

CLASSES OF OPERATORS ON VECTOR VALUED INTEGRATION SPACES

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(Received 30 April; revised 20 October 1976)

Communicated by E. Strzelecki

Abstract

Let $L^p(\Omega, K)$ denote the Banach space of weakly measurable functions F defined on a finite measure space and taking values in a separable Hilbert space K for which $\|F\|_p = (\int |F(\omega)|^p)^{1/p} < +\infty$. The bounded Hermitian operators on $L^p(\Omega, K)$ (in the sense of Lumer) are shown to be of the form

$$AF(\omega) = B(\omega)F(\omega),$$

where $B(\omega)$ is a uniformly bounded Hermitian operator valued function on K . This extends the result known for classical L^p spaces. Further, this characterization is utilized to obtain a new proof of Cambern's theorem describing the surjective isometries of $L^p(\Omega, K)$. In addition, it is shown that every adjoint abelian operator on $L^p(\Omega, K)$ is scalar.

Subject classification (Amer. Math. Soc. (MOS), 1970): primary 46 E 40, 46 E 30, 47 B 99

1. Introduction

The concept of a hermitian operator on a Banach space is due to G. Lumer (1961). Lumer (1963) characterized all bounded hermitian operators on a non-atomic reflexive Orlicz space as multiplications by real L^∞ functions and used this characterization to determine the surjective isometries. The atomic case was treated by Tam (1969). Similar characterizations have been undertaken by the present authors for certain direct sums of Hilbert spaces (Fleming and Jamison, 1974a, b) and for l^p sums of Banach spaces by Berkson and Sourour (1974). It is of interest then to consider the problem of finding the bounded hermitian operators on $L^p(\mu)$ spaces of vector valued functions when the measure is not necessarily purely atomic.

In the present paper we characterize the bounded hermitian operators on the spaces $L^p(\Omega, k)$ of weakly measurable functions with values in a separable Hilbert space. From these results we obtain a characterization of the surjective isometries which has been obtained also by Cambern (1974) with different methods. The

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results on hermitian operators are also used to show that the adjoint abelian operators on $L^p(\Omega, k)$ are scalar. This is related to an open problem of Stampfli (1969).

Some comments are in order concerning the generality of the theorems proved in the paper. In the first place, standard adjustments in the arguments will allow the measure space Ω to be σ -finite. More delicate is the assumption that K is a separable Hilbert space. Although the theorem is true if K is assumed to be a separable smooth Banach space, we feel that the essential elements of the proof are best revealed and understood by assuming K to be a Hilbert space. However, this last assumption seems to be essential for our proof of Theorem 4.14. More will be said on this subject in Section 4.

2. Hermitian operators and the spaces $L^p(\Omega, K)$

For the definition of a semi-inner-product (abbreviated s.i.p.) we refer the reader to Lumer’s paper (1961). A s.i.p. $[\cdot, \cdot]$ on a complex Banach space X is said to be compatible with the norm if $[F, F] = \|F\|^2$ for every $F \in X$. A bounded linear operator T on a complex Banach space is said to be hermitian if there is a s.i.p. compatible with the norm, such that $[TF, F]$ is real for every $F \in X$. In the case that X is a Hilbert space this definition yields the usual class of bounded hermitian operators.

Let (Ω, Σ, μ) be a finite measure space and let K be a separable Hilbert space with norm and inner product denoted by $|\cdot|_2$ and (\cdot, \cdot) respectively. A K valued function F on Ω is said to be *weakly measurable* (Hille and Phillips, 1957) if the complex valued function $(F(\cdot), z)$ is measurable for each $z \in K$. For $1 \leq p < \infty$, we denote by $L^p(\Omega, K)$ the Banach space of (equivalence classes of) weakly measurable K valued functions on Ω for which $\|F\|_p < \infty$, where

$$\|F\|_p = (\int |F(\omega)|_2^p d\mu)^{1/p}, \quad p < \infty,$$

$$\|F\|_\infty = \text{ess sup } |F(\omega)|_2 \quad \text{for } p = \infty.$$

It is important to note that the vector simple functions of the form

$$\psi = \sum_{i=1}^n \chi_{E_i} x_i,$$

where the E_i are disjoint measurable sets and the x_i are distinct elements of K , are dense in $L^p(\Omega, K)$. For further facts concerning $L^p(\Omega, K)$ we refer the reader to Dunford and Schwartz (1958), Hille and Phillips (1957) and Lang (1969).

