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A BOUND ON THE p-LENGTH OF P-SOLVABLE GROUPS

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Abstract. Let G be a finite p-solvable group and P a Sylow p-subgroup of G. Suppose that $\gamma_{\ell(p-1)}(P) \subseteq \gamma_r(P)^{p^s}$ for $\ell(p-1) < r + s(p-1)$, then the p-length is bounded by a function depending on ℓ.

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1. Introduction. All the groups considered are finite. In the following, p will be a prime number.

A group G is p-solvable if it has a sequence of subgroups

$$G = N_1 \supset N_2 \supset \ldots \supset N_k = 1$$

such that N_{i+1} is a normal subgroup of N_i and the index $|N_i : N_{i+1}|$ is either coprime to p or a power of p. In such a case the minimal number of factors N_i/N_{i+1}, which are p-groups, is called the p-length of G. Alternatively, a group G is p-solvable if the upper series

$$1 \subseteq O_{p'}(G) \subseteq O_{p',p}(G) \subseteq O_{p',p,p'}(G) \subseteq \ldots$$

ends in G and we call the p-length of G to the number of symbols p appearing in the series. Recall that the previous series is constructed as follows: $O_{p'}(G)$ is the maximal normal p'-subgroup of G; $O_{p',p}(G)$ denotes the inverse image of the maximal normal p-subgroup of $G/O_{p'}(G)$; $O_{p',p,p'}(G)$ denotes the inverse image of the maximal normal p'-subgroup of $G/O_{p',p}(G)$ and so on.

Much is known about the p-length of p-solvable groups. For example, by a result of Hall and Higman [**4**], explicit bounds of the p-length are known in terms of the derived length or the exponent of the Sylow p-subgroup. More recently González-Sánchez and Weigel [**2**] proved that if p is odd and the elements of order p of the Sylow subgroups are contained in the (p-2)-centre of the Sylow p-subgroup, then the p-length is equal to 1. This result was generalised by Khukhro in [**5**] by proving that if the elements of order p (or 4 if $p = 2$) of the Sylow p-subgroup are contained in the ℓ-centre of the Sylow p-subgroup, then the p-length of the group is bounded in terms of ℓ. In the same paper, Khukhro [**5**] also proved that if the Sylow p-subgroup is powerful, then the p-length is 1, and he posted the question on whether this result can be generalised in the same

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way as the result for the Sylow p-central subgroup. In this short paper we will try to answer this question by giving a bound on the p-length of a p-solvable group in terms of some power-commutator conditions in the Sylow p-subgroup.

MAIN THEOREM. *Let G be a p-solvable group and P a Sylow p-subgroup of G. Suppose that*

$$\gamma_{\ell(p-1)}(P) \subseteq \gamma_r(P)^{p^s}$$

for $\ell(p-1) < r + s(p-1)$. Then the p-length of G is bounded in terms of l. This condition holds in particular when $\gamma_{\ell(p-1)}(P) \subseteq P^{p^l}$.

The notation is standard in the group theory. The subgroups $O_p(G)$ and $O_{p'}(G)$ denote the maximal normal subgroup of order a power of p and coprime to p respectively. $[N,_k M]$ denotes the commutator subgroup $[N, M, \dots, M]$, where M appears k times.

Consider now a group G and a normal elementary abelian p-group V. Let φ be the action by conjugation of G on V:

$$\varphi : G \times V \to V \ / \ (g, v) \to v^g.$$

We can regard V as a vector space over the field $\mathbb{F}_p$. In this case, the action by conjugation of G on V can be regarded as an action by linear transformation on this vector space, and we denote this action as follows:

$$(g, v) \in G \times V \to vT(g),$$

where $T(g)$ denotes the action of an element g of G on V. If we denote by 1_V the identity transformation on V, then

$$v(T(g) - 1_V) = [v, g].$$

2. Potent filtrations. Let P be a p-group, we say that a sequence of normal subgroups $(N_i)_{i=1}^k$ is a *potent filtration of type ℓ* of P if

(1) $N_i \subseteq N_j$, for all $i > j$,
(2) $N_k = 1$,
(3) $[N_i, P] \subseteq N_{i+1}$, for $i = 1, \dots, k-1$,
(3) $[N_i,_l P] \subseteq N_{i+1}^p$, for $i = 1, \dots, k-1$.

