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CONTACT METRIC THREE-MANIFOLDS WITH CONSTANT SCALAR TORSION

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Abstract

In this paper we study three-dimensional contact metric manifolds satisfying $\|\tau\| = \text{constant}$. The local description, as well as several global results and new examples of such manifolds are given.

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1. Introduction

In contact geometry, the tensor field $\tau = \mathcal{L}_\xi g$, introduced by Chern and Hamilton [4], and the Jacobi operator $l = R(., \xi)\xi$ play a fundamental role. In a $(2m + 1)$-dimensional contact metric manifold M, the function $\operatorname{Tr} l$ and the scalar torsion $\|\tau\|$ are related by the relation $\|\tau\|^2 = 4(2m - \operatorname{Tr} l) \geq 0$ [4, 13]. So the constancy of each $\operatorname{Tr} l$ and $\|\tau\|$ implies the constancy of the other. (Thus we will be using the constancy of $\|\tau\|$ or $\operatorname{Tr} l$ interchangeably in this paper.) It is well known that there exist a lot of classes of contact metric manifolds with $\|\tau\| = \text{constant}$, such as the Sasakian manifolds, the K-contact manifolds, the tangent sphere bundle equipped with the Sasaki metric of a Riemannian manifold of constant curvature, or more generally the (κ, μ)-contact manifolds [2], the normal bundle of a maximal dimension integral submanifold of a Sasakian manifold [1, page 189], the homogeneous contact Riemannian three-manifolds [12], the three-dimensional pseudosymetric of constant type contact metric manifolds which satisfy one more condition [6, 7], and the Jacobi (κ, μ)-contact manifolds [5]. For more information about contact metric manifolds with $\|\tau\| = \text{constant}$, see [9, 11]. So it is natural to look for the potential existence of more contact metric manifolds with $\|\tau\| = \text{constant}$ beyond the aforementioned well-known classes.

In this paper, we study the condition $\|\tau\| = \text{constant}$ in the three-dimensional case and the content is organized in the following way. Section 2 is devoted mainly to preliminaries on contact metric manifolds and to some new examples. In Section 3, several global results of three-dimensional contact metric manifolds with $\|\tau\| =$ constant are given. Finally, Section 4 is concerned with the local description of such manifolds. In particular, in this section, in terms of contact metric manifolds with $\|\tau\| =$ constant, we distinguish between and characterize the (κ, μ)-contact manifolds and the Jacobi (κ, μ)-contact manifolds.

2. Preliminaries

A contact manifold is a differentiable manifold M^{2m+1} together with a global 1-form η (a contact form) such that $\eta \wedge (d\eta)^m \neq 0$ everywhere. Since $d\eta$ is of rank $2m$, there exists a unique vector field ξ (the Reeb or the characteristic vector field of the contact structure η) satisfying $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for all vector fields X. The distribution D defined by the subspace $X \epsilon T_p M : \eta(X) = 0$ for all $p \epsilon M$ is called the contact distribution. Every contact manifold has an underlying almost contact structure $(\eta, \xi.\phi)$, where ϕ is a global tensor field of type $(1, 1)$ such that

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \quad \phi^2 = -I + \eta \otimes \xi. \tag{2.1}$$

A Riemannian metric g (the associated metric) can be defined such that

$$\eta(X) = g(X, \xi) \quad \text{and} \quad d\eta(X, Y) = g(X, \phi Y) \tag{2.2}$$

for all vector fields X and Y on M^{2m+1}. We note that g and ϕ are not unique for a given contact form η, but g and ϕ are canonically related to each other by

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.3}$$

We refer to (η, ξ, ϕ, g) as a contact metric structure (c.m.s. in short) and to the manifold M^{2m+1} carrying such a structure as a contact metric manifold (c.m.m. in short) and this will be denoted by $M^{2m+1}(\eta, \xi, \phi, g)$. Denoting Lie differentiation and the curvature tensor by $\mathcal{L}$ and R, respectively, we define the operators l, h and τ by

$$l = R(., \xi)\xi, \quad h = \tfrac{1}{2}\mathcal{L}_\xi \phi, \quad \tau = \mathcal{L}_\xi g = 2g(h\phi, .).$$

On every c.m.m. $M^{2m+1}(\eta, \xi, \phi, g)$ we have many important formulas,

$$l\xi = h\xi = 0, \quad \eta \circ h = 0, \quad \operatorname{Tr} h = \operatorname{Tr} \phi h = 0, \quad h\phi = -\phi h,$$
$$hX = \lambda X \quad \text{implies } h\phi X = -\lambda \phi X.$$

Moreover, if ∇ is the Riemannian connection of g, S is the Ricci tensor of type $(0, 2)$, Q is the corresponding Ricci operator satisfying $g(QX, Y) = S(X, Y)$ and $r = \operatorname{Tr} Q$ is the scalar curvature, then

$$\nabla_\xi \phi = 0, \quad \nabla_X \xi = -\phi X - \phi h X, \quad \operatorname{Tr} l = g(Q\xi, \xi) = 2m - \operatorname{Tr} h^2 \leq 2m$$
$$\tau = 2g(\phi., h.), \quad \nabla_\xi \tau = 2g(\phi., (\nabla_\xi h).)$$
$$\nabla_\xi h = \phi - \phi l - \phi h^2.$$

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The conditions $\|\tau\| =$ constant, $\operatorname{Tr} l =$ constant and $\operatorname{Tr} h^2 =$ constant are equivalent. A c.m.m. $M^{2m+1}(\eta, \xi, \phi, g)$ for which ξ is a Killing vector field, that is, for which $\mathcal{L}_\xi g = 0$, is called a K-contact manifold. A c.m.m. $M^{2m+1}(\eta, \xi, \phi, g)$ is K-contact manifold if and only if $h = 0$ (or, equivalently, $\tau = 0$). If we take the product $M^{2m+1} \times \mathbb{R}$, the c.m.s. on M^{2m+1} gives rise to an almost complex structure J on $M^{2m+1} \times \mathbb{R}$ given by $J(X, f(d/dt)) = (\phi X - f\xi, \eta(X)(d/dt))$. If this structure is integrable, then M^{2m+1} is called Sasakian. A c.m.m. is Sasakian if and only if $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$ for all vector fields X, Y on the manifold. If $\dim M^{2m+1} = 3$, then a K-contact manifold is Sasakian. A c.m.m. $M(\eta, \xi, \phi, g)$ is said H-contact manifold if the characteristic vector field ξ is harmonic or, equivalently, if ξ is an eigenvector of the Ricci operator [13]. Sasakian and K-contact manifolds are H-contact manifolds. More details on contact manifolds are found in [1].

A generalization of Sasakian manifolds are the (κ, μ)-contact manifolds [2], the curvature tensor of which satisfies the condition

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2.4}$$

for all vector fields X, Y, where $\kappa = \operatorname{Tr} l/2$ and μ are constant. If κ, μ in (2.4) are nonconstant smooth functions, then M^{2m+1} is called a generalized (κ, μ)-contact manifold [10].

Moreover generalizations of (κ, μ)-contact manifolds and K-contact manifolds are the Jacobi (κ, μ)-contact manifolds, which satisfy the condition

$$l = -\kappa\phi^2 + \mu h, \tag{2.5}$$

where κ, μ are constant [5]. From (2.5), $\operatorname{Tr} h = 0$ and $\operatorname{Tr} \phi^2 = -2m$, it follows that $\operatorname{Tr} l = 2m\kappa =$ constant.

We note that all manifolds are assumed to be connected and smooth. The set of the vector fields on the manifold M will be denoted by $\mathcal{X}(M)$.

In the next proposition, an essential characteristic of the class of contact metric manifolds with $\operatorname{Tr} l =$ constant is proved.

Proposition 2.1. *For a contact metric* $(2m + 1)$*-manifold, the condition* $\operatorname{Tr} l =$ constant *is invariant under a D-homothetic deformation.*

Proof. By a D-homothetic deformation [14] on $M(\eta, \xi, \phi, g)$ we mean a change of structure tensors of the form

$$\bar{\eta} = \alpha\eta, \quad \bar{\xi} = \frac{1}{\alpha}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = \alpha g + \alpha(\alpha - 1)\eta \otimes \eta,$$

where α is a positive constant. It is well known that $M(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is also a c.m.m. By direct computation, we see that the tensor h is transformed in the following way.

$$\bar{h} = \frac{1}{\alpha}h.$$

Moreover, using this and $\operatorname{Tr} l =$ constant (equivalently, $\operatorname{Tr} h^2 =$ constant), we get $\operatorname{Tr}\bar{h}^2 =$ constant and so $\operatorname{Tr}\bar{l} =$ constant for any positive number α.

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In the following, we give new examples of contact metric manifolds with $\mathrm{Tr}\, l =$ constant $\neq 2$. Examples (1)–(3) concern Jacobi (κ, μ)-contact manifolds, which, for appropriate choices of the function $f = f(y, z)$, degenerate into (κ, μ)-contact manifolds. Example (4) concerns a Jacobi (κ, μ)-contact manifold. Examples (5) and (6) concern contact metric manifolds with $\mathrm{Tr}\, l =$ constant, which are not Jacobi (κ, μ)-contact manifolds. □

Examples. In all six examples, the three-dimensional manifold M is always the same contact manifold $(\mathbb{R}^3,\ \eta = dx - y\,dz)$, and only the associated metric g defines the different examples.

