

IV

Hot hadronic matter

10 Relativistic gas

10.1 Relation of statistical and thermodynamic quantities

The first law of thermodynamics describes the change in energy dE of a system in terms of a change in volume dV and entropy dS :

$$dE(V, S) = -P dV + T dS, \quad (10.1)$$

$$T = \left(\frac{\partial E}{\partial S} \right)_V, \quad P = - \left(\frac{\partial E}{\partial V} \right)_S. \quad (10.2)$$

The coefficients of the first law are the temperature T and the pressure P . Both can be introduced as the partial derivatives of the energy $E(V, S)$. E is a function of the extensive variables V and S , i.e., variables that increase with the size of the system. Below, we include into this consideration the baryon number, see Eq. (10.12), which is also an extensive variable.

The free energy,

$$F(V, T) \equiv E - TS, \quad (10.3)$$

is the quantity in which, as indicated, the dependence on the entropy is replaced by the dependence on temperature, an intensive variable that does not change with the size of the system. Namely,

$$dF(V, T) = dE - T dS - S dT = -P dV - S dT, \quad (10.4)$$

and, as a consequence of the transformation Eq. (10.3), we obtain in analogy to Eq. (10.2)

$$S = - \left(\frac{\partial F}{\partial T} \right)_V, \quad P = - \left(\frac{\partial F}{\partial V} \right)_T. \quad (10.5)$$

For an extensive system with $F \propto V$, a very useful relation for the entropy density σ follows from Eq. (10.5):

$$\boxed{\sigma \equiv \frac{S}{V} = \frac{\partial P}{\partial T}.} \quad (10.6)$$

Since the free energy F depends on $T = 1/\beta$, we relate it to the statistical functions introduced in chapter 4. To establish the connection in a more quantitative manner, we obtain the energy E in terms of the free energy F . We substitute S into Eq. (10.4), in order to arrive at a relation for $E(V, T)$:*

$$\begin{aligned} E &= F(V, T) + TS(V, T), \\ &= F(V, T) - T \frac{\partial}{\partial T} F(V, T) \equiv -\frac{\partial}{\partial(1/T)} \left(\frac{-F}{T} \right). \end{aligned} \quad (10.7)$$

This relation has the same form as Eq. (4.13), and we deduce that

$$F(V, T) = -\beta^{-1} [\ln Z(V, \beta) + f]. \quad (10.8)$$

The integration constant f could be a function of V . However, by definition of P , Eq. (10.5), and using Eq. (10.8),

$$P = \frac{\partial}{\partial V} (\beta^{-1} \ln Z) + T \frac{\partial f}{\partial V}. \quad (10.9)$$

In the definition of the partition function Eq. (4.14), the individual energies E_i depend on the volume V . We obtain

$$P = \frac{-\sum_i (\partial E_i / \partial V) e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} + T \frac{\partial f}{\partial V} = -\left\langle \frac{\partial E_i}{\partial V} \right\rangle + T \frac{\partial f}{\partial V}. \quad (10.10)$$

Only if $\partial f / \partial V$ vanishes does the proper relation between work and pressure arise: the work done on the system, when the volume V is decreased by dV (dV is negative), is equivalent to the mean value of the change of all the energy levels brought about by a change of the volume. A constant f is a physically irrelevant ambiguity in the relationship Eq. (10.8) between the free energy F and the canonical partition function Z , and can be discarded. Thus, Eqs. (10.8) and (10.9) read

$$\boxed{F(V, T) = -\beta^{-1} \ln Z(V, \beta), \quad \beta P = \frac{\partial \ln Z}{\partial V}, \quad \beta^{-1} = T.} \quad (10.11)$$

This well-known equation establishes the bridge between the thermal (T , F , and P) and statistical (β and $\ln Z$) quantities. The volume V is present in both formulations, but, in fact, since $\ln Z$ and F are extensive in V for infinite volumes, V in general disappears from many further considerations.

* Clearly $E(V, S)$ is not the same function as $E(V, T)$, which is indicated by stating the variables on which E depends, rather than introducing a new symbol.

We now allow for the presence of a conserved (baryon) number b , i.e., we consider $E(V, S, b)$. This necessitates the introduction of the (baryo)chemical potential μ . μ is the incremental energy cost required to change the baryon number at fixed pressure and entropy:

$$dE(V, S, b) = -P dV + T dS + \mu db, \tag{10.12}$$

$$P = - \left(\frac{\partial E}{\partial V} \right)_{S,b}, \quad T = \left(\frac{\partial E}{\partial S} \right)_{V,b}, \quad \mu = \left(\frac{\partial E}{\partial b} \right)_{V,S}. \tag{10.13}$$

The coefficients P , T , and μ are, as before, fixed in terms of the partial differentials of $E(V, S, b)$ with respect to its three variables.

The familiar generalization of the free energy Eq. (10.4), often called the thermodynamic potential, $\mathcal{F}(V, T, \mu)$, is defined by the transformation

$$\mathcal{F}(V, T, \mu) \equiv E(V, S, b) - ST - \mu b. \tag{10.14}$$

On evaluating the differentials as in Eq. (10.4), we indeed see that, as suggested on the left-hand side of Eq. (10.14), the thermodynamic potential is a function of V , T , and μ ,

$$d\mathcal{F} = -P dV - S dT - b d\mu, \tag{10.15}$$

where

$$P = - \left(\frac{\partial \mathcal{F}}{\partial V} \right)_{T,\mu}, \quad S = - \left(\frac{\partial \mathcal{F}}{\partial T} \right)_{V,\mu}, \quad b = - \left(\frac{\partial \mathcal{F}}{\partial \mu} \right)_{V,T}. \tag{10.16}$$

A series of arguments that has allowed us to establish Eq. (10.11) fixes a relation between $\mathcal{F}(V, T, \mu)$ and the grand partition function $\mathcal{Z}(V, T, \lambda)$:

$$\boxed{\mathcal{F}(V, T, \mu) = -\beta^{-1} \ln \mathcal{Z}(V, \beta, \lambda), \quad \beta = 1/T, \quad \lambda = e