3. Hermitian operators on $L^p(\Omega, K)$

In what follows, the support of a K valued function F on Ω is the set $\{\omega \mid F(\omega) \neq 0\}$ where 0 is the zero vector in K , and if $z \in K$, ε is the function which is equal to z for all $\omega \in \Omega$. We use script letters to denote operators on $L^p(\Omega, K)$.

Since the dual space of $L^p(\Omega, K)$ for $1 \leq p < \infty$ is $L^q(\Omega, K)$ where $1/p + 1/q = 1$ (Lang, 1969), it is easy to show that a s.i.p. compatible with the norm on $L^p(\Omega, K)$, $1 \leq p < \infty$, is given by

$$(3.1) \quad [F, G] = \int (F(\omega), G(\omega)) \left(\frac{|G(\omega)|_2}{\|G\|_p} \right)^{p-2} d\mu.$$

Before we state the main theorem on hermitian operators on $L^p(\Omega, K)$ we offer the following example of a bounded hermitian operator on $L^p(\Omega, K)$.

EXAMPLE: Let $H(\omega)$ be a $\mathcal{B}(K)$ valued function on Ω such that

$$(3.2) \quad H(\omega) = H(\omega)^* \text{ almost everywhere,}$$

$$(3.3) \quad H(\omega)z \text{ is weakly measurable for each } z \in K, \text{ and}$$

$$(3.4) \quad \|H(\omega)\| \leq M < \infty \text{ almost everywhere, where } \|\cdot\| \text{ denotes the uniform norm on } \mathcal{B}(K). \text{ If we define } (\mathcal{M}_H F)(\omega) = H(\omega)F(\omega) \text{ then in view of (3.3) and (3.4) it is clear that } \mathcal{M}_H \text{ is a bounded linear operator on } L^p(\Omega, K). \text{ Furthermore, since}$$

$$(3.5) \quad [\mathcal{M}_H F, F] = \int (H(\omega)F(\omega), F(\omega)) \left(\frac{|F(\omega)|_2}{\|F\|_p} \right)^{p-2} d\mu$$

it follows from (3.2) that \mathcal{M}_H is hermitian.

In this section we will show that every bounded hermitian operator on $L^p(\Omega, K)$ is of the form $\mathcal{M}_{H(\cdot)}$ for some $\mathcal{B}(K)$ valued function $H(\cdot)$ satisfying the conditions of the example. To do this we first need to establish some preliminary results. Henceforth, $1 \leq p < \infty$ and $p \neq 2$.

(3.6) LEMMA. Let $F_1, F_2 \in L^p(\Omega, K)$ with disjoint supports E_1 and E_2 . If \mathcal{A} is a bounded hermitian operator on $L^p(\Omega, K)$ then

$$(3.7) \quad \int_{E_1} ((\mathcal{A}F_1)(\omega), F_2(\omega)) |F_2(\omega)|_2^{p-2} d\mu = \int_{E_2} ((\mathcal{A}F_2)(\omega), F_1(\omega)) |F_1(\omega)|_2^{p-2} d\mu.$$

The proof of this lemma follows immediately by writing out the requirement that $[\mathcal{A}(F_1 + e^{i\varphi} F_2), F_1 + e^{i\varphi} F_2]$ is real for every choice of $\varphi \in \mathcal{R}$. We omit the details.

