WEAKLY SEMI-SIMPLE FINITE-DIMENSIONAL ALGEBRAS

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Let A be a finite-dimensional (associative) algebra over an arbitrary field F. We shall say that a semi-group S is a *translate* of A if there exist an algebra B over F and an epimorphism $\phi: B \to F$ such that $A = 0\phi^{-1}$ and $S = 1\phi^{-1}$. It is shown in **(2)** that any such semi-group S has a kernel (defined below) that is completely simple in the sense of Rees. Following Stefan Schwarz **(4)**, we define the *radical* $R(S)$ of S to be the union of all ideals I of S such that some power I^n of I lies in the kernel K of S. First we prove that the radical of a translate of A is a translate of the radical of A. It follows that A is nilpotent if and only if it has a translate S such that $R(S) = S$. We then investigate the opposite extreme, i.e., the case in which $R(S) = K$. If $R(S) = K$, we shall say that S is *K-semi-simple*. We declare that A is *weakly semi-simple* if some translate S of A is K-semi-simple. It is shown that A is weakly semi-simple if and only if fAf is semi-simple for some (hence every) principal idempotent f in A; equivalently, $A = fAf \oplus R(A)$ (as vector spaces) where $R(A)$ is the radical of A. This result enables us to give a characterization without the use of idempotents of the algebras of class Q studied by R. M. Thrall in **(5)**.

1. Preliminaries.

1.1. A non-empty subset I of a semi-group S is said to be an *ideal* of S if $SI \cup IS \subset I$. The intersection K of all ideals of S, if not empty, is a minimal ideal of S called the *kernel* of S. K is *completely simple* if it is a union of groups and has no proper ideals. For further information concerning completely simple semi-groups see **(3)**.

1.2. Let S be a semi-group with kernel K. An ideal I of S is said to be K-*potent* if some power I^n lies in K. The *radical* $R(S)$ of S is the union of all K-potent ideals of S.

1.3. We shall assume that the reader is familiar with the basic theory of finite-dimensional algebras as expounded for example in **(1)**. However, we wish to emphasize that by *ideal* of A where A is an algebra we shall mean as usual a subspace of the underlying vector space which is at the same time an "ideal" (in the sense of 1.1) of the multiplicative semi-group of A. On the other hand, when we speak of an *ideal* of a translate of A (see 1.5) we imply no more than the definition of 1.1.

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It is easily shown that if S is the multiplicative semi-group of a finite-dimensional algebra, then $R(S)$ as defined in 1.2 coincides with the usual definition of the radical of an algebra. We shall accordingly denote the radical of an algebra A by $R(A)$.

1.4. An idempotent e (possibly zero) in an algebra A is said to be a *principal idempotent* if $u^2 = u$ and $ue = eu = 0$ together imply $u = 0$.

1.5. A semi-group S will be said to be a *translate* of an algebra A over F if there exists an algebra B over F and an epimorphism $\phi: B \to F$ such that $S = 1\phi^{-1}$ and $A = 0\phi^{-1}$. Alternatively, one may see that a semi-group S is a translate of A if there exists an algebra B containing A as an ideal such that $S = A + x$ for some $x \in B, x \notin A$; the multiplication in S is, of course, assumed to coincide with that in B Note that if $S = A + x$, then $S = A + s$ for any $s \in S$.

1.6. We shall need the following facts from **(2)** concerning a translate S of a *finite-dimensional* algebra A over a field F:

(i) S has a completely simple kernel K.

(ii) Some power of every element of S lies in a subgroup of S.

(iii) If $\alpha_1, \ldots, \alpha_n \in F$ are such that $\sum \alpha_i = 1$, then $\sum \alpha_i s_i \in S$ for any $s_1, \ldots, s_n \in S$.

(iv) The kernel K of S is not in general a linear variety (i.e., a translate of a subspace of A). However, if we let $M(K)$ be the smallest linear variety containing K, then $M(K)$ is a K-potent ideal.

(v) Let Γ be a faithful representation of the algebra B as an algebra of matrices. Then all elements of $\Gamma(K)$ have the same rank k. Moreover, an element s of S lies in K if and only if the rank of $\Gamma(s)$ is k.