(1) Consider on M an arbitrary smooth function $f = f(y, z)$ of variables y, z. The tensor fields (η, ξ, ϕ, g), where

$$\xi = \frac{\partial}{\partial x},$$
$$g = (g_{ij}) : g_{11} = g_{22} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -y,$$
$$g_{23} = g_{32} = \frac{1}{2}(\rho x - f), \quad g_{33} = y^2 + \frac{1 + (\rho x - f)^2}{4}, \quad \rho = \text{constant} > 0$$

and

$$\phi = (\phi_{ij}) : \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2y, \quad \phi_{13} = y(\rho x - f), \quad \phi_{22} = f - \rho x,$$
$$\phi_{23} = -\frac{1 + (\rho x - f)^2}{2}, \quad \phi_{32} = 2, \quad \phi_{33} = \rho x - f,$$

define a contact metric structure on M. Moreover, $M(\eta, \xi, \phi, g)$ is generally a non(κ, μ)-contact manifold, Jacobi (κ, μ)-contact manifold with $\kappa = \mathrm{Tr}\, l/2 = 1 - \rho^2/4 < 1$ and $\mu = 2 - \rho < 2$. In particular, if we choose $f = f(z)$, then $M(\eta, \xi, \phi, g)$ is a (κ, μ)-contact manifold.

(2) Consider on M an arbitrary smooth function $f = f(y, z)$ of variables y, z. The tensor fields (η, ξ, ϕ, g), where

$$\xi = \frac{\partial}{\partial x},$$
$$g = (g_{ij}) : g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -y, \quad g_{22} = e^{\rho x}$$
$$g_{23} = g_{32} = -\frac{1}{2} f e^{\rho x}, \quad g_{33} = y^2 + \frac{1 + f^2 e^{2\rho x}}{4e^{\rho x}}, \quad \rho = \text{constant} > 0$$

and

$$\phi = (\phi_{ij}) : \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2ye^{\rho x}, \quad \phi_{13} = -yfe^{\rho x},$$
$$\phi_{22} = fe^{\rho x}, \quad \phi_{32} = 2e^{\rho x}, \quad \phi_{23} = -\frac{1 + f^2 e^{2\rho x}}{2e^{\rho x}}, \quad \phi_{33} = -fe^{\rho x},$$

define a contact metric structure on M. Moreover, $M(\eta, \xi, \phi, g)$ is generally a non(κ, μ)-contact manifold, Jacobi (κ, μ)-contact manifold with $\kappa = \mathrm{Tr}\, l/2 = 1 - \rho^2/4$ and $\mu = 2$.

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In particular, M is a (κ,μ)-contact manifold if we choose $f(y,z) = -\frac{1}{2}\rho y^2 + d(z)$, where $d(z)$ is a smooth function of z.

(3) Consider on M an arbitrary smooth function $f = f(y,z)$ of variables y, z. The tensor fields (η, ξ, ϕ, g), where

$$\xi = \frac{\partial}{\partial x},$$
$$g = (g_{ij}) : g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -y, \quad g_{23} = g_{32} = -\tfrac{1}{2}(f + \rho x),$$
$$g_{22} = 1, \quad g_{33} = y^2 + \frac{1 + (f + \rho x)^2}{4}, \quad \rho = \text{constant} > 0$$

and

$$\phi = (\phi_{ij}) : \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2y, \quad \phi_{13} = -y(f + \rho x),$$
$$\phi_{22} = f + \rho x, \quad \phi_{32} = 2, \quad \phi_{23} = -\frac{1 + (f + \rho x)^2}{2}, \quad \phi_{33} = -(f + \rho x)$$

define a contact metric structure on M. Moreover, $M(\eta, \xi, \phi, g)$ is generally a non(κ,μ)-contact manifold, Jacobi (κ,μ)-contact manifold with $\kappa = \mathrm{Tr}\, l/2 = 1 - \rho^2/4$ and $\mu = \rho + 2 > 2$. In particular, M is a (κ,μ)-contact manifold if we choose $f = f(z)$.

(4) The tensor fields (η, ξ, ϕ, g), where

$$\xi = \frac{\partial}{\partial x},$$
$$g = (g_{ij}) : g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -y, \quad g_{23} = g_{32} = \frac{\rho x}{2},$$
$$g_{22} = \rho^2 x^2 + 1, \quad g_{33} = y^2 + \tfrac{1}{4}, \quad \rho = \text{constant} > 0$$

and

$$\phi = (\phi_{ij}) : \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2y(\rho^2 x^2 + 1), \quad \phi_{13} = \rho x y,$$
$$\phi_{22} = -\rho x, \quad \phi_{32} = 2(\rho^2 x^2 + 1), \quad \phi_{23} = -\tfrac{1}{2}, \quad \phi_{33} = \rho x$$

define a contact metric structure on M. Moreover, $M(\eta, \xi, \phi, g)$ is a non(κ,μ)-contact manifold, Jacobi (κ,μ)-contact manifold, with $\kappa = \mathrm{Tr}\, l/2 = 1 - \rho^2/4$ and $\mu = \rho + 2 > 2$.

(5) In the open subset $U = \{(x,y,z) \epsilon \mathbb{R}^3 : 0 < y < \pi\}$ of M, the tensor fields (η, ξ, ϕ, g), where

$$\xi = \frac{\partial}{\partial x},$$
$$g = (g_{ij}) : g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -y, \quad g_{22} = e^{\rho x \sin y}$$
$$g_{23} = g_{32} = \frac{1}{2} \cot y, \quad g_{33} = y^2 + \frac{1 + \cot^2 y}{4 e^{\rho x \sin y}}, \quad \rho = \text{constant} > 0$$

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and

$$\phi = (\phi_{ij}) : \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2ye^{\rho x \sin y}, \quad \phi_{13} = y \cot y,$$
$$\phi_{22} = -\cot y, \quad \phi_{23} = -\frac{1 + \cot^2 y}{2e^{\rho(\sin y)x}}, \quad \phi_{32} = 2e^{\rho x \sin y}, \quad \phi_{33} = \cot y$$

define a contact metric structure. Moreover, $U(\eta, \xi, \phi, g)$ is a non-Jacobi (κ, μ)-manifold, contact metric manifold with $\mathrm{Tr}\, l = 2(1 - \rho^2/4) =$ constant and $\mu = 2 + \rho \cos y$ is the nonconstant smooth function of Proposition 3.1. This follows a comparison of the Lie brackets $[\xi, e]$ of Lemma 3.2 and Theorem 4.4 and using $\mu = -2A$, $\lambda = \rho/2$.

(6) Consider on M the function $F = \int e^{\rho \cos x} \cos x \, dx$, $\rho =$ constant > 0. The tensor fields (η, ξ, ϕ, g), where

$$\xi = \frac{\partial}{\partial x},$$
$$g = (g_{ij}) : g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{13} = g_{31} = -y, \quad g_{22} = e^{-\rho \cos x}$$
$$g_{23} = g_{32} = -\frac{F}{2}\rho e^{-\rho \cos x}, \quad g_{33} = y^2 + \frac{1 + \rho^2 F^2 e^{-2\rho \cos x}}{4e^{-\rho \cos x}}$$

and

$$\phi = (\phi_{ij}) : \phi_{11} = \phi_{21} = \phi_{31} = 0, \quad \phi_{12} = 2ye^{-\rho \cos x}, \quad \phi_{13} = -\rho y e^{-\rho \cos x} F,$$
$$\phi_{22} = \rho e^{-\rho \cos x} F, \quad \phi_{23} = -\frac{1 + \rho^2 F^2 e^{-2\rho \cos x}}{2e^{-\rho \cos x}}, \quad \phi_{32} = 2e^{-\rho \cos x},$$
$$\phi_{33} = -\rho e^{-\rho \cos x} F,$$

define a contact metric structure on M. Moreover, $M(\eta, \xi, \phi, g)$ is a non-Jacobi (κ, μ)-manifold, contact metric manifold with $\mathrm{Tr}\, l = 2(1 - \rho^2/4) =$ constant and $\mu = 1 + \rho \cos x$ is the nonconstant smooth function of Proposition 3.1. This follows a comparison of the Lie brackets $[\xi, e]$ of Lemma 3.2 and Theorem 4.4 and using $\mu = -2A$, $\lambda = \rho/2$.

The claims of examples 1–6 could follow from Theorems 4.2, 4.5 and 4.6 by properly choosing the functions $t = t(x, y, z)$, $c_1 = c_1(y, z)$ and $c_2 = c_2(y, z)$. Specifically, examples 1, 2 and 3 follow choosing $t = \pi, \pi/2, 0$, respectively, and $c_1 = f(y, z)$, $c_2 = 2$. Example 4 follows choosing $t = 2 \cot^{-1} \rho x$, $c_1 = 0$, $c_2 = 2$. Examples 5 and 6 follow choosing $t = y$, $t = x$, respectively, and $c_1 = 0$, $c_2 = 2$.

From the above examples, it follows that the class of contact metric manifolds with $\mathrm{Tr}\, l =$ constant is a proper generalization of classes of (κ, μ)-contact manifolds and Jacobi (κ, μ)-contact manifolds. In particular, the following diagram is valid.

(κ, μ)-contact manifolds	$\subset$	Jacobi (κ, μ)-contact manifolds	$\subset$	contact metric manifolds with $\mathrm{Tr}\, l =$ constant.

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3. Global results

As we have seen, the (κ,μ)-contact manifolds are characterized by the relation (2.4), where $\kappa = \operatorname{Tr} l/2$ and μ are constant. In the next proposition, an expression of $R(X, Y)\xi$ is given for an arbitrary three-dimensional c.m.m. with $\operatorname{Tr} l =$ constant.