(3.8) LEMMA. Let $E \in \Sigma$ with $\mu(E) > 0$ and $z \in K$. If \mathcal{A} is a bounded hermitian operator on $L^p(\Omega, K)$, $1 \leq p < \infty$, $p \neq 2$, then

$$(3.9) \quad \mathcal{A}(\chi_E z) = \chi_E \mathcal{A}(z).$$

PROOF. It is enough to prove this for vector z with $|z|_2 = 1$. Thus we let $z \in K$ with norm 1 and E be a set in Σ with positive measure. Let E_1 be any subset of $\Omega \setminus E$ with positive measure. Let $F_1 = \chi_{E_1} z$ and $F_2 = \alpha F$ where α is a non-zero real number and $F \in L^p(\Omega, K)$ with support in E_1 and $|F(\omega)|_2 = 1$ almost everywhere

on E_1 . Then from (3.7) we have

$$(3.10) \quad \int_{E_1} (\mathcal{A}(\chi_E z)(\omega), \alpha F(\omega)) |\alpha F(\omega)|_2^{p-2} d\mu = \int_E (\alpha(\mathcal{A}F)(\omega), \chi_E(\omega)z) |\chi_E(\omega)z|_2^{p-2} d\mu.$$

For $\alpha = 1$ then we have

$$(3.11) \quad \int_{E_1} (\mathcal{A}(\chi_E z)(\omega), F(\omega)) d\mu = \int_E (\mathcal{A}(F)(\omega), \chi_E(\omega)z) d\mu.$$

Combining (3.10) and (3.11) we obtain

$$(3.12) \quad (1 - |\alpha|^{p-2}) \int_{E_1} (\mathcal{A}(\chi_E z)(\omega), F(\omega)) d\mu = 0.$$

Since $p \neq 2$ we see from (3.12) that

$$(3.13) \quad \int_{E_1} (\mathcal{A}(\chi_E z)(\omega), F(\omega)) d\mu = 0.$$

If we now suppose that $E_1 = \text{supp}(\mathcal{A}(\chi_E z)) \cap (\Omega \setminus E)$ has positive measure, then from (3.13) we are forced to conclude that

$$(3.14) \quad \int_{E_1} |\mathcal{A}(\chi_E z)(\omega)|_2 d\mu = 0.$$

Clearly this is impossible and we must conclude that the set E_1 has zero measure. Whence, the support of $\mathcal{A}(\chi_E z)$ is contained in E . The conclusion of the lemma is now easily seen.

With what we have established we will now show that a bounded hermitian operator on $L^p(\Omega, K)$, $1 \leq p < \infty$, $p \neq 2$, is induced by a $\mathcal{B}(K)$ valued multiplication.

In what follows we shall assume that \mathcal{A} is a bounded hermitian operator on $L^p(\Omega, K)$, $1 \leq p < \infty$, $p \neq 2$. Thus

$$(3.15) \quad \int |\mathcal{A}(F)(\omega)|_2^p d\mu \leq \|\mathcal{A}\|^p \|F\|_p^p$$

for each $F \in L^p(\Omega, K)$. From (3.15) and (3.9) we see that for each $z \in K$ there exists a set $E_z \in \Sigma$ such that $\mu(E_z) = 0$ and

$$(3.16) \quad |\mathcal{A}(z)(\omega)|_2 \leq \|\mathcal{A}\| \|z\|_2 \quad \text{for every } \omega \in \Omega \setminus E_z.$$

Let L be a countable dense subset of K . Then from (3.16) we see that there is a set $E_0 \in \Sigma$ with $\mu(E_0) = 0$ such that for any z and z' in $L \cup \{0\}$

$$(3.17) \quad |\mathcal{A}(z - z')(\omega)|_2 \leq \|\mathcal{A}\| \|z - z'\|_2 \quad \text{for } \omega \in \Omega \setminus E_0.$$