(vi) Let $e^2 = e \in K$. Then **(2**, 1.7, 1.8, and 2.4**)** imply that $eM(K)e - e$ and $(1 - e)M(K)(1 - e)$ are both nilpotent subalgebras of $A = S - e$, where by $(1 - e)m(1 - e)$ we mean $m - em - me + eme$.

2. The radical of a translate of an algebra.

2.0. LEMMA. *Let S be a translate of a finite-dimensional algebra A, and let K be the kernel of S. Then, if $e^2 = e \in K$,*

$$M(K) - M(K) = M(K) - e$$

is a nilpotent ideal of A.

Proof. Since $M(K)$ is a linear variety (see 1.6 (iv)), it is clear that $M(K) - x = M(K) - y$ for any $x, y \in M(K)$. Thus

$$M(K) - e = M(K) - M(K)$$

for $e = e^2 \in K \subseteq M(K)$.

To see that $M(K) - e$ is a left ideal, note that

$$A(M(K) - e) = (S - e)(M(K) - e) = SM(K) - eM(K) - M(K)e + e \subseteq M(K) - M(K) - M(K) + e.$$

Now since $M(K)$ is a linear variety, $M(K) + M(K) - M(K) = M(K)$, whence

$$A(M(K) - e) \subseteq e - M(K) = M(K) - e.$$

Similarly, $M(K) - e$ is a right ideal of A.

Now let $M = M(K) - e$. It is easily seen that

$$(1) \qquad M = eMe + eM(1 - e) + (1 - e)Me + (1 - e)M(1 - e)$$

is a direct sum decomposition of M as a vector space. Since $M(K)$ is an ideal of S as well as a linear variety, each summand of (1) is indeed contained in M. By 1.6 (vi), $eMe = eM(K)e - e$ and

$$(1 - e)M(1 - e) = (1 - e)M(K)(1 - e)$$

are both nilpotent. Since $(1 - e)M(1 - e)$ is nilpotent, it follows that $M(1 - e)$ and $(1 - e)M$ are both nilpotent ideals of M and therefore contained in the radical $R(M)$ of M. Now clearly the last three summands of (1) are contained in $M(1 - e) + (1 - e)M$ and hence lie in the radical. This together with the fact that eMe is nilpotent implies that $R(M) = M$, i.e., M is nilpotent.

2.1. Lemma. *Let S, K, and A be as in* 2.0. *If I is a K-potent ideal of S, then $I - x \subseteq R(A)$ for all $x \in I$.*

Proof. Since K is the minimal ideal of S, we have $K \subseteq I$. We may assume without loss of generality that $I = M(I)$, the smallest linear variety containing I. For since

$$M(I) = \{\sum \alpha_i s_i : \sum \alpha_i = 1 \text{ and } s_i \in I\},$$

it is clear that $M(I)^n \subseteq M(K)$ if $I^n \subseteq K$; then since $M(K)$ is K-potent (1.6 (iv)), it follows that $M(I)$ is also K-potent.

Since $K \subseteq I$, we have $M(K) \subseteq M(I) = I$. Let $e = e^2 \in K$; then

$$I - e = I - x$$

for any $x \in I$ since I is a linear variety. Now

$$A(I - e) = (S - e)(I - e) \subseteq SI - eI - Ie + e \subseteq (I - I - I) + e = e - (I + I - I) \subseteq e - I = I - e.$$

Hence $A(I - e) \subseteq (I - e)$ and $I - e$ is a left ideal of A. Similarly, one may show that $I - e$ is a right ideal.

Now to complete the proof we need only show that $I - e$ is nilpotent. We first claim that

$$(I - e)^n = I^n - M(K) \qquad \text{for } n = 1, 2, \ldots,$$

This is obvious for $n = 1$ since $e \in M(K)$. Suppose our claim holds for $n = k$; then

$$\begin{aligned}(I - e)^{k+1} &= (I - e)(I - e)^k \\ &\subseteq (I - e)(I^k - M(K)) \\ &\subseteq I^{k+1} - eI^k + eM(K) - IM(K) \\ &\subseteq I^{k+1} - (M(K) - M(K) + M(K)) \\ &\subseteq I^{k+1} - M(K).\end{aligned}$$

Now if $I^n \subseteq K$, then

$$(I - e)^n \subseteq I^n - M(K) \subseteq K - M(K) \subseteq M(K) - M(K) = M(K) - e$$

which by 2.0 is nilpotent. Consequently, $I - e$ must also be nilpotent.