PROPOSITION 3.1. *Let $M(\eta, \xi, \phi, g)$ be a three-dimensional c.m.m. with* $\operatorname{Tr} l =$ constant.

(i) *If $\operatorname{Tr} l = 2$, then M is a Sasakian manifold and so $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$.*

(ii) *If $\operatorname{Tr} l \neq 2$, then*

$$R(X, Y)\xi = g(X, \phi Y)\phi Q\xi + \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for any $X, Y \epsilon \mathcal{X}(M)$, where $\kappa = \operatorname{Tr} l/2$, and

$$\mu = -\frac{1}{2}\Big(r - 2\kappa - \frac{1}{1-\kappa}\operatorname{div}\phi hQ\xi\Big)$$

is a smooth function, not necessarily constant (compare with examples 5 and 6) where div *denotes the divergence.*

To prove Proposition 3.1, we will need the following lemma [3, 8].

LEMMA 3.2. *Let $M(\eta, \xi, \phi, g)$ be a three-dimensional c.m.m. and let U be the open set of M, where $h \neq 0$. Then, for any point $P \epsilon U$, there exists a smooth local orthonormal basis $\{\xi, e, \phi e\}$, such that $he = \lambda e$, $h\phi e = -\lambda\phi e$, where λ is a nonvanishing smooth function. Therefore, in U,*

$$\left.\begin{aligned}
&\nabla_e\xi = -(1+\lambda)\phi e, \quad \nabla_{\phi e}\xi = (1-\lambda)e, \quad \nabla_\xi\xi = 0,\\
&\nabla_\xi e = A\phi e, \quad \nabla_e e = B\phi e, \quad \nabla_{\phi e}\phi e = Ce, \quad \nabla_\xi\phi e = -Ae,\\
&\nabla_{\phi e}e = -C\phi e + (\lambda - 1)\xi, \quad \nabla_e\phi e = -Be + (1+\lambda)\xi,\\
&[\xi, e] = (A + \lambda + 1)\phi e, \quad [\xi, \phi e] = -(A - \lambda + 1)e,\\
&[e, \phi e] = -Be + C\phi e + 2\xi,
\end{aligned}\right\} \tag{3.1}$$

where A, B, C are smooth functions on U. Moreover,

$$\left.\begin{aligned}
&R(e, \phi e)\xi = (2\lambda C - e\lambda)e + (\phi e\lambda - 2\lambda B)\phi e,\\
&R(e, \xi)\xi = (1 - \lambda^2 - 2\lambda A)e, \quad R(\phi e, \xi)\xi = (1 - \lambda^2 + 2\lambda A)\phi e,
\end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned}
&Qe = \Big(\frac{r}{2} + \lambda^2 - 1 - 2\lambda A\Big)e + (\xi\lambda)\phi e + (2\lambda B - \phi e\lambda)\xi,\\
&Q\phi e = (\xi\lambda)e + \Big(\frac{r}{2} + \lambda^2 - 1 + 2\lambda A\Big)\phi e + (2\lambda C - e\lambda)\xi,\\
&Q\xi = (2\lambda B - \phi e\lambda)e + (2\lambda C - e\lambda)\phi e + (\operatorname{Tr} l)\xi,
\end{aligned}\right\} \tag{3.3}$$

$$r = 2(eC + \phi eB - B^2 - C^2 + 2A + 1 - \lambda^2), \tag{3.4}$$

$$B = -\operatorname{div}\phi e, \quad C = -\operatorname{div} e, \tag{3.5}$$

$$\left.\begin{aligned}
&\xi B = -C(A - \lambda + 1) + e(A - \lambda),\\
&\xi C = B(A + \lambda + 1) - \phi e(A + \lambda).
\end{aligned}\right\} \tag{3.6}$$

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REMARK 3.3. When $\mathrm{Tr}\, l = \text{constant} \neq 2$, then, from the relation $h^2 = (\mathrm{Tr}\, l/2 - 1)\phi^2$, which is valid on any three-dimensional c.m.m., we have $h \neq 0$ (that is, $\lambda \neq 0$) in any point of the manifold and so Lemma 3.2 is applied around any point of the manifold. We suppose that $\lambda > 0$.

PROOF OF PROPOSITION 3.1. (i) If $\mathrm{Tr}\, l = 2$, then $h = 0$ and so M is a Sasakian manifold.

(ii) If $\mathrm{Tr}\, l \neq 2$, let

$$X = p_1 e + p_2 \phi e + \eta(X)\xi \quad \text{and} \quad Y = \mu_1 e + \mu_2 \phi e + \eta(Y)\xi \quad \text{for all } X, Y \epsilon \mathcal{X}(M),$$

where p_i, μ_i are smooth functions on the manifold. Using the basic properties of the curvature tensor and (3.2), we calculate

$$\begin{aligned} R(X,Y)\xi &= (p_1\mu_2 - p_2\mu_1)R(e,\phi e)\xi + (p_1\eta(Y) - \mu_1\eta(X))R(e,\xi)\xi \\ &\quad + (p_2\eta(Y) - \mu_2\eta(X))R(\phi e,\xi)\xi \\ &= (p_1\mu_2 - p_2\mu_1)((2\lambda C)e - (2\lambda B)\phi e) \\ &\quad + (p_1\eta(Y) - \mu_1\eta(X))(1 - \lambda^2 - 2\lambda A)e \\ &\quad + (p_2\eta(Y) - \mu_2\eta(X))(1 - \lambda^2 + 2\lambda A)\phi e \\ &= (p_1\mu_2 - p_2\mu_1)((2\lambda C)e - (2\lambda B)\phi e) \\ &\quad + (1 - \lambda^2)\{\eta(Y)(p_1 e + p_2\phi e) - \eta(X)(\mu_1 e + \mu_2 \phi e)\} \\ &\quad - 2\lambda A\{\eta(Y)(p_1 e - p_2 \phi e) - \eta(X)(\mu_1 e - \mu_2\phi e)\}. \end{aligned}$$

Now, using the relations $\phi\xi = 0$, $h\xi = 0$, $he = \lambda e$, $h\phi e = -\lambda\phi e$, $\phi^2 e = -e$ and (3.3), we find

$$\begin{aligned} g(X,\phi Y) &= g(p_1 e + p_2\phi e + \eta(X)\xi, \mu_1\phi e - \mu_2 e) = -(p_1\mu_2 - p_2\mu_1) \\ \phi Q\xi &= \phi((2\lambda B)e + (2\lambda C)\phi e + (\mathrm{Tr}\, l)\xi) = -((2\lambda C)e - (2\lambda B)\phi e) \\ hX &= \lambda(p_1 e - p_2\phi e), \quad hY = \lambda(\mu_1 e - \mu_2\phi e). \end{aligned}$$

Substituting the above and $\mathrm{Tr}\, l = 2(1 - \lambda^2) = 2\kappa$ in $R(X,Y)\xi$ gives

$$\begin{aligned} R(X,Y)\xi &= g(X,\phi Y)\phi Q\xi \\ &\quad + (1 - \lambda^2)\{\eta(Y)(X - \eta(X)\xi) - \eta(X)(Y - \eta(Y)\xi)\} \\ &\quad - 2\lambda A\Big\{\eta(Y)\frac{1}{\lambda}hX - \eta(X)\frac{1}{\lambda}hY\Big\} \\ &= g(X,\phi Y)\phi Q\xi + \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \end{aligned}$$

where $\mu = -2A$.

Moreover, using $\phi h = -h\phi$, (3.3), (3.5), (3.4) and $\lambda^2 = 1 - \mathrm{Tr}\, l/2 = 1 - \kappa$, we calculate

$$\begin{aligned} \phi hQ\xi &= -h\phi Q\xi = -h\{-(2\lambda C)e + (2\lambda B)\phi e\} = 2\lambda^2(Ce + B\phi e) \\ \mathrm{div}\,\phi hQ\xi &= 2\lambda^2 \mathrm{div}(Ce + B\phi e) = 2\lambda^2(C\mathrm{div}\, e + eC + B\mathrm{div}\,\phi e + \phi eB) \\ &= 2\lambda^2(-C^2 + eC - B^2 + \phi eB) = 2\lambda^2\Big(\frac{r}{2} - 2A - (1 - \lambda^2)\Big). \end{aligned}$$

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So $(1/2\lambda^2)\mathrm{div}\,\phi hQ\xi = r/2 + \mu - \kappa$ and thus

$$\mu = -\frac{1}{2}\Big(r - 2\kappa - \frac{1}{1-\kappa}\mathrm{div}\,\phi hQ\xi\Big).$$

This completes the proof of Proposition 3.1. □

An immediate consequence of Proposition 3.1 and the definition of a Jacobi (κ, μ)-contact manifold (see (2.5)) is the following corollary.

COROLLARY 3.4. *A three-dimensional c.m.m. $M(\eta, \xi, \phi, g)$ with $\mathrm{Tr}\, l = \text{constant} \neq 2$ is a Jacobi (κ, μ)-contact manifold if and only if the function $r - (1/(1-\kappa))\mathrm{div}\,\phi hQ\xi$ is constant. In this case, $\kappa = \mathrm{Tr}\, l/2$ and $\mu = -\frac{1}{2}(r - 2\kappa - (1/(1-\kappa))\mathrm{div}\,\phi hQ\xi)$.*

Another immediate consequence of Proposition 3.1 and of the divergence theorem is the following theorem.

