K. Write $M = A_0H \cap B_0H$. Then M/K lies in the Hirsch–Plotkin radical of G/K, and hence M is ascendant in G, as the Hirsch–Plotkin radical of G/K is hypercentral. Since $H < M$, this proves that M is not locally nilpotent. The factorizer $X(H)$ of H in $G = AB$ has a triple factorization

$$X(H) = \bar{A}\bar{B} = \bar{A}H = \bar{B}H,$$

where $\bar{A} = A \cap BH$ and $\bar{B} = B \cap AH$. If $\bar{A}_0 = A_0 \cap \bar{A} = A_0 \cap BH$ and $\bar{B}_0 = B_0 \cap \bar{B} = B_0 \cap AH$, then $\bar{A}_0$ and $\bar{B}_0$ are contained in the Hirsch–Plotkin radicals of $\bar{A}$ and $\bar{B}$, respectively. Moreover

$$\bar{A}_0H \cap \bar{B}_0H = (A_0 \cap BH)H \cap (B_0 \cap AH)H = A_0H \cap B_0H = M,$$

so that $\bar{A}_0H \cap \bar{B}_0H$ is not locally nilpotent. Therefore $X(H) = \bar{A}\bar{B}$ is also a minimal counterexample, and without loss of generality we may suppose that G has a triple factorization

$$G = AB = AH = BH.$$

Then the subgroups A_0H and B_0H are normal in G, and hence also M is a normal subgroup of G. The structure of soluble groups with finite abelian section rank (see [**6**]) allows us to investigate only the following possible choices for K.

Case 1: *K is finite*. By induction on the order of K be may suppose that K is a minimal normal subgroup of G. As M is not locally nilpotent, we have that $[K, M] \neq 1$ and hence $[K, M] = K$. Then $H_0(M/K, K) = 0$, and it follows from Theorem 3.4 of [**9**] that $H^2(G/K, K) = 0$. Therefore there exists a subgroup J of G such that $G = KJ$ and $K \cap J = 1$. The centralizer $C_J(K)$ is normal in G, and Lemma 1 shows that the theorem holds for the finite factor group $G/C_J(K)$. In particular $MC_J(K)/C_J(K)$ is locally nilpotent, and so M is locally nilpotent since $K \cap C_J(K) = 1$. This contradiction proves that the subgroup K cannot be finite.

Case 2: *K is periodic and residually finite*. Each primary component K_p of K is finite, and so by Case 1 the group $M/K_{p'}$ is locally nilpotent for every prime p. As the groups K_p and $K/K_{p'}$ are G-isomorphic, it follows that K_p is hypercentrally embedded in M. Then K is hypercentrally embedded in M, and M is locally nilpotent, a contradiction.

Case 3: *K is a radicable p-group* (*p prime*). By induction on the rank of K we may suppose that every proper G-invariant subgroup of K is finite. In particular, as K is not hypercentrally embedded in M, the intersection $\bar{Z}(M) \cap K$ is finite. It follows from Case 1 that also the factor group $G/(\bar{Z}(M) \cap K)$ is a counterexample, and hence it can be assumed that $Z(M) \cap K = 1$. Thus $H^0(M/K, K) = 0$. Moreover, $G/C_G(K)$ is isomorphic with an irreducible linear group by Lemma 5 of [**4**], and hence it is abelian-by-finite (see [**6**] Part 1, Theorem 3.21). Then $M/C_M(K)$ is FC-hypercentrally embedded in G, and Theorem 3.5 of [**9**] yields that $H^2(G/K, K) = 0$. Therefore there exists a subgroup J of G such that $G = KJ$ and $K \cap J = 1$. The centralizer $C_J(K)$ is normal in G, and $MC_J(K)/C_J(K)$ is not locally nilpotent. Put $\tilde{G} = G/C_J(K)$. As K and $\tilde{K}$ are isomorphic M-modules, we obtain that $Z(\tilde{M}) \cap \tilde{K} = 1$. Moreover $C_{\tilde{G}}(\tilde{K}) = \tilde{K}$, and replacing G by $\tilde{G}$ we may suppose that $C_G(K) = K$ and $Z(M) \cap K = 1$. In particular K is the Fitting