If (z_n) is a Cauchy sequence from $L \cup \{0\}$ then it is clear from (3.17) that $(\mathcal{A}(z_n)(\omega))$ is a Cauchy sequence in K for each $\omega \in \Omega \setminus E_0$. Since L is dense any $z \in K$ is the limit of some sequence (z_n) from L . Thus we define for each $\omega \in \Omega \setminus E_0$ and $z \in K$

$$(3.18) \quad H(\omega)z = \lim \mathcal{A}(z_n)(\omega),$$

where (z_n) is a sequence from L converging to z . It is easy to see that $H(\omega)z$ is well defined and we note that for each $z \in K, H(\cdot)z$ is a weakly measurable K valued function. Furthermore, from (3.17) we have

$$(3.19) \quad |H(\omega)z|_2 \leq \| \mathcal{A} \| |z|_2 \quad \text{for } \omega \in \Omega \setminus E_0.$$

Whence $H(\cdot)z$ belongs to $L^p(\Omega, K)$ for each $z \in K$.

If $z \in K$ and (z_n) is a sequence from L which converges to z then $|H(\omega)z - \mathcal{A}(z_n)|_2^p$ converges to zero for almost all ω and since it is dominated by an integrable function it follows by Lebesgue's dominated convergence theorem that

$$(3.20) \quad \lim \| H(\cdot)z - \mathcal{A}(z_n) \|_p = 0.$$

But since $\lim \| z - z_n \|_p = 0$ we must have

$$(3.21) \quad H(\cdot)z = \mathcal{A}(z)(\cdot).$$

In view of (3.21) we have established that for each $\omega \in \Omega \setminus E_0, H(\cdot)$ is a linear transformation on K . Moreover from (3.19) we have that $H(\omega)$ is bounded for each $\omega \in \Omega \setminus E_0$ and in fact

$$(3.22) \quad \| H(\omega) \| \leq \| \mathcal{A} \| \quad \text{for } \omega \in \Omega \setminus E_0.$$

Therefore if we define $\mathcal{M}_{H(\cdot)}$ on $L^p(\Omega, K)$ by $(\mathcal{M}_{H(\cdot)}F)(\omega) = H(\omega)F(\omega)$ it will follow that $\mathcal{M}_{H(\cdot)} = \mathcal{A}$. To see this let $\psi = \sum_{i=1}^n \chi_{E_i} z_i$, where the z_i are distinct elements of K and the E_i are disjoint measurable sets. By the linearity of \mathcal{A} and the definition of $\mathcal{M}_{H(\cdot)}$ it is clear that

$$(3.23) \quad \mathcal{M}_{H(\cdot)}\psi = \mathcal{A}\psi.$$

Since vector simple functions are dense in $L^p(\Omega, K)$ it follows that

$$(3.24) \quad \mathcal{M}_{H(\cdot)}F = \mathcal{A}F \quad \text{for each } F \in L^p(\Omega, K).$$

(3.25) **THEOREM.** *Let $1 \leq p < \infty, p \neq 2$. Then \mathcal{A} is a bounded hermitian operator on $L^p(\Omega, K)$ if and only if there is a $\mathcal{B}(K)$ valued function $H(\cdot)$ defined on Ω such that*

$$(3.26) \quad H(\omega)z \text{ is weakly measurable for each } z \in K.$$

$$(3.27) \quad \| H(\omega) \| \leq \| \mathcal{A} \| \text{ almost everywhere,}$$

$$(3.28) \quad H(\omega) = H(\omega)^* \text{ almost everywhere,}$$

$$(3.29) \quad \mathcal{A}F = \mathcal{M}_{H(\cdot)}F \text{ for } F \in L^p(\Omega, K).$$