Furthermore, we say that a group N is *PF-embedded of type ℓ* in P if there is a potent filtration of P beginning at a subgroup N.

PROPOSITION 1. *Let P be a pro-p group, and let $\{N_i\}_{i=1}^k$ a potent filtration of P of type ℓ. Then:*

(1) *$[N_i^p, P] = [N_i, P]^p$ for all i,*
(2) *$[N_i, P]_{i=1,\dots k}$ is a potent filtration of type ℓ of P,*
(3) *N_i^p is a potent filtration of type ℓ of P.*

Proof. See [**1**, Proposition 3.2]. □

We continue with the case where N is PF-embedded subgroup of type $p-2$.

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PROPOSITION 2. *Let G be a p-solvable group and P a Sylow p subgroup of G. Suppose that N is a PF-embedded subgroup of type $p-2$ of P. Then $N \subseteq O_{p'p}(G)$.*

Proof. We can assume that $O_{p'}(G) = 1$. In order to simplify notations, we put $O_p(G) = V$. Modding out by the Frattini subgroup, we can also assume that V is an elementary abelian p-group. Furthermore, $V = C_G(V)$ [**3**, Theorem 6.3.2]. Therefore, G/V acts faithfully on V; so we can regard V as a $\mathbb{F}_p(G/V)$–module. Moreover, we can consider the Jordan–Holder series of V as an $\mathbb{F}_p[G/V]$-module:

$$V = V_1 \supseteq V_2 \supseteq V_3 \supseteq \ldots \supseteq V_n$$

such that V_i/V_{i+1} is simple for every i. Without loss of generality, we can take the quotient of the $\mathbb{F}_p[G/V]$–module V over the second term of Jordan–Holder series and assume that V is a simple $\mathbb{F}_p[G/V]$-module.

Take $(N_i)_{i=1}^k$ as a potent filtration starting at N. We will prove by reverse induction on i that for all i, $N_i \subseteq V$. For i large enough it is clear. Suppose now that $N_{i+1} \subseteq V$. Take $v \in V$ and $n \in N_i$. Then

$$[v, \underbrace{n, \ldots, n}_{p-2}] \in [V, \underbrace{N_i, \ldots N_i}_{p-2}] \subseteq [N_i,_{p-2} P] \subseteq N_i^p \subseteq V^p = 1.$$

Therefore, for all $v \in V$ and $n \in N_i$, we have

$$v(T(n) - 1_V)^{p-2} = 0.$$

By [**2**, Corollary 5.2] the size of Jordan blocks of $T(n)$ can only be 1. Therefore $N_i \subseteq \ker(T)$ and by [**3**, Theorem 6.3.2] $N_i \subseteq C_G(V) = V$. $\square$

In order to prove the Main Theorem, we will need a weaker result for PF-embedded subgroups of type $p-1$.

PROPOSITION 3. *Let G be a p-solvable group, P a Sylow p subgroup of G and N a PF-embedded subgroup of type $p-1$ of P. Then,*

(a) *If $p \geq 5$, then $N^p \subseteq O_{p'p}(G)$.*
(b) *If $p = 3$, then $N^{p^2} \subseteq O_{p'p}(G)$.*
(c) *If $p = 2$, then $N \subseteq O_{p'p}(G)$.*

Proof. As in the proof of Proposition 3, we can assume that $O_{p'}(G) = 1$ and $O_p(G) = V$ is a simple $\mathbb{F}_p(G/V)$-module. Note that $V = C_G(V)$ (see [**3**, Theorem 6.3.2]). Take $(N_i)_{i=1}^k$ as a potent filtration starting at N.

(a) We will prove by reverse induction on i that $N_i^p \subseteq V$. For i large enough it is clear. Let us suppose that $N_{i+1}^p \subseteq V$. Take $n^p \in N_i^p$ and $v \in V$. Then by Proposition 1,

$$[v, n^p, n^p] \in [V, N_i^p, N_i^p] \subseteq [P, N_i, N_i]^{p^2} \subseteq N_{i+1}^{p^2} \subseteq V^p = 1.$$

Therefore,

$$v(T(n^p) - 1_V)^2 = 1.$$

Then, as in Proposition 3 and since $p \geq 5$, the Jordan block size can only be 1. So $N_i^p \subseteq C_G(V) = V$.