THEOREM 3.5. *Let $M(\eta, \xi, \phi, g)$ be a three-dimensional closed (compact without boundary) c.m.m. with $\mathrm{Tr}\, l = \text{constant} \neq 2$. Then*

$$\int_M (r - 2(\kappa - \mu))\, dM = 0.$$

We recall that on any three-dimensional, non-Sasakian, (κ, μ)-contact manifold, $r = 2(\kappa - \mu)$ [2] is valid.

Now, in order to prove the next theorem, recall that on any three-dimensional Riemannian manifold the well-known formula

$$\sum_i (\nabla_{e_i} Q)e_i = \tfrac{1}{2}\,\mathrm{grad}\, r \tag{3.7}$$

is valid, where e_i, $i = 1, 2, 3$ is a local orthonormal frame.

THEOREM 3.6. *On any three-dimensional c.m.m. with $\mathrm{Tr}\, l = \text{constant} \neq 2$, the following formula is valid.*

$$\mathrm{div}\, Q\xi = \xi\frac{r}{2}.$$

In particular, if M is closed, then

$$\int_M (\xi r)\, dM = 0.$$

PROOF. Using the relations (3.1), (3.3), (3.5) and (3.6) of Lemma 3.2, we calculate

$$\begin{aligned}(\nabla_e Q)e &= \nabla_e Qe - Q\nabla_e e = \nabla_e\Big\{\Big(\frac{r}{2} + \lambda^2 - 1 - 2\lambda A\Big)e + (2\lambda B)\xi\Big\} - Q(B\phi e)\\
&= \Big(e\frac{r}{2} - 2\lambda eA\Big)e + \Big(\frac{r}{2} + \lambda^2 - 1 - 2\lambda A\Big)B\phi e + 2\lambda(eB)\xi\\
&\quad - 2\lambda B(1+\lambda)\phi e - B\Big\{\Big(\frac{r}{2} + \lambda^2 - 1 + 2\lambda A\Big)\phi e + (2\lambda C)\xi\Big\}\\
&= \Big(e\frac{r}{2} - 2\lambda eA\Big)e - \{4\lambda AB + 2\lambda(1+\lambda)B\}\phi e + 2\lambda(eB - BC)\xi,\end{aligned}$$

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$$
\begin{aligned}
(\nabla_{\phi e}Q)\phi e &= \nabla_{\phi e}Q\phi e - Q\nabla_{\phi e}\phi e \\
&= \nabla_{\phi e}\Big\{\Big(\frac{r}{2} + \lambda^2 - 1 + 2\lambda A\Big)\phi e + (2\lambda C)\xi\Big\} - CQe \\
&= \Big(\phi e\frac{r}{2} + 2\lambda\phi eA\Big)\phi e + \Big(\frac{r}{2} + \lambda^2 - 1 + 2\lambda A\Big)C\phi e + (2\lambda\phi eC)\xi \\
&\qquad + 2\lambda C(1-\lambda)e - C\Big\{\Big(\frac{r}{2} + \lambda^2 - 1 - 2\lambda A\Big)e + (2\lambda B)\xi\Big\} \\
&= (4\lambda AC + 2\lambda(1-\lambda)C)e + \Big(\phi e\frac{r}{2} + 2\lambda\phi eA\Big)\phi e \\
&\qquad + (2\lambda\phi eC - 2\lambda BC)\xi, \\
(\nabla_\xi Q)\xi &= \nabla_\xi Q\xi - Q\nabla_\xi\xi = \nabla_\xi\{(2\lambda B)e + (2\lambda C)\phi e + (\mathrm{Tr}\, l)\xi\} \\
&= 2\lambda(\xi B)e + 2\lambda AB\phi e + 2\lambda(\xi C)\phi e - (2\lambda AC)e \\
&= 2\lambda\{-C(A-\lambda+1) + eA - AC\}e \\
&\qquad + 2\lambda\{AB + B(A+\lambda+1) - \phi eA\}\phi e.
\end{aligned}
$$

From the above,

$$
(\nabla_e Q)e + (\nabla_{\phi e}Q)\phi e + (\nabla_\xi Q)\xi = \Big(e\frac{r}{2}\Big)e + \Big(\phi e\frac{r}{2}\Big)\phi e + 2\lambda(eB + \phi eC - 2BC)\xi. \tag{3.8}
$$

On the other hand (3.7), for $e_1 = e$, $e_2 = \phi e$, $e_3 = \xi$, is written as

$$
(\nabla_e Q)e + (\nabla_{\phi e}Q)\phi e + (\nabla_\xi Q)\xi = \tfrac{1}{2}\{(er)e + (\phi er)\phi e + (\xi r)\xi\}. \tag{3.9}
$$

Comparing (3.8) and (3.9),

$$
\xi r = 4\lambda(eB + \phi eC - 2BC). \tag{3.10}
$$

Also

$$
\begin{aligned}
\mathrm{div}\, Q\xi &= \mathrm{div}\{(2\lambda B)e + (2\lambda C)\phi e + (\mathrm{Tr}\, l)\xi\} \\
&= (2\lambda B)\mathrm{div}\, e + e(2\lambda B) + (2\lambda C)\mathrm{div}\,\phi e + \phi e(2\lambda C) + (\mathrm{Tr}\, l)\mathrm{div}\,\xi \\
&= -2\lambda BC + 2\lambda eB - 2\lambda BC + 2\lambda\phi eC \\
&= 2\lambda(eB + \phi eC - 2BC).
\end{aligned} \tag{3.11}
$$

From (3.10) and (3.11),

$$
2\mathrm{div}\, Q\xi = \xi r.
$$

Moreover, if M is closed, then

$$
\int_M (\xi r)\, dM = 2\int_M (\mathrm{div}\, Q\xi)\, dM = 0.
$$

This completes the proof of Theorem 3.6. □

Next, we provide two cases when a three-dimensional c.m.m. with $\mathrm{tr}\, l = \text{constant} \neq 2$ reduces to a (κ, μ)-contact manifold.

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Proposition 3.7. *A three-dimensional c.m.m.* $M(\eta, \xi, \phi, g)$ *with* $\operatorname{Tr} l = \text{constant} \neq 2$ *is an H-contact manifold if and only if* $M(\eta, \xi, \phi, g)$ *is a* (κ, μ)*-contact manifold. In particular, M is locally isometric to one of the following unimodular Lie groups* SU(2), SL(2, *R*), *E*(2), *E*(1, 1) *equipped with a left invariant metric.*

Proof. If M is an H-contact manifold, then $\phi Q\xi = 0$ and so, from Proposition 3.1,

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY),$$

where $\kappa = \operatorname{Tr} l/2 = \text{constant} \neq 2$ and μ is a function. This means that M is a generalized (κ, μ)-contact manifold. Therefore, from [10, Theorem 3.6], we have that the function μ is constant and so M is a (κ, μ)-contact manifold. The inverse is an immediate consequence of Proposition 3.1. For the rest of the proof, see [2, Theorem 3].

We note that Proposition 3.7 extends [5, Proposition 1.3]. □

Proposition 3.8. *If the Ricci operator Q of a three-dimensional c.m.m. with* $\operatorname{Tr} l = \text{constant} \neq 2$ *is parallel* ($\nabla Q = 0$)*, then M is flat, that is, a* $(0, 0)$*-contact manifold.*

Proof. At first from (3.7) we get $r = \text{constant}$. Moreover, using the formulas (3.1), (3.3) and (3.6) of Lemma 3.2, we calculate

$$\begin{aligned} 0 &= (\nabla_\xi Q)e = \nabla_\xi Qe - Q\nabla_\xi e \\ &= \nabla_\xi\Big\{\Big(\frac{r}{2} + \lambda^2 - 1 - 2\lambda A\Big)e + (2\lambda B)\xi\Big\} - Q(A\phi e) \\ &= -2\lambda(\xi A)e + \Big(\frac{r}{2} + \lambda^2 - 1 - 2\lambda A\Big)A\phi e + 2\lambda(\xi B)\xi \\ &\qquad - A\Big\{\Big(\frac{r}{2} + \lambda^2 - 1 + 2\lambda A\Big)\phi e + (2\lambda C)\xi\Big\} \\ &= -2\lambda(\xi A)e - 4\lambda A^2\phi e + 2\lambda\{-C(2A - \lambda + 1) + eA\}\xi. \end{aligned}$$

Thus

$$A = 0 \quad \text{and} \quad (1 - \lambda)C = 0. \tag{3.12}$$

Following this method and using (3.12) we get, from $(\nabla_{\phi e}Q)e = 0$ and $(\nabla_e Q)\phi e = 0$, the following relations.

$$\begin{aligned} &(1 - \lambda)B = 0, \quad (1 + \lambda)B = 0, \quad (1 + \lambda)C = 0 \\ &(\lambda - 1)\Big(\frac{r}{2} + \lambda^2 - 1\Big) + 2\lambda\phi eB + 2\lambda C^2 - 2(\lambda - 1)(1 - \lambda^2) = 0, \\ &(\lambda + 1)\Big(\frac{r}{2} + \lambda^2 - 1\Big) + 2\lambda eC + 2\lambda B^2 - 2(\lambda + 1)(1 - \lambda^2) = 0. \end{aligned} \tag{3.13}$$

So, from (3.12), (3.13), (3.4) and $\operatorname{Tr} l = 2(1 - \lambda^2)$, we finally find $Q = 0$, and from the well-known formula

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)QX - g(X, Z)QY + g(QY, Z)Y - g(QX, Z)Y \\ &\qquad - \frac{r}{2}\{g(Y, Z)X - g(X, Z)Y\}, \end{aligned}$$

which is valid on any three-dimensional Riemannian manifold, we have $R = 0$. This implies that the manifold is flat. □

4. The local description

In order to describe locally the three-dimensional contact metric manifolds (c.m.m.) with $\|\tau\|$ = constant (equivalently, $\mathrm{Tr}\, l$ = constant), we will use the following classical theorem of Darboux [1, page 24] for the 3-dimensional case.