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subgroup of G by Lemma 3, and the factor group G/K is abelian-by-finite. Let L/K be an abelian normal subgroup of G/K such that G/L is finite. For each positive integer n, the n-th term $Z_n(H)$ of the upper central series of H is a nilpotent normal subgroup of G, so that $Z_n(H) \leq K$. On the other hand, K lies in $Z_\omega(H)$, since H is hypercentral, and so $K = Z_\omega(H)$. Assume that $Z(A_0) \cap K$ contains a non-trivial element a, and let m be the least positive integer such that $a \in Z_m(H)$. Then $Z_{m-1}(H)$ is properly contained in K, and hence is finite. Write $\bar{G} = G/Z_{m-1}(H)$. Then $\bar{a}$ centralizes $\bar{A}_0$ and $\bar{H}$, so $\bar{a} \in Z(\bar{M}) \cap \bar{K}$ and $Z(\bar{M}) \cap \bar{K} \neq 1$. As $Z_{m-1}(H)$ is finite and $Z(M) \cap K = 1$, this contradicts Lemma 2.3 of [2]. Therefore $Z(A_0) \cap K = 1$ and hence also $A_0 \cap K = 1$. But $A \cap K$ is contained in A_0, so that $A \cap K = 1$. The same argument shows that $B \cap K = 1$. Then the subgroups A and B are abelian-by-finite, and in particular the indices $|A : A_0|$ and $|B : B_0|$ are finite. The factorizer $X = X(K)$ of K in $G = AB$ has a triple factorization

$$X = A^*B^* = A^*K = B^*K,$$

where $A^* = A \cap BK$ and $B^* = B \cap AK$. It follows from Lemma 2 that $A_0K \cap B_0K = (A_0 \cap BK)K \cap (B_0 \cap AK)K$ is nilpotent-by-finite and hence X is also. Thus the Fitting subgroup Y of X is nilpotent and X/Y is finite. As $K \leq Y \cap L \leq L$, we have that $Y \cap L$ is a nilpotent normal subgroup of L. Clearly K is the Fitting subgroup of L, so that $Y \cap L = K$, and K has finite index in X. But $A^* \cap K = B^* \cap K = 1$, so that A^* and B^* are finite, and $X = A^*B^*$ is also finite. This contradiction completes the proof of this case.

Case 4: *K is a periodic radicable group.* Each primary component K_p of K is radicable, so that Case 3 shows that $M/K_{p'}$ is locally nilpotent for every prime p. Then $K/K_{p'}$ is hypercentrally embedded in M, and hence K_p lies in the hypercentre of M. It follows that K is hypercentrally embedded in M, and M is locally nilpotent.

Case 5: *K is torsion-free.* Let T be the maximum periodic normal subgroup of G. As $K \cap T = 1$, we have that MT/T is not locally nilpotent, and hence the factor group G/T is also a counterexample. Thus we may suppose that G has no non-trivial periodic normal subgroups, so that in particular the set of primes $\pi(G)$ is finite (see [6] Part 2, Lemma 9.34). It follows that G is nilpotent-by-polycyclic-by-finite (see [6] Part 2, Theorem 10.33). If F is the Fitting subgroup of G, then $K \cap Z(F) \neq 1$. Consider a non-trivial element x of $K \cap Z(F)$, and let N be the normal closure of x in G. Thus N is a cyclic module over the polycyclic-by-finite group G/F, and hence it contains a free abelian subgroup E such that N/E is a π-group, where π is a finite set of primes (see [6] Part 2, Corollary 1 to Lemma 9.53). Clearly

$$\Big(\bigcap_{p \notin \pi} N^p\Big) \cap E = \bigcap_{p \notin \pi} (N^p \cap E) = \bigcap_{p \notin \pi} E^p = 1,$$

so that $\bigcap_{p \notin \pi} N^p$ is periodic, and $\bigcap_{p \notin \pi} N^p = 1$ since $N \leq K$ is torsion-free. Let p be any prime which does not belong to π. As $N^p \neq 1$, by induction on the torsion-free rank of G we may suppose that the theorem holds for G/N^p. Therefore M/N^p is locally nilpotent. Let r be the Prüfer rank of N. Then $|N/N^p| = p^r$, so that N/N^p lies in the r-th term of the upper central series of M/N^p. It follows that

$$[N, \underbrace{M, \ldots, M}_{r}] \leq \bigcap_{p \notin \pi} N^p = 1,$$

and so $N \leq Z_r(M)$. Thus M is locally nilpotent, and this last contradiction completes the proof of Theorem A.