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(b) As in (a), we prove by reverse induction on i that $N_i^{p^2} \subseteq V$. For i large enough it is clear. Let us suppose that $N_{i+1}^{p^2} \subseteq V$. Take $n^{p^2} \in N_i^{p^2}$ and $v \in V$. Then, by [**1**, Theorem 2.4],

$$[v, n^{p^2}] \in [V, N_i^{p^2}] \subseteq [V,_{p^2} N_i] \subseteq N_{i+1}^{p^3} \subseteq V^p = 1.$$

Therefore, $N_i^{p^2} \subseteq C_G(V) = V$.

(c) As in (a) we prove by reverse induction on i that $N_i \subseteq V$. For i large enough, it is clear. Let us suppose that $N_{i+1} \subseteq V$. Take $n \in N_i$ and $v \in V$. Then,

$$[n, v] \in [N_i, V] \subseteq N_{i+1}^2 \subseteq V^p = 1.$$

Therefore, $n \in C_G(V) = V$ and $N_i \subseteq V$. □

3. Proof of the main theorem. Now we introduce a family $E_{k,r}(P)$ of subgroups of a finite p-group P as in [**1**].

Definition 4. Let P be a finite p-group. For any pair k, r of positive integers, we define the subgroup

$$E_{k,r}(P) = \prod_{i+j(p-1)\geq k \text{ and } i\geq r} \gamma_i(P)^{p^j}.$$

The next theorem is a stronger version of the Main Theorem.

Theorem 5. *Let G be a p-solvable group, and let P be a Sylow p-subgroup of G. Suppose that $\gamma_{\ell(p-1)}(P) \subseteq E_{\ell(p-1)+1,1}(P)$, then the p-length of G is bounded in terms of ℓ.*

Proof. Put $E = E_{(\ell-1)(p-1),1}(P)$. By [**1**, Theorem 4.8], the subgroup E is PF-embedded of type $p-1$ in P. So $E^{p^2} \subseteq O_{p',p}(G)$.

By construction, the exponent of $P/O_{p',p}(G)$ is at most the exponent of P/E^{p^2} which is bounded by $p^{\ell+1}$. Therefore, by [4, Theorem A], the p-length of G is bounded in terms of ℓ. □

4. A question and a lemma. In Section 2, we proved that if G is a 2-solvable group, P is a Sylow 2-subgroup of P and N is a PF-embedded subgroup of P, then $N \subseteq O_{p,p'}(G)$. For the case were p is an odd prime we have only a weaker version of the result. Therefore, the following question arises naturally.

Question 1. Let G be a p-solvable group, P is a Sylow p-subgroup of P and N is a PF-embedded subgroup of P. Then $N \subseteq O_{p,p'}(G)$.

A positive answer to this question will provide an improvement of the implicit bound in Theorem 5. The following lemma could be helpful. It is a generalisation of [**3**, Theorem 6.3.2].

Lemma 6. *Let G be a p-solvable group such that $O_{p'}(G) = 1$. Let N be a normal subgroup of G such that there exists an integer l for which $[O_p(G),_l N] = 1$. Then $N \subseteq O_p(G)$.*

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Proof. Put $V = O_p(G)$, and denote by p^r the exponent of V. Note that $O_{p'}(N) = 1$ and put $M = O_{p',p}(N)$. Then by [**1**, Theorem 2.4],

$$[V, M^{p^{r+l}}] \subseteq [V, M]^{p^{r+l}} \prod_{i=1}^{r+l} [V,_{p^i} M]^{p^{r+l-i}}.$$

In the previous equation, either $p^i \geq l$ or $r + l - i \geq r$. Therefore, $[V, M^{p^{r+l}}] = 1$.

Let H be a non-trivial p'-Hall subgroup of M. Then,

$$[V, H] = [V, H^{p^{r+l}}] \subseteq [V, M^{p^{r+l}}] = 1.$$

Therefore, H centralises V and $M = H \times V$. In particular, $H \subseteq O_{p'}(G) = 1$ and therefore $O_{p',p}(N) = O_p(N)$, which implies that $N \subseteq O_p(G)$. □

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