THEOREM 4.1. *For each point P of a three-dimensional contact manifold (M, η) there exist local coordinates $(U, (x, y, z))$, $P \epsilon U$, such that*

$$\eta = dx - y\, dz. \tag{4.1}$$

Now, let $M(\eta, \xi, \phi, g)$ be a three-dimensional c.m.m. Our initial goal is to describe (η, ξ, ϕ, g) in this Darboux coordinate system.

We have $\xi = \partial/\partial x$ and, from (2.1), $\phi(\partial/\partial x) = 0$. Let

$$\phi \frac{\partial}{\partial y} = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}, \tag{4.2}$$

where a, b, c are smooth functions on U. From (4.1),

$$\eta\Big(\frac{\partial}{\partial y}\Big) = 0, \quad \eta\Big(\frac{\partial}{\partial z}\Big) = -y. \tag{4.3}$$

From (2.1), (4.2) and (4.3), it follows that

$$c\phi \frac{\partial}{\partial z} = -ab \frac{\partial}{\partial x} - (1 + b^2) \frac{\partial}{\partial y} - bc \frac{\partial}{\partial z} \tag{4.4}$$

and $a = cy$. So (4.2) is written as

$$\phi \frac{\partial}{\partial y} = cy \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z}. \tag{4.5}$$

From (4.5), it immediately follows that $c \neq 0$ everywhere on U and so (4.4) is written as

$$\phi \frac{\partial}{\partial z} = -by \frac{\partial}{\partial x} - \frac{1 + b^2}{c} \frac{\partial}{\partial y} - b \frac{\partial}{\partial z}. \tag{4.6}$$

Consequently, the matrix of the components of ϕ in this system is given by

$$\phi = \begin{pmatrix} 0 & yc & -yb \\ 0 & b & -\dfrac{1 + b^2}{c} \\ 0 & c & -b \end{pmatrix}. \tag{4.7}$$

Now, for the calculation of the metric tensor g, using (2.1), (2.2), (4.3), (4.5), (4.6) and $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])$ (see [1, page 69]), we finally get

$$g_{11} = g\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big) = 1, \quad g_{12} = g_{21} = g\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big) = 0,$$
$$g_{13} = g_{31} = g\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\Big) = -y$$

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and

$$bg_{22} + cg_{23} = 0, \quad \frac{1+b^2}{c}g_{22} + bg_{23} = \frac{1}{2}, \quad bg_{23} + cg_{33} = \frac{1}{2} + cy^2.$$

From the last three equations, we have $g_{22} = c/2 > 0$, $g_{23} = g_{32} = -b/2$ and $g_{33} = y^2 + (1+b^2)/2c$.

So the matrix of components of g is

$$g = \begin{pmatrix} 1 & 0 & -y \\ 0 & \frac{c}{2} & -\frac{b}{2} \\ -y & -\frac{b}{2} & y^2 + \frac{1+b^2}{2c} \end{pmatrix} \quad \text{with } \det g = \frac{1}{4}. \tag{4.8}$$

We will now calculate, in the Darboux coordinates system, the tensor field $h = \frac{1}{2}\mathcal{L}_\xi \phi$. Using (4.5),

$$\begin{aligned} 2h\frac{\partial}{\partial y} &= (\mathcal{L}_\xi \phi)\frac{\partial}{\partial y} = \Big[\xi, \phi\frac{\partial}{\partial y}\Big] - \phi\Big[\xi, \frac{\partial}{\partial y}\Big] \\ &= \Big[\frac{\partial}{\partial x}, cy\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\Big] - \phi\Big[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big] = yc_x\frac{\partial}{\partial x} + b_x\frac{\partial}{\partial y} + c_x\frac{\partial}{\partial z}, \end{aligned}$$

where $A_x = \partial A/\partial x$. So

$$2h\frac{\partial}{\partial y} = yc_x\frac{\partial}{\partial x} + b_x\frac{\partial}{\partial y} + c_x\frac{\partial}{\partial z}. \tag{4.9}$$

Analogously, using (4.6),

$$2h\frac{\partial}{\partial z} = -yb_x\frac{\partial}{\partial x} - \Big(\frac{1+b^2}{c}\Big)_x\frac{\partial}{\partial y} - b_x\frac{\partial}{\partial z}. \tag{4.10}$$

Consequently, the matrix of h is

$$h = \begin{pmatrix} 0 & \frac{1}{2}yc_x & -\frac{1}{2}yb_x \\ 0 & \frac{1}{2}b_x & -\frac{1}{2}\Big(\frac{1+b^2}{c}\Big)_x \\ 0 & \frac{1}{2}c_x & -\frac{1}{2}b_x \end{pmatrix}. \tag{4.11}$$

From (4.9) and (4.10), it follows that $h = 0$ if and only if $b_x = c_x = 0$. So the metric g is Sasakian (that is, $\operatorname{Tr} l = 2$) if and only if the functions b and c are independent of x (see [1, page 230]). From now on we suppose that the three-dimensional c.m.m. $M(\eta, \xi, \phi, g)$ has $\operatorname{Tr} l = \text{constant} \neq 2$. From (4.11), we have that the eigenvalues of h satisfy the equation

$$0 = \begin{vmatrix} -\lambda & \frac{1}{2}yc_x & -\frac{1}{2}yb_x \\ 0 & \frac{1}{2}b_x - \lambda & -\frac{1}{2}\Big(\frac{1+b^2}{c}\Big)_x \\ 0 & \frac{1}{2}c_x & -\frac{1}{2}b_x - \lambda \end{vmatrix} = \lambda\Big\{\lambda^2 - \frac{b_x^2}{4} + \frac{1}{4}c_x\Big(\frac{1+b^2}{c}\Big)_x\Big\}.$$

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So, since $\lambda \neq 0$ (Remark 3.3), it follows that

$$4\lambda^2 = b_x^2 - c_x\Big(\frac{1+b^2}{c}\Big)_x = \frac{(cb_x - bc_x)^2 + c_x^2}{c^2}. \tag{4.12}$$

We note that at any point of the manifold is $(b_x, c_x) \neq (0,0)$.

Equation (4.12) is written as

$$\Big(b_x - b\frac{c_x}{c}\Big)^2 + \Big(\frac{c_x}{c}\Big)^2 = \rho^2, \quad \rho^2 = 4\lambda^2 \quad \text{for all } \rho > 0. \tag{4.13}$$

Putting $b_x - b(c_x/c) = \rho\cos t$ and $c_x/c = \rho\sin t$ for any smooth function $t = t(x,y,z)$, the differential equation (4.13) is reduced to the system of two differential equations given by

$$\Big\{b_x - b\frac{c_x}{c} - \rho\cos t = 0 \text{ and } c_x - \rho(\sin t)c = 0\Big\}. \tag{4.14}$$

The solutions of this system are

$$\begin{gathered} 0 < c = c(x,y,z) = c_2(y,z)e^{\rho\int(\sin t)\,dx}, \\ b = b(x,y,z) = e^{\rho\int(\sin t)\,dx}\Big\{c_1(y,z) + \rho\int e^{-\rho\int(\sin t)\,dx}(\cos t)\,dx\Big\}, \end{gathered} \tag{4.15}$$

where $c_1(y,z)$ and $c_2(y,z) > 0$ are arbitrary smooth functions of y and z.

So we have proved the following theorem.

Theorem 4.2. *Let $M(\eta,\xi,\phi,g)$ be a three-dimensional contact metric manifold. Then, around any point of M, there exist coordinates (x,y,z) so that the tensor fields η,ξ,ϕ,g and h are given by (4.1), $\xi = \partial/\partial x$, (4.7), (4.8) and (4.11), respectively, where $b = b(x,y,z)$ and $c = c(x,y,z) > 0$ are arbitrary smooth functions. In particular:*

(i) $\operatorname{Tr} l = 2$ *(that is, M is a Sasakian manifold) if and only if the functions b and c are independent of x; and*

(ii) $\operatorname{Tr} l = \text{constant} \neq 2$ *if and only if the functions b and c satisfy (4.15), where $c_1(y,z)$, $c_2(y,z) > 0$ and $t(x,y,z)$ are arbitrary smooth functions.*

The eigenvector of h when $\operatorname{Tr} l = \text{constant} \neq 2$. Let us suppose now that $X = \rho_1(\partial/\partial x) + \rho_2(\partial/\partial y) + \rho_3(\partial/\partial z)$ is a nonzero eigenvector of h with $hX = \lambda X$, $\lambda > 0$, where ρ_i, $i = 1,2,3$ are smooth functions. Then, using (4.9), (4.10) and $h(\partial/\partial x) = 0$,

$$\begin{aligned} hX &= \rho_1 h\frac{\partial}{\partial x} + \rho_2 h\frac{\partial}{\partial y} + \rho_3 h\frac{\partial}{\partial z} \\ &= \frac{1}{2}\rho_2\Big(yc_x\frac{\partial}{\partial x} + b_x\frac{\partial}{\partial y} + c_x\frac{\partial}{\partial z}\Big) + \frac{1}{2}\rho_3\Big(-yb_x\frac{\partial}{\partial x} - \Big(\frac{1+b^2}{c}\Big)_x\frac{\partial}{\partial y} - b_x\frac{\partial}{\partial z}\Big). \end{aligned}$$