2.2. Corollary. *Let S, A, and K be as in* 2.0. *Then $R(S) \subseteq R(A) + e$ for any $e = e^2 \in K$.*

Proof. By definition, $R(S)$ is the union of all K-potent ideals I of S. By the preceding lemma, $I \subseteq R(A) + e$ for any e in I; and since $K \subset I$, we may choose e to be any idempotent in K. Hence $R(S) \subseteq R(A) + e$.

2.3. Lemma. *Let S, A, and K be as in* 2.0. *Then $R(A) + e$ is a K-potent ideal of S for any idempotent e in K.*

Proof. Since $S = A + e$,

$$\begin{aligned}(R(A) + e)S &= (R(A) + e)(A + e) \\ &\subseteq R(A)e + eA + R(A)A + e \\ &\subseteq Ae + eA + R(A) + e.\end{aligned}$$

Hence, to show that $R(A) + e$ is an ideal of S, it suffices to show that $Ae \cup eA \subseteq R(A)$. This follows immediately from 2.0, since

$$Ae + e = (A + e)e = Se \subset K,$$

implying that

$$Ae \subseteq K - e \subseteq M(K) - e \subseteq R(A).$$

Similarly, $eA \subseteq K - e \subseteq R(A)$.

It remains to show that $R(A) + e$ is K-potent. To do this, first we establish that

(2) $$M(K)A \cup AM(K) \subseteq M(K) - e.$$

If $x \in M(K)$, then $x - e \in M(K) - e$, which by 2.0 is an ideal of A. Let $a \in A$. Then $xa - ea = (x - e)a \in M(K) - e$, and since by the first paragraph of this proof $eA \subseteq M(K) - e$, we obtain that $xa \in M(K) - e$. Thus $xA \subseteq M(K) - e$. Similarly $Ax \subseteq M(K) - e$, and (2) holds.

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We now use (2) to obtain

(3) $$(R(A) + e)^n \subseteq R(A)^n + M(K), \qquad \text{for } n = 1, 2, \ldots .$$

Suppose this holds for n, then

$$\begin{aligned}(R(A) + e)^{n+1} &= (R(A) + e)(R(A)^n + M(K)) \\ &\subseteq R(A)^{n+1} + eR(A)^n + R(A)M(K) + eM(K) \\ &\subseteq R(A)^{n+1} + M(K)A + AM(K) + M(K) \\ &\subseteq R(A)^{n+1} + M(K) - e + M(K) \\ &\subseteq R(A)^{n+1} + M(K).\end{aligned}$$

Thus (3) holds, and if n exceeds the index of nilpotency of $R(A)$, we obtain $(R(A) + e)^n \subseteq M(K)$. Now since $M(K)$ is K-potent, $R(A) + e$ must be also.

2.4. THEOREM. *Let S be a translate of a finite-dimensional algebra A, and let e be an idempotent in the kernel of S. Then*

$$R(S) = R(A) + e.$$

Proof. By 2.3, $R(A) + e$ is a K-potent ideal of S and is therefore contained in $R(S)$. On the other hand, we know from 2.2 that $R(S) \subseteq R(A) + e$.

2.5. COROLLARY. *If S is as above, then $R(S)$ is a K-potent ideal of S and hence the unique maximal K-potent ideal of S.*

2.6. COROLLARY. *If S and A are as above, and if $R(S) = S$, then A is nilpotent.*

3. Weakly semi-simple algebras.

3.0. If a translate S of the algebra A has a multiplicative zero, i.e., if $K = \{z\}$, then $s \to s - z$ is an isomorphism from S onto the multiplicative semi-group of A. Hence the only translates of S that are of interest are those for which K is non-trivial. Corollary 2.6 above deals with the case $R(S) = S$. In this section, we single out for consideration those algebras that have translates S whose radical and kernel coincide. To this end, we shall say that a semi-group S with kernel K is *K-semi-simple* if $R(S) = K$; and that an algebra A is *weakly semi-simple* if it possesses a K-semi-simple translate.