PROOF. The sufficiency is clear and in view of the preceding argument all that remains to be shown is (3.28). To this end let $z \in L$ with $|z|_2 = 1$. Then

$$(3.30) \quad [\mathcal{A}(z), z] = \int (H(\omega)z, z) d\mu.$$

Since \mathcal{A} is hermitian, there is a set $E_z \in \Sigma$ with $\mu(E_z) = 0$ such that $(H(\omega)z, z)$ is real for $\omega \in \Omega \setminus E_z$. But since L is countable it follows that there exists a set $E_0 \in \Sigma$

with $\mu(E_0) = 0$ such that for each $z \in L$

$$(3.31) \quad (H(\omega)z, z) \text{ is real for } \omega \in \Omega \setminus E_0.$$

Furthermore, since L is dense and since the inner product is continuous on both variables (3.31) holds for each $z \in K$ and hence $H(\omega) = H(\omega)^*$ for $\omega \in \Omega \setminus E_0$. This completes the proof.

4. The isometries of $L^p(\Omega, K)$

In what follows we shall assume that \mathcal{W} is a surjective isometry of $L^p(\Omega, K)$. If $T \in \mathcal{B}(K)$ we denote by \mathcal{M}_T the operator defined by $(\mathcal{M}_T F)(\omega) = T[F(\omega)]$ for $F \in L^p(\Omega, K)$. \mathcal{M}_T is a bounded linear operator and is hermitian on $L^p(\Omega, K)$ whenever $T = T^*$. We note that $\mathcal{M}_T = \mathcal{M}_{T_1} + i\mathcal{M}_{T_2}$ where $T_1 = T_1^*$, $T_2 = T_2^*$ and $T = T_1 + iT_2$. Similarly if $E \in \Sigma$ we denote by \mathcal{C}_E the operator on $L^p(\Omega, K)$ defined by

$$(4.1) \quad (\mathcal{C}_E F)(\omega) = \chi_E F(\omega).$$

The operator \mathcal{C}_E is a hermitian projection on $L^p(\Omega, K)$. It is clear that $\mathcal{W}\mathcal{C}_E\mathcal{W}^{-1}$ is a projection and by a result of Koehler and Rosenthal (1970) it is also hermitian. We can say more.

(4.2) LEMMA. For each $E \in \Sigma$ there exists a set $\Phi(E) \in \Sigma$ such that

$$\mathcal{W}\mathcal{C}_E\mathcal{W}^{-1} = \mathcal{C}_{\Phi(E)}.$$

PROOF: Let $E \in \Sigma$ with $\mu(E) > 0$. Since $\mathcal{W}\mathcal{C}_E\mathcal{W}^{-1}$ is a hermitian projection, there exists by Theorem (3.25) a $\mathcal{B}(K)$ valued function $P_E(\cdot)$ on Ω such that $P_E(\omega)$ is projection for almost all ω and $\mathcal{W}\mathcal{C}_E\mathcal{W}^{-1} = \mathcal{M}_{P_E(\cdot)}$.

Since \mathcal{C}_E commutes with every operator of the form $\mathcal{M}_{H(\cdot)}$, where $H(\cdot)$ satisfies the conditions of Theorem (3.25), and since

$$\mathcal{W}\mathcal{M}_T\mathcal{W}^{-1} = \mathcal{W}\mathcal{M}_{T_1}\mathcal{W}^{-1} + i\mathcal{W}\mathcal{M}_{T_2}\mathcal{W}^{-1},$$

it follows from (3.25) that

$$(4.3) \quad \mathcal{M}_{P_E(\cdot)}\mathcal{M}_T = \mathcal{M}_T\mathcal{M}_{P_E(\cdot)} \text{ for each } T \in \mathcal{B}(K).$$

Thus for each $T \in \mathcal{B}(K)$ and $F \in L^p(\Omega, K)$ there exists a set $E(F, T) \in \Sigma$ with $\mu(E(F, T)) = 0$ such that

$$(4.4) \quad P_E(\omega)T[F(\omega)] = T[P_E(\omega)F(\omega)] \text{ for } \omega \in \Omega \setminus E(F, T).$$