So

$$2hX = y(\rho_2 c_x - \rho_3 b_x)\frac{\partial}{\partial x} + \Big(\rho_2 b_x - \rho_3\Big(\frac{1+b^2}{c}\Big)_x\Big)\frac{\partial}{\partial y} + (\rho_2 c_x - \rho_3 b_x)\frac{\partial}{\partial z}.$$

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From this and from $2hX = 2\lambda(\rho_1(\partial/\partial x) + \rho_2(\partial/\partial y) + \rho_3(\partial/\partial z))$, we get the system

$$\Big\{2\lambda\rho_1 = y(\rho_2 c_x - \rho_3 b_x), 2\lambda\rho_2 = \rho_2 b_x - \rho_3\Big(\frac{1+b^2}{c}\Big)_x, 2\lambda\rho_3 = \rho_2 c_x - \rho_3 b_x\Big\}.$$

From the first and the third equations of the system, we get $\rho_1 = y\rho_3$ and so $X = y\rho_3(\partial/\partial x) + \rho_2(\partial/\partial y) + \rho_3(\partial/\partial z)$. Hence the above system is reduced to the homogeneous system

$$\Big\{(2\lambda - b_x)\rho_2 + \Big(\frac{1+b^2}{c}\Big)_x\rho_3 = 0, -c_x\rho_2 + (2\lambda + b_x)\rho_3 = 0\Big\} \tag{4.16}$$

with determinant $d = 0$. So, using (4.5) and (4.6), the eigenvectors of h are $\xi, X, \phi X$, where

$$\begin{gathered}\xi = \frac{\partial}{\partial x}, \quad X = y\rho_3\frac{\partial}{\partial x} + \rho_2\frac{\partial}{\partial y} + \rho_3\frac{\partial}{\partial z} \quad \text{and} \\ \phi X = y(\rho_2 c - \rho_3 b)\frac{\partial}{\partial x} + \Big(\rho_2 b - \frac{1+b^2}{c}\rho_3\Big)\frac{\partial}{\partial y} + (\rho_2 c - \rho_3 b)\frac{\partial}{\partial z},\end{gathered} \tag{4.17}$$

with eigenvalues $0, \lambda$ and $-\lambda$, respectively, where ρ_2 and ρ_3 are solutions of the system (4.16) and $(b_x, c_x) \neq (0,0)$ everywhere.

Special cases. In this paragraph, we will look for conditions that characterize the (κ,μ)-contact manifolds and the Jacobi (κ,μ)-contact manifolds as subclasses of the class of contact metric manifolds with $\mathrm{Tr}\, l = \text{constant} \neq 2$.

First, we state the following lemma, the proof of which immediately follows from relations (4.14).

LEMMA 4.3. *Let $M(\eta,\xi,\phi,g)$ be a three-dimensional contact metric manifold with* $\mathrm{Tr}\, l = \text{constant} \neq 2$ *($\Leftrightarrow \lambda = \rho/2 = \text{constant} > 0$). Then, at any point P of the manifold, there exists a neighborhood U of P so that at least one of the functions $b_x + \rho$ and $b_x - \rho$ does not vanish anywhere in U. Moreover:*

if $b_x + \rho \neq 0$, then $\cos t \neq -1$ everywhere in U; and
if $b_x - \rho \neq 0$, then $\cos t \neq 1$ everywhere in U.

Now, we will examine, separately, the cases $b_x + \rho \neq 0$ everywhere in U and $b_x - \rho \neq 0$ everywhere in U. In each case, we will find at each point of the manifold a local orthonormal frame $(\xi, e, \phi e)$ of eigenvectors of h. Next, we will compute the Lie brackets $[\xi, e]$, $[\xi, \phi e]$ and $[e, \phi e]$ in order to compare these with the corresponding ones of Lemma 3.2.

The case $b_x + \rho \neq 0$ everywhere in U. From the second equation of (4.16), we have $\rho_3 = (c_x/(b_x + \rho))\rho_2$. Substituting ρ_3 in (4.17) and using (4.14), (4.7) and Lemma 4.3, we calculate

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$$\begin{aligned}
X &= \frac{yc_x\rho_2}{b_x+\rho}\frac{\partial}{\partial x} + \rho_2\frac{\partial}{\partial y} + \frac{c_x\rho_2}{b_x+\rho}\frac{\partial}{\partial z}\\
&= \frac{\rho_2}{b_x+\rho}\Big\{yc\rho(\sin t)\frac{\partial}{\partial x} + (\rho + b\rho\sin t + \rho\cos t)\frac{\partial}{\partial y} + c\rho(\sin t)\frac{\partial}{\partial z}\Big\}\\
&= \frac{\rho_2\rho}{b_x+\rho}\Big\{(\sin t)\Big(yc\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\Big) + (1+\cos t)\frac{\partial}{\partial y}\Big\}.
\end{aligned}$$

Choosing $\rho_2 = (b_x+\rho)/\rho \neq 0$, we have the nonzero eigenvectors

$$X = (\sin t)\phi\frac{\partial}{\partial y} + (1+\cos t)\frac{\partial}{\partial y} \quad \text{and} \quad \phi X = -(\sin t)\frac{\partial}{\partial y} + (1+\cos t)\phi\frac{\partial}{\partial y}.$$

Moreover, using (2.3), (4.3) and (4.8),

$$\begin{aligned}
|X|^2 = |\phi X|^2 &= (\sin^2 t)g\Big(\phi\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial y}\Big) + (1+\cos t)^2 g\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big)\\
&= (\sin^2 t + 1 + \cos^2 t + 2\cos t)g\Big(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big)\\
&= c(1+\cos t).
\end{aligned}$$

Hence, the vector fields $(\xi, e, \phi e)$, where $\xi = \partial/\partial x$,

$$\begin{aligned}
e &= \sqrt{\frac{1}{c(1+\cos t)}}\Big((1+\cos t)\frac{\partial}{\partial y} + (\sin t)\phi\frac{\partial}{\partial y}\Big)\\
\phi e &= \sqrt{\frac{1}{c(1+\cos t)}}\Big(-(\sin t)\frac{\partial}{\partial y} + (1+\cos t)\phi\frac{\partial}{\partial y}\Big)
\end{aligned}$$

define, at any point P of U, a smooth local orthonormal frame, such that $h\xi = 0$, $he = \lambda e$, $h\phi e = -\lambda\phi e$ and $\lambda = \rho/2 > 0$. Putting $E = \sin t/\sqrt{c(1+\cos t)} \neq 0$, $F = (1+\cos t)/\sqrt{c(1+\cos t)} \neq 0$, we have $\xi = \partial/\partial x$, $e = F(\partial/\partial y) + E\phi(\partial/\partial y)$ and $\phi e = -E(\partial/\partial y) + F\phi(\partial/\partial y)$ with $\begin{vmatrix} F & E \\ -E & F\end{vmatrix} = F^2 + E^2 = 2/c \neq 0$. Now, using the above expressions of $\xi, e, \phi e$ and relations (4.7) and (4.14), we will calculate the Lie brackets $[\xi, e]$, $[\xi, \phi e]$ and $[e, \phi e]$.

$$\begin{aligned}
[\xi, e] &= \Big[\frac{\partial}{\partial x}, E\phi\frac{\partial}{\partial y} + F\frac{\partial}{\partial y}\Big] = E_x\phi\frac{\partial}{\partial y} + F_x\frac{\partial}{\partial y} + E\Big[\frac{\partial}{\partial x}, \phi\frac{\partial}{\partial y}\Big] + F\Big[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big]\\
&= E_x\phi\frac{\partial}{\partial y} + F_x\frac{\partial}{\partial y} + E\Big[\frac{\partial}{\partial x}, yc\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\Big]\\
&= E_x\phi\frac{\partial}{\partial y} + F_x\frac{\partial}{\partial y} + E\Big(yc_x\frac{\partial}{\partial x} + b_x\frac{\partial}{\partial y} + c_x\frac{\partial}{\partial z}\Big)\\
&= E_x\phi\frac{\partial}{\partial y} + F_x\frac{\partial}{\partial y} + E\frac{c_x}{c}\Big(yc\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} - b\frac{\partial}{\partial y}\Big) + Eb_x\frac{\partial}{\partial y}\\
&= E_x\phi\frac{\partial}{\partial y} + F_x\frac{\partial}{\partial y} + E\frac{c_x}{c}\phi\frac{\partial}{\partial y} - E\frac{c_x}{c}b\frac{\partial}{\partial y} + Eb_x\frac{\partial}{\partial y}
\end{aligned}$$

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$$= \left(E_x + E\frac{c_x}{c}\right)\phi\frac{\partial}{\partial y} + \left(F_x - E\frac{c_x b}{c} + Eb_x\right)\frac{\partial}{\partial y}$$
$$= \left(E_x + E\frac{c_x}{c}\right)\phi\frac{\partial}{\partial y} + \left(F_x + E\frac{cb_x - bc_x}{c}\right)\frac{\partial}{\partial y}. \qquad (*)$$