Proof of Theorem B. Assume that the result is false, and choose a counterexample $G = AB$ such that the radicable part R of the maximum periodic normal subgroup of G has minimal total rank. Put $F_0 = A_0F \cap B_0F$. Then Theorem A proves that F_0 lies in the Hirsch–Plotkin radical of G, and hence is locally nilpotent. The periodic subgroups of the factor group G/R are finite, so that the Hirsch–Plotkin radical and the Fitting subgroup of G/R coincide (see [**6**] Part 2, p. 35), and it follows again from Theorem A that F_0/R is contained in the Fitting subgroup of G/R. As the Fitting subgroup of an $\mathcal{S}_1$-group is nilpotent, we obtain that F_0 is subnormal in G and F_0/R is nilpotent. Also, in an $\mathcal{S}_1$-group each nilpotent subnormal subgroup lies in the Fitting subgroup, so F_0 is not nilpotent and $R \neq 1$. Since F_0 is locally nilpotent, we have also that F_0 is not finite-by-nilpotent. Let S be an infinite G-invariant subgroup of R with minimal total rank, Then S is a radicable abelian p-group for some prime p, and all its proper G-invariant subgroups are finite. Thus $G/C_G(S)$ is isomorphic with an irreducible linear group by Lemma 5 of [**4**], and hence it is abelian-by-finite. Moreover G/S is an $\mathcal{S}_1$-group, so that its Fitting subgroup F_1/S is nilpotent and $F_0 \leq F_1$ by the minimal choice of G. Therefore $[S, F_1] \neq 1$, and hence $[S, F_1] = S$. Thus $H_0(F_1/S, S) = 0$, and Theorem C of [**8**] yields that $H^2(G/S, S)$ has finite exponent. Then there exists a subgroup J of G such that $G = SJ$ and $S \cap J$ is finite. The subgroup $S \cap J$ is normal in G, and the factor group $G/(S \cap J)$ is also a counterexample, since F_0 is not finite-by-nilpotent. Therefore we may suppose that $S \cap J = 1$, so that $R = S \times (J \cap R)$, where $J \cap R$ is a radicable normal subgroup of G. Clearly $F_0(J \cap R)/(J \cap R)$ is not nilpotent, so $J \cap R = 1$ and R has no infinite proper G-invariant subgroups. The centralizer $C_J(R)$ is normal in G, and the periodic subgroups of $J/C_J(R)$ are finite (see [**6**] Part 1, Corollary to Lemma 3.28), so that $G/C_J(R)$ is an $\mathcal{S}_1$-group. As $C_J(R) \cap R = 1$, the group $F_0C_J(R)/C_J(R)$ is not nilpotent, and the theroem is false for the group $G/C_J(R)$. Clearly R is G-isomorphic with the radicable part of the maximum periodic normal subgroup of $G/C_J(R)$, so that $G/C_J(R)$ is also a minimal counterexample. Moreover

$$C_{J/C_J(R)}(RC_J(R)/C_J(R)) = 1,$$

and hence we may suppose that $C_J(R) = 1$ and $C_G(R) = R$. Thus it follows from Lemma 3 that R is the Fitting subgroup of G. The factorizer $X = X(R)$ of R in G has the triple factorization

$$X = A^*B^* = A^*R = B^*R,$$

where $A^* = A \cap BR$ and $B^* = B \cap AR$. Write $A_0^* = A_0 \cap BR$ and $B_0^* = B_0 \cap AR$. Then A_0^* and B_0^* are nilpotent normal subgroups of A^* and B^*, respectively, and Lemma 2 shows that $A_0^*R \cap B_0^*R$ is nilpotent. Since

$$A_0^*R \cap B_0^*R = (A_0 \cap BR)R \cap (B_0 \cap AR)R = A_0R \cap B_0R = A_0F \cap B_0F = F_0,$$

we have that F_0 is nilpotent. This contradiction completes the proof of Theorem B.

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