By considering constant functions and using the separability of K we can conclude from (4.4) that for each $T \in \mathcal{B}(K)$ there exists a set $E(T) \in \Sigma$ with $\mu(E(T)) = 0$ such that

$$(4.5) \quad P_E(\omega)T(z) = TP_E(\omega)(z) \text{ for each } z \in K \text{ and } \omega \in \Omega \setminus E(T).$$

Finally, since K is separable, every rank 1 operator in $\mathcal{B}(K)$ can be approximated by a sequence of rank 1 operators from a given countable class. Hence there exists $E_1 \in \Sigma$ with $\mu(E_1) = 0$ such that for every rank 1 operator

$$(4.6) \quad P_E(\omega)T(z) = TP_E(\omega)(z) \quad \text{for each } z \in K \text{ and } \omega \in \Omega \setminus E_1.$$

The important thing to note about (4.6) is that E_1 does not depend on z or T .

Now let us define $\Phi(E) = \{\omega \mid P_E(\omega) \neq 0\}$. Suppose there exists $E' \subset \Phi(E) \cap (\Omega \setminus E_1)$ with $\mu(E') > 0$ such that $P_E(\omega) \neq I$ for each $\omega \in E'$. If $\omega_1 \in E'$ then there exists z_1 and $z_2 \in K$, each with norm 1, such $P_E(\omega_1)z_1 = z_1$ and $P_E(\omega_1)z_2 = 0$. But if we define T by $Tz = (z, z_1)z_2$ we find that $P_E(\omega_1)Tz_1 = 0$ while $TP_E(\omega_1)z_1 = z_2$! This contradicts (4.6) and consequently $P_E(\omega)$ must be the identity operator on K for $\omega \in \Phi(E)$. This completes the proof.

Lamperti (1958) calls a map Φ of Σ onto Σ (defined modulo sets of measure zero) a regular set isomorphism if (i) $\Phi(\tilde{E}) = \widetilde{\Phi(E)}$, (ii) $\Phi(UE_i) = U\Phi(E_i)$ and (iii) $\mu(\Phi(E)) = 0$ if and only if $\mu(E) = 0$.

The following lemma is easy to verify.

(4.7) LEMMA. *The map $E \rightarrow \Phi(E)$ is a regular set isomorphism.*

We will now show how the preceding results enable us to obtain a representation of \mathcal{W} . To that end let $z \in K$ with $\|z\|_2 = 1$ and E a measurable set. If we define $z_E = \mathcal{C}_E z$, then

$$(4.8) \quad \mathcal{W} z_E = \mathcal{C}_{\Phi(E)} \mathcal{W}(z).$$

Since \mathcal{W} is an isometry of $L^p(\Omega, K)$ we have

$$(4.9) \quad \mu(E) = \|z_E\|_p^p = \int_{\Phi(E)} |\mathcal{W}(z)|_2^p d\mu.$$

Moreover, since the measure $\nu(E) = \mu(\Phi^{-1}(E))$ is absolutely continuous with respect to μ we have

$$(4.11) \quad \mu(E) = \nu(\Phi(E)) = \int_{\Phi(E)} \left(\frac{d\nu}{d\mu}\right) d\mu.$$

Comparing (4.9) and (4.11) we have by the uniqueness of the Radon–Nikodym derivative that there exists $E_2 \subset \Omega$ with $\mu(E_2) = 0$ such that

$$(4.12) \quad |\mathcal{W}(z)|_2^p = \left(\frac{d\nu}{d\mu}\right) \quad \text{for } \omega \in \Omega \setminus E_2.$$

By an argument analogous to that preceding Theorem (3.25) it can be shown that there exists a $\mathcal{B}(K)$ valued function $U(\cdot)$ such that $U(\omega)$ is a surjective isometry for almost all ω and

$$(4.13) \quad \mathcal{W}(z)(\omega) = \left(\frac{d\nu}{d\mu}\right)^{1/p}(\omega) U(\omega)z.$$