But

$$\begin{aligned}
E_x + E\frac{c_x}{c} &= \left(\frac{\sin t}{\sqrt{c(1+\cos t)}}\right)_x + \frac{\rho\sin^2 t}{\sqrt{c(1+\cos t)}} \\
&= \left\{t_x(\cos t)\sqrt{c(1+\cos t)} - \frac{1}{2}(\sin t)\frac{1}{\sqrt{c(1+\cos t)}}(c_x(1+\cos t)\right. \\
&\qquad \left. - c(\sin t)t_x)\right\}\frac{1}{c(1+\cos t)} + \frac{\rho\sin^2 t}{\sqrt{c(1+\cos t)}} \\
&= \left\{t_x(\cos t)\sqrt{c(1+\cos t)} - \frac{1}{2}\frac{\sin t}{\sqrt{c(1+\cos t)}}(\rho c(\sin t)(1+\cos t)\right. \\
&\qquad \left. - c(\sin t)t_x)\right\}\frac{1}{c(1+\cos t)} + \frac{\rho\sin^2 t}{\sqrt{c(1+\cos t)}} \\
&= \frac{1}{\sqrt{c(1+\cos t)}}\left\{\rho\sin^2 t + t_x(\cos t)\right. \\
&\qquad \left. - \frac{1}{2}\frac{\sin t}{c(1+\cos t)}(\rho c(\sin t)(1+\cos t) - c(\sin t)t_x)\right\} \\
&= \frac{1}{\sqrt{c(1+\cos t)}}\left\{\rho\sin^2 t + t_x(\cos t) - \frac{1}{2}\rho\sin^2 t + \frac{1}{2}\frac{\sin^2 t}{1+\cos t}t_x\right\} \\
&= \frac{1}{\sqrt{c(1+\cos t)}}\left\{\frac{1}{2}\rho\sin^2 t + t_x(\cos t) + \frac{1}{2}(1-\cos t)t_x\right\} \\
&= \frac{1}{\sqrt{c(1+\cos t)}}\left\{\frac{1}{2}\rho\sin^2 t + \frac{1}{2}(\cos t)t_x + \frac{1}{2}t_x\right\} \\
&= \frac{1}{2}\frac{1}{\sqrt{c(1+\cos t)}}\left\{\rho(1-\cos^2 t) + t_x(1+\cos t)\right\} \\
&= \frac{1+\cos t}{2\sqrt{c(1+\cos t)}}(\rho - \rho\cos t + t_x) = \frac{1}{2}F(\rho - \rho\cos t + t_x).
\end{aligned}$$

Also,

$$\begin{aligned}
F_x + E\frac{cb_x - bc_x}{c} &= F_x + E\rho\cos t = \left(\frac{1+\cos t}{\sqrt{c(1+\cos t)}}\right)_x + \rho\frac{\sin t\cos t}{\sqrt{c(1+\cos t)}} \\
&= \frac{1}{2}\sqrt{\frac{c}{1+\cos t}}\frac{-t_x(\sin t)c - (1+\cos t)c_x}{c^2} + \frac{\rho\sin t\cos t}{\sqrt{c(1+\cos t)}} \\
&= \frac{1}{2}\frac{\sin t}{\sqrt{c(1+\cos t)}}\{-t_x - (1+\cos t)\rho + 2\rho\cos t\} \\
&= -E\frac{\rho - \rho\cos t + t_x}{2}.
\end{aligned}$$

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Substituting the two last relations in (∗), we finally get

$$[\xi, e] = \tfrac{1}{2}(\rho - \rho\cos t + t_x)\phi e. \tag{4.18}$$

Comparing (4.18) with $[\xi, e] = (A + \lambda + 1)\phi e$ of (3.1), we have $A + \lambda + 1 = \frac{1}{2}(\rho - \rho\cos t + t_x)$, where $\lambda = \rho/2$, and so $A = -1 - (\rho/2)\cos t + \frac{1}{2}t_x$. Substituting A in $[\xi, \phi e] = -(A - \lambda + 1)$ of (3.1) gives

$$[\xi, \phi e] = \tfrac{1}{2}(\rho + \rho\cos t - t_x)e. \tag{4.19}$$

Now, we will compute $[e, \phi e]$, using the properties of the Lie bracket.

$$\begin{aligned}
[e, \phi e] &= \Big[E\phi\frac{\partial}{\partial y} + F\frac{\partial}{\partial y}, F\phi\frac{\partial}{\partial y} - E\frac{\partial}{\partial y}\Big] \\
&= \Big[E\phi\frac{\partial}{\partial y}, F\phi\frac{\partial}{\partial y}\Big] - \Big[E\phi\frac{\partial}{\partial y}, E\frac{\partial}{\partial y}\Big] + \Big[F\frac{\partial}{\partial y}, F\phi\frac{\partial}{\partial y}\Big] - \Big[F\frac{\partial}{\partial y}, E\frac{\partial}{\partial y}\Big] \\
&= EF\Big[\phi\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial y}\Big] + E\phi\frac{\partial}{\partial y}F\phi\frac{\partial}{\partial y} - F\phi\frac{\partial}{\partial y}E\phi\frac{\partial}{\partial y} \\
&\quad - \Big(E^2\Big[\phi\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big] + E\phi\frac{\partial}{\partial y}E\frac{\partial}{\partial y} - E\frac{\partial}{\partial y}E\phi\frac{\partial}{\partial y}\Big) \\
&\quad + \Big(F^2\Big[\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial y}\Big] + F\frac{\partial}{\partial y}F\phi\frac{\partial}{\partial y} - F\phi\frac{\partial}{\partial y}F\frac{\partial}{\partial y}\Big) \\
&\quad - \Big(EF\Big[\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big] + F\frac{\partial}{\partial y}E\frac{\partial}{\partial y} - E\frac{\partial}{\partial y}F\frac{\partial}{\partial y}\Big) \\
&= \Big(E\phi\frac{\partial}{\partial y}F - F\phi\frac{\partial}{\partial y}E\Big)\phi\frac{\partial}{\partial y} - \Big(F\frac{\partial}{\partial y}E - E\frac{\partial}{\partial y}F\Big)\frac{\partial}{\partial y} \\
&\quad - \Big(E\phi\frac{\partial}{\partial y}E\frac{\partial}{\partial y} - E\frac{\partial}{\partial y}E\phi\frac{\partial}{\partial y} + E^2\Big[\phi\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big]\Big) \\
&\quad + \Big(F\frac{\partial}{\partial y}F\phi\frac{\partial}{\partial y} - F\phi\frac{\partial}{\partial y}F\frac{\partial}{\partial y} + F^2\Big[\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial y}\Big]\Big) \\
&= \Big(E\phi\frac{\partial}{\partial y}F - F\phi\frac{\partial}{\partial y}E + E\frac{\partial}{\partial y}E + F\frac{\partial}{\partial y}F\Big)\phi\frac{\partial}{\partial y} \\
&\quad + \Big(-F\frac{\partial}{\partial y}E + E\frac{\partial}{\partial y}F - E\phi\frac{\partial}{\partial y}E - F\phi\frac{\partial}{\partial y}F\Big)\frac{\partial}{\partial y} \\
&\quad + (F^2 + E^2)\Big[\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial y}\Big] \\
&= \Big(-F^2\phi\frac{\partial}{\partial y}\frac{E}{F} + \frac{1}{2}\frac{\partial}{\partial y}(E^2 + F^2)\Big)\phi\frac{\partial}{\partial y} \\
&\quad + \Big(-F^2\frac{\partial}{\partial y}\frac{E}{F} - \frac{1}{2}\phi\frac{\partial}{\partial y}(E^2 + F^2)\Big)\frac{\partial}{\partial y} + (E^2 + F^2)\Big[\frac{\partial}{\partial y}, \phi\frac{\partial}{\partial y}\Big].
\end{aligned} \tag{$**$}$$

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But using (4.5), we compute

$$-F^2\phi\frac{\partial}{\partial y}\frac{E}{F}+\frac{1}{2}\frac{\partial}{\partial y}(E^2+F^2)=-\frac{1+\cos t}{c}\phi\frac{\partial}{\partial y}\frac{\sin t}{1+\cos t}+\frac{1}{2}\frac{\partial}{\partial y}\frac{2}{c}$$
$$=-\frac{1+\cos t}{c}\frac{1}{1+\cos t}\phi\frac{\partial}{\partial y}t+\frac{\partial}{\partial y}\frac{1}{c}$$
$$=-\frac{1}{c}\Big(yc\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial z}\Big)t-\frac{c_y}{c^2}$$
$$=-\frac{1}{c}(yct_x+bt_y+ct_z)-\frac{c_y}{c^2}.$$

Also,

$$-F^2\frac{\partial}{\partial y}\frac{E}{F}-\frac{1}{2}\phi\frac{\partial}{\partial y}(E^2+F^2)=-\frac{1+\cos t}{c}\frac{\partial}{\partial y}\frac{\sin t}{1+\cos t}-\frac{1}{2}\phi\frac{\partial}{\partial y}\frac{2}{c}$$
$$=-\frac{1+\cos t}{c}\frac{1}{1+\cos t}t_y-\Big(yc\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial z}\Big)\frac{1}{c}$$
$$=-\frac{1}{c}t_y+\frac{1}{c^2}(ycc_x+bc_y+cc_z)$$

and

$$(E^2+F^2)\Big[\frac{\partial}{\partial y},\phi\frac{\partial}{\partial y}\Big]=\frac{2}{c}\Big[\frac{\partial}{\partial y},yc\frac{\partial}{\partial x}+b\frac{\partial}{\partial y}+c\frac{\partial}{\partial z}\Big]$$
$$=\frac{2}{c}\Big\{c\frac{\partial}{\partial x}+yc_y\frac{\partial}{\partial x}+b_y\frac{\partial}{\partial y}+c_y\frac{1}{c}\Big(\phi\frac{\partial}{\partial y}-yc\frac{\partial}{\partial x}-b\frac{\partial}{\partial y}\Big)\Big\}$$
$$=2\frac{\partial}{\partial x}+\frac{2}{c}\Big(b_y-\frac{bc_y}{c}\Big)\frac{\partial}{\partial y}+\frac{2}{c^2}c_y\phi\frac{\partial}{\partial y}$$
$$=2\xi+\frac{2}{c^2}(cb_y-bc_y)\frac{\partial}{\partial y}+\frac{2c_y}{c^2}\phi\frac{\partial}{\partial y}.$$