3.1. LEMMA. *Let S be a K-semi-simple translate of a finite-dimensional algebra. A. Let K be the kernel of S and e an idempotent in K. Then*

(i) $M(K) = K$

and

(ii) $x = ex + xe - exe$ *for all* $x \in K$.

Proof. Since by 1.6 (iv) $M(K)$ is K-potent, we have $M(K) \subset R(S) = K$; whence $K = M(K)$.

Let Γ be a faithful matrix representation of a super-algebra of A which contains S as a translate of A, and let e be an idempotent in K. Now by choosing a suitable basis for the representation space, we may assume that

$$\Gamma(e) = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix. Now let $x \in K$. Since $K = M(K)$, the element $y = x - ex - xe + exe + e$ lies in K. Thus

$$\Gamma(y) = \begin{bmatrix} I_k & 0 \\ 0 & C \end{bmatrix} \in \Gamma(K)$$

where C is some $(n - k) \times (n - k)$ matrix; here n denotes the degree of the representation. By 1.6 (v), since y and e both lie in K, the rank of $\Gamma(e)$ must equal the rank of $\Gamma(y)$. This implies that $C = 0$. Hence $\Gamma(y) = \Gamma(e)$. Since Γ is faithful, this shows that $y = e$. This clearly implies (ii).

3.2. LEMMA. *Let S, A, K, and e be as in* 3.1. *Then*

$$R(A) = eA + Ae + eAe = eA + Ae.$$

Proof. We know from the first paragraph of the proof of 2.3 that eA and Ae are always contained in $R(A)$. It follows that

$$eAe \subseteq eR(A) \subseteq eA \subseteq R(A).$$

Hence $eAe + eA + Ae \subseteq R(A)$.

Now let $r \in R(A)$. Since S is K-semi-simple, $K = R(S)$; hence 2.4 implies that $R(A) = K - e$ for some idempotent e in K. Thus $r = x - e$ for $x \in K$. By 3.1 (ii), $x = ex + xe - exe$, and so

$$r = ex + xe - exe - e = e(x - e) + (x - e)e - e(x - e)e,$$

which is an element of $eA + Ae + eAe$ since $x - e \in S - e = A$. This shows that $R(A) = eA + Ae + eAe$. Since eA is contained in A, $eAe \subset Ae$ and therefore $Ae + eA = Ae + eA + eAe$.

3.3. LEMMA. *Let A be a subalgebra of a finite-dimensional algebra B, and let e be an idempotent of B such that $R(A) = eA + Ae$. Then there exists a principal idempotent f in A such that fAf is semi-simple.*

Proof. First, since $eA \cup Ae \subseteq R(A)$, we have

$$eAe \subseteq eR(A) \subseteq eA \subset R(A).$$

Whence $R(A) = eA + Ae + eAe$. It follows that if we let

$$A_0 = (1 - e)A(1 - e) = \{a - ea - ae + eae : a \in A\},$$

we obtain that $A = R(A) + A_0$ is a direct sum as vector spaces, i.e., A_0 is complementary to the radical of A. It follows that A_0 must be semi-simple. Since A_0 is semi-simple, it contains an identity, say f. Now

$$fAf = f(eA + Ae + A_0)f = fA_0f = A_0,$$

since $f \in (1 - e)A(1 - e)$ implies $fe = ef = 0$. To see that f is principal in A, let $g^2 = g \in A$ such that $fg = gf = 0$. Then $g = r + a$ where $a \in A_0$ and $r \in R(A)$, whence $0 = fgf = faf = a$. This implies that $g = r$, and therefore $g = 0$ since $R(A)$ contains no non-zero idempotents.

3.4. LEMMA. *Let A be a finite-dimensional algebra such that $A = fAf + R(A)$ is a direct sum as vector spaces for some idempotent f in A. Then every principal idempotent in A is of the form $f + n$ where $n \in R(A)$ and $n = nf + fn + nfn$.*

Proof. Let h be a principal idempotent in A. Then $h = g + n$ where $g \in fAf$ and $n \in R(A)$. The idempotency of h easily implies that $g^2 = g$ and

$$gn + ng + n^2 = n.$$

Let $k = f - g$. Since $g \in fAf$, we have $k^2 = k$. Our aim is to show that $k = 0$. Since h is a principal idempotent, it suffices to show that $kh = hk = 0$. From

$$kh = (f - g)(g + n) = (f - g)n = fn - gn$$

and

$$hk = (g + n)(f - g) = n(f - g) = nf - ng,$$

it is clear that we need only show that $fn = gn$ and $nf = ng$. To do this, we first note that

$$fn = f(gn + ng + n^2) = gn + fng + fn^2 = gn + fn^2$$

since $fng = fngf \in fAf \cap R(A) = (0)$. Similarly, $nf = ng + n^2f$. We now show that $fn^2 = n^2f = 0$. Since fng and fn^2g lie in $fAf \cap R(A) = (0)$, we have $fn^2 = fn(gn + ng + n^2) = fn^3$ implying that $fn^2 = fn^3 = \ldots = fn^k = 0$ if k exceeds the index of nilpotency of n. A similar argument shows that $n^2f = 0$. We have therefore established that $g = f$.

From the above paragraph we know that $n = nf + fn + n^2$ and that $n^2f = fn^2 = 0$. Now

$$n(nf + fn + n^2) = n^2f + nfn + n^3 = nfn + n^3$$

since $n^2f = 0$. Thus

$$n^2 = nfn + n^3. \tag{4}$$

This implies that

$$n^3 = n(n^2) = n(nfn + n^3) = n^2fn + n^4 = n^4,$$

since $n^2f = 0$. Now since n is nilpotent, $n^3 = n^4$ implies that $n^3 = 0$. From (4) we now conclude that $n^2 = nfn$; whence

$$n = nf + fn + n^2 = nf + fn + nfn.$$

3.5. LEMMA. *If A and f are as in 3.4, and h is a principal idempotent of A, then hAh is semi-simple.*

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Proof. By 3.4, we know that $h = f + n$ where $n \in R(A)$. It is well known **(1**, p. 25**)** that $R(hAh) = hR(A)h$. Therefore to show that hAh is semi-simple, it suffices to show that $hR(A)h = 0$. Let $hah \in hR(A)h$. Then

$$hah = (f+n)a(f+n) = fan + naf + nan,$$

since $faf \in fAf \cap R(A) = (0)$. It remains to show that $fan = naf = nan = 0$. From 3.4 we know that $n = nf + fn + nfn$, whence

$$fan = fa(nf + fn + nfn) = fanf + fafn + fanfn = 0$$

since $fR(A)f = 0$. Similarly, naf and $nan = 0$.

3.6. Lemma. *Let A be a finite-dimensional algebra over F which contains a principal idempotent f such that fAf is semi-simple. Then A is weakly semi-simple.*

Proof. Let B be the algebra obtained by adjoining an identity to A in the usual way. Then A is an ideal of B and B/A is isomorphic to F; so, clearly, $S = A + 1$ is a translate of A. Let K denote the kernel of S. We must show that $R(S) = K$. By 2.4 we have $R(S) = R(A) + e$ for any idempotent e in K. Thus it suffices to show that $R(A) + e = K$.

Now since f is a principal idempotent, it follows from **(1**, Lemma 9, p. 26**)** that

$$R(A) = (1-f)Af + fA(1-f) + (1-f)A(1-f),$$

and $A = fAf + R(A)$ is a vector-space direct sum. Whence

$$R(A) \subseteq (1-f)A + A(1-f);$$

furthermore

$$(1-f)A = (1-f)(fAf + R(A)) = (1-f)R(A).$$

Similarly, $A(1-f) = R(A)(1-f)$; consequently,

$$R(A) = (1-f)R(A) + R(A)(1-f).$$

Now $R(A) + (1-f) = (R(A) - f) + 1$ is a left ideal of $S = A + 1$, for

$$\begin{aligned}(A+1)(R(A) + (1-f)) &\subseteq AR(A) + R(A) + A(1-f) + (1-f)\\ &\subseteq R(A) + (1-f)\end{aligned}$$

since $AR(A) \subseteq R(A)$ and $A(1-f) = R(A)(1-f) \subseteq R(A)$ as shown above. Similarly, $R(A) + (1-f)$ is a right ideal of S and therefore an ideal of S.

Let $g = 1 - f$. We now show that $R(A) + g$ has no proper ideals. First note that $G = g(R(A) + g)g = gR(A)g + g$ is a group; for if $y = gng + g \in G$, then $yz = zy = g$ where

$$z = g(-n + n^2 - \ldots - n^i)g + g,$$

for any i exceeding the index of nilpotency of n. Now if I is an ideal of $R(A) + g$, $gIg \subseteq I \cap G$. Since G is a group, G must be contained in I. In particular, $g \in I$. Now let y be any element of $R(A) + g$. Let $g\bar{y}g$ be the inverse of gyg in the group G, and set $z = ygg\bar{y}gy$. We now have $gz = gy$, $zg = yg$, and $gzg = gyg$. By the above, we know that $R(A)g + gR(A) = R(A)$; it follows that $w = gw + wg - gwg$ for any w in $R(A) + g$. This implies that $y = z$. Now from $z = ygg\bar{y}gy$ and $g \in I$, we obtain $y = z \in I$. This shows that $I = R(A) + g$.

Since $R(A) + g$ is an ideal, it follows immediately from the preceding paragraph that $K = R(A) + g$. Clearly, then, $R(A) + g = R(A) + e$ for any idempotent e of K.

3.8. THEOREM. *If A is a finite-dimensional algebra, the following are equivalent:*
(i) *A is weakly semi-simple.*
(ii) *There exists a principal idempotent f in A such that fAf is semi-simple.*
(iii) *For all principal idempotents f in A, fAf is semi-simple.*
(iv) *$A = fAf + R(A)$ is a vector-space direct sum for some idempotent f in A.*
Moreover, if A is weakly semi-simple, then every subalgebra of A that complements the radical is of the form fAf for some principal idempotent f in A.

Proof. The equivalence of (i), (ii), (iii), and (iv) follow immediately from the foregoing lemmas together with the fact that (ii) is equivalent to (iv), which is a direct result of the Peirce decomposition of A with respect to the principal idempotent f; cf. (**1**, pp. 25 ff.).

To establish the last sentence of the theorem, assume that A is weakly semi-simple and that $A = D + R(A)$ is a vector-space direct sum for some subalgebra D. D must be semi-simple and therefore has an identity, say f. Then $D \subset fAf$ trivially. To show the converse we need only show f to be principal, for then by (**1**, §9, p. 25) we have $\dim(fAf) = \dim D$. Let $g = g^2 \in A$ such that $gf = fg = 0$. Now $g = d + r$ when $d \in D$ and $r \in R(A)$. Hence

$$0 = fg = f(d + r) = d + fr,$$

implying $d = 0$. Therefore $g = r \in R(A)$; since g is idempotent and r is nilpotent, $g = 0$.

3.9. COROLLARY. *If a finite-dimensional algebra A is a direct sum of a semi-simple algebra and a nilpotent algebra, then A is weakly semi-simple.*

After R. M. Thrall (**5**), we shall say that an algebra A is of class Q if it possesses an idempotent e satisfying

(i) eAe is semi-simple,
(ii) $AeA = A$,
and (iii) $A = eAe \oplus R(A)$ (as vector spaces).

We observe that (iii) always implies (i) and that they are equivalent if e is a principal idempotent in (i). In any case, it is clear from the above theorem that (i) and (iii) are equivalent to weak semi-simplicity. This fact enables us to characterize without idempotents the algebras of class Q:

3.9. THEOREM. *A weakly semi-simple finite-dimensional algebra A is of class Q if and only if* $A^2 = A$ *and* $R(A)^3 = 0$.

Proof. The necessity of these conditions follows immediately from (5, Corollary 1) and Condition (ii). To show their sufficiency, let e be a principal idempotent such that $A = eAe + R(A)$. Then

$$A = A^2 = A^3 = (eAe + R(A))^3 \subseteq AeA + R(A)^3 = AeA;$$

whence $A = AeA$.

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