Now using the fact that the vector simple functions are dense in $L^p(\Omega, K)$ we have:

(4.14) **THEOREM (Cambern).** *Let \mathcal{W} be a surjective isometry of $L^p(\Omega, K)$, $1 \leq p < \infty$, $p \neq 2$. Then there exists a regular set isomorphism Φ of the σ -algebra Σ of measurable sets onto itself (defined modulo null sets), a scalar valued function h defined on Ω satisfying $|h|^p = dv/d\mu$, and a $\mathcal{B}(K)$ valued function U defined on Ω such that $U(\omega)z$ is measurable for each $z \in K$ and $U(\omega)$ is a surjective isometry of K for almost all ω such that for $F \in L^p(\Omega, K)$,*

$$(4.15) \quad (\mathcal{W}F)(\omega) = U(\omega)h(\omega)\Phi(F)(\omega).$$

Conversely, every map \mathcal{W} of this form is a surjective isometry of $L^p(\Omega, K)$.

REMARKS. Theorem (4.14) generalizes the theorem of Lamperti (1958). The proof given by Cambern does not rely on the theory of hermitian operators but does require the use of Lamperti's theorem.

Camborn asked whether a similar characterization can be given for the surjective isometries of $L^p(\Omega, B)$ for the case that B belongs to a suitable class of finite dimensional Banach spaces. The following example shows that even when B is two dimensional, the theorem does not hold.

EXAMPLE. Let (Ω, Σ, μ) be a finite measure space and consider $L^p(\Omega, l^p_2)$. Let Φ_1, Φ_2 be distinct regular set isomorphisms with $h_k = (dv_k/d\mu)^{1/p}$ where $v_k(E) = \mu(\Phi_k^{-1}(E))$ for $E \in \Sigma$ and $k = 1, 2$. If $F \in L^p(\Omega, l^p_2)$ then $F(\cdot) = (F_1(\cdot), F_2(\cdot))$. Let $(TF)(\omega) = (h_1(\omega)\Phi_1(F)(\omega), h_2(\omega)\Phi_2(F)(\omega))$. It follows from Lamperti's theorem (1958) that T is an onto isometry of $L^p(\Omega, l^p_2)$. Clearly T is not of the form (4.15).

The techniques used to prove (4.14) can be extended to the case $L^p(\Omega, B)$ when B is a separable smooth Banach space having the following property: If P is a projection such that $PT = TP$ for every hermitian T then P is the identity on B . We note, however, that this condition seems to be very restrictive. It does not hold even when $B = l^p(2)$. This possibly accounts for the existence of an isometry such as the one given in the example above. It is tempting to conjecture that this restrictive condition holds only when B is a Hilbert space. In any case it would be interesting to characterize the class of Banach spaces for which the condition holds.

5. Adjoint abelian operators on $L^p(\Omega, K)$

Stampfli (1969) introduced the notion of an adjoint abelian operator on a Banach space. An operator \mathcal{S} on a complex Banach space X is said to be adjoint abelian if there is a semi-inner product $[,]$ compatible with the norm on X such that

$$(5.1) \quad [\mathcal{S}F, G] = [F, \mathcal{S}G] \quad \text{for all } F, G \in X.$$

If X is a Hilbert space, the adjoint abelian operators coincide with the usual hermitian operators.

Stampfli (1969) left as an open question as to whether every adjoint abelian operator on a weakly complete Banach space is scalar. In Fleming and Jamison (1976) we showed that this is indeed the case for the classical spaces L^p and $\mathcal{C}(K)$, K compact metric. We now show that this is the case for $L^p(\Omega, K)$.