Substituting the three last relations in (**) gives

$$[e,\phi e]=\Big\{-\frac{1}{c}(yct_x+bt_y+ct_z)-\frac{c_y}{c^2}\Big\}\phi\frac{\partial}{\partial y}+\Big\{-\frac{1}{c}t_y+\frac{1}{c^2}(ycc_x+bc_y+cc_z)\Big\}\frac{\partial}{\partial y}$$
$$+2\xi+\frac{2}{c^2}(cb_y-bc_y)\frac{\partial}{\partial y}+\frac{2c_y}{c^2}\phi\frac{\partial}{\partial y}$$
$$=-\frac{1}{c^2}\{-c_y+c^2(yt_x+t_z)+bct_y\}\phi\frac{\partial}{\partial y}$$
$$+\frac{1}{c^2}\{-bc_y+c(-t_y+yc_x+c_z+2b_y)\}\frac{\partial}{\partial y}+2\xi.$$

So

$$[e,\phi e]=2\xi-\frac{1}{c^2}\{-c_y+c^2(yt_x+t_z)+bct_y\}\phi\frac{\partial}{\partial y}$$
$$+\frac{1}{c^2}\{-bc_y+c(-t_y+yc_x+c_z+2b_y)\}\frac{\partial}{\partial y}. \tag{4.20}$$

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The case $b_x - \rho \neq 0$ everywhere in U. From the first of (4.16), we have $\rho_2 = ((((1+b^2)/c)_x)/(b_x - \rho))\rho_3$. Substituting ρ_2 in (4.17) and using (4.14), (4.7) and Lemma 4.3, we finally get

$$X = \frac{\rho_3}{c(1-\cos t)}\Big((\sin t)\frac{\partial}{\partial y} + (1-\cos t)\phi\frac{\partial}{\partial y}\Big).$$

Choosing $\rho_3 = c(1-\cos t) \neq 0$, we have the nonzero eigenvectors of h

$$X = (\sin t)\frac{\partial}{\partial y} + (1-\cos t)\phi\frac{\partial}{\partial y} \quad \text{and} \quad \phi X = -(1-\cos t)\frac{\partial}{\partial y} + (\sin t)\phi\frac{\partial}{\partial y}.$$

Moreover, using (2.3), (4.3) and (4.8),

$$|X| = |\phi X| = \sqrt{c(1-\cos t)}.$$

Hence, the vector fields $\xi = \partial/\partial x$,

$$e = \frac{1}{\sqrt{c(1-\cos t)}}\Big((\sin t)\frac{\partial}{\partial y} + (1-\cos t)\phi\frac{\partial}{\partial y}\Big)$$

and

$$\phi e = \frac{1}{\sqrt{c(1-\cos t)}}\Big(-(1-\cos t)\frac{\partial}{\partial y} + (\sin t)\phi\frac{\partial}{\partial y}\Big)$$

define on U an orthonormal frame of eigenvectors of h, such that $h\xi = 0$, $he = \lambda e$, $h\phi e = -\lambda\phi e$ ($\lambda = \rho/2 > 0$). Working as in the case $b_x + \rho \neq 0$, we finally get, for the Lie brackets $[\xi, e]$, $[\xi, \phi e]$, $[e, \phi e]$, the formulas (4.18), (4.19) and (4.20), respectively. So we have proved the following theorem.

THEOREM 4.4. *Let $M(\eta, \xi, \phi, g)$ be a three-dimensional c.m.m. with* $\mathrm{Tr}\, l = \text{constant} \neq 2$. *Then, at any point P of M, there exists a neighborhood U of P so that at least one of the functions $b_x + \rho$ and $b_x - \rho$ does not vanish anywhere on U.*

(i) *If $b_x + \rho \neq 0$ everywhere in U, then the* triad$(\xi, e, \phi e)$, *where $\xi = \partial/\partial x$,*

$$e = \frac{1}{\sqrt{(1+\cos t)}}\Big((1+\cos t)\frac{\partial}{\partial y} + (\sin t)\phi\frac{\partial}{\partial y}\Big)$$
$$\phi e = \frac{1}{\sqrt{c(1+\cos t)}}\Big(-(\sin t)\frac{\partial}{\partial y} + (1+\cos t)\phi\frac{\partial}{\partial y}\Big)$$

defines a smooth orthonormal frame of eigenvectors of h in U, such that $h\xi = 0$, $he = \lambda e$, $h\phi e = -\lambda\phi e$ ($\lambda = \rho/2 > 0$).

(ii) *If $b_x - \rho \neq 0$ everywhere in U, then the* triad$(\xi, e, \phi e)$, *where $\xi = \partial/\partial x$,*

$$e = \frac{1}{\sqrt{c(1-\cos t)}}\Big((\sin t)\frac{\partial}{\partial y} + (1-\cos t)\phi\frac{\partial}{\partial y}\Big)$$
$$\phi e = \frac{1}{\sqrt{c(1-\cos t)}}\Big(-(1-\cos t)\frac{\partial}{\partial y} + (\sin t)\phi\frac{\partial}{\partial y}\Big)$$

defines a smooth orthonormal frame of eigenvectors of h in U, such that $h\xi = 0$, $he = \lambda e$, $h\phi e = -\lambda\phi e$ ($\lambda = \rho/2 > 0$).

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Moreover, in any case ((i) or (ii)), the Lie brackets $[\xi, e]$, $[\xi, \phi e]$ *and* $[e, \phi e]$ *are given by*

$$\begin{aligned}
[\xi, e] &= \tfrac{1}{2}(\rho - \rho\cos t + t_x)\phi e \\
[\xi, \phi e] &= \tfrac{1}{2}(\rho + \rho\cos t - t_x)e \\
[e, \phi e] &= 2\xi - \frac{1}{c^2}\{-c_y + c^2(yt_x + t_z) + bct_y\}\phi\frac{\partial}{\partial y} \\
&\quad + \frac{1}{c^2}\{-bc_y + c(-t_y + yc_x + c_z + 2b_y)\}\frac{\partial}{\partial y}.
\end{aligned}$$

When is a three-dimensional c.m.m. with Tr l = constant ≠ 2 a Jacobi (κ, μ)-contact manifold? Comparing relations $[\xi, e] = \frac{1}{2}(\rho - \rho\cos t + t_x)\phi e$ of Theorem 4.4 and $[\xi, e] = (A + \lambda + 1)\phi e$ of (3.1), we get $A + \rho/2 + 1 = \frac{1}{2}(\rho - \rho\cos t + t_x)$ or $A = \frac{1}{2}(t_x - \rho\cos t) - 1$. So $\mu = -2A = \rho\cos t - t_x + 2$. Hence, using the definition of a Jacobi (κ, μ)-contact manifold (see relation (2.5)), we state the following theorem.

THEOREM 4.5. *Let* $M(\eta, \xi, \phi, g)$ *be a three-dimensional c.m.m. with* Tr l = constant ≠ 2. *Then* $M(\eta, \xi, \phi, g)$ *is a Jacobi* (κ, μ)*-contact manifold if and only if the function* $t = t(x, y, z)$ *satisfies the equation*

$$t_x - \rho\cos t + \nu = 0, \tag{4.21}$$

where ν = constant. *Therefore, in this case,* $\kappa = \mathrm{Tr}\, l/2 = 1 - \rho^2/4$ *and* $\mu = \nu + 2$.

Comment. Obviously, the function $t = t(x, y, z)$ = constant is a solution of (4.21).

When is a three-dimensional c.m.m. with Tr l = constant ≠ 2 a (κ, μ)-contact manifold? Let $M(\eta, \xi, \phi, g)$ be a three-dimensional c.m.m. with Tr l = constant ≠ 2. According to Proposition 3.7, M is a (κ, μ)-contact manifold if and only if the vector field ξ is an eigenvector of the Ricci operator Q, or, equivalently, from (3.3), $B = C = 0$, or from (3.1), $[e, \phi e] = 2\xi$. Therefore, using $[e, \phi e] = 2\xi$ and the last relation of Theorem 4.4, we get that $M(\eta, \xi, \phi, g)$ is a (κ, μ)-contact manifold if and only if

$$c^2(yt_x + t_z) + bct_y - c_y = 0 \quad \text{and} \quad c(t_y - c_z - 2b_y - yc_x) + bc_y = 0. \tag{4.22}$$

So, we have proved the following theorem.

THEOREM 4.6. *Let* $M(\eta, \xi, \phi, g)$ *be a three-dimensional c.m.m. with* Tr l = constant ≠ 2. *Then* $M(\eta, \xi, \phi, g)$ *is a* (κ, μ)*-contact manifold (equivalently, it is H-contact) if and only if the functions* $b = b(x, y, z)$, $c = c(x, y, z)$ *and* $t = t(x, y, z)$ *satisfy conditions (4.22).*

It is obvious that, on a (κ, μ)-contact manifold, the function $t = t(x, y, z)$ satisfies condition (4.21).

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