If \mathcal{S} is an adjoint abelian operator on $L^p(\Omega, K)$ then \mathcal{S}^2 is hermitian as well as adjoint abelian and hence there is a $\mathcal{B}(K)$ valued function $H(\cdot)$ satisfying the conditions of Theorem (3.25) such that $\mathcal{S}^2 = \mathcal{M}_{H(\cdot)}$. Since $\mathcal{M}_{H(\cdot)}$ is adjoint abelian it follows (5.1) and (3.1) that for each $G \in L^p(\Omega, K)$ there exists a set $E_G \in \Omega$ with $\mu(E_G) = 0$ and

$$(5.2) \quad \frac{|H(\omega)G(\omega)|_2}{|G(\omega)|_2} = \frac{\|\mathcal{M}_{H(\cdot)}G\|_p}{\|G\|_p} \quad \text{for } \omega \in \text{supp}(G) \setminus E_G \cup E_H,$$

where

$$E_H = \{\omega \mid H(\omega) \neq 0\}.$$

By considering constant functions and using the separability of K we can conclude from (5.2) that there exists $E_0 \subset \Omega$ with $\mu(E_0) = 0$ such that

$$(5.3) \quad |H(\omega)z|_2 = \|\mathcal{M}_{H(\cdot)}z\|_p \quad \text{for } \omega \in \Omega \setminus (E_0 \cup E_H).$$

It is now an easy matter to show that there is a fixed constant λ such that

$$(5.4) \quad |H(\omega)z|_2 = \lambda \quad \text{for every } z \in K \text{ with } |z|_2 = 1 \\ \text{and for all } \omega \in \Omega \setminus (E_0 \cup E_H).$$

Hence $\lambda^{-1}H(\omega)$ is an isometry for $\omega \in \Omega \setminus (E_0 \cup E_H)$. Furthermore, it follows from (5.2) and (5.4) that $\mu(E_H) = 0$. Therefore, if an operator is adjoint abelian on $L^p(\Omega, K)$, its square must be a multiple of an isometry. There is more that can be said.

(5.5) THEOREM. *A linear operator \mathcal{S} on $L^p(\Omega, K)$ is adjoint abelian on $L^p(\Omega, K)$, $1 < p < \infty$, $p \neq 2$ if and only if there is a real constant λ and an isometry \mathcal{W} such that $\mathcal{W}^2 = I$ and*

$$(5.6) \quad \mathcal{S} = \lambda\mathcal{W}.$$

PROOF. The sufficiency follows from Theorem 6 of Fleming and Jamison (1976). To prove the necessity, let \mathcal{S} be an adjoint abelian operator. We may suppose that $\|\mathcal{S}\| = 1$. Then by the arguments given above there exists an isometry \mathcal{W}_1 such that $\mathcal{S}^2 = \mathcal{W}_1$ and by Theorem 1 of Stampfli (1969) it follows that $\|\mathcal{S}\|^2 = \|\mathcal{S}^2\| = 1$. Thus $\|\mathcal{S}F\|_p \leq 1$ for any F in $L^p(\Omega, K)$ of norm 1. If we suppose that there exists F in $L^p(\Omega, K)$ of norm 1 such that $\|\mathcal{S}F\|_p < 1$ then we see that

$$(5.7) \quad 1 = \|\mathcal{S}^2F\|_p \leq \|\mathcal{S}\| \|\mathcal{S}F\|_p \leq \|\mathcal{S}F\|_p < 1$$

which is absurd. Therefore \mathcal{S} is an isometry. From Theorem 6 of Fleming and Jamison (1976) there exists an isometry \mathcal{W} of $L^p(\Omega, K)$ such that $\mathcal{W}^2 = I$ and $\mathcal{S} = \mathcal{W}$. This completes the proof.

The next result follows from Corollary 4(a) of Stampfli (1969).

(5.8) COROLLARY. *Every adjoint abelian operator on $L^p(\Omega, K)$ is a scalar operator.*

REMARK: It is possible to give a more complete description of adjoint abelian operators on $L^p(\Omega, K)$ using Theorem (4.14) and arguments similar to those used in Fleming and Jamison (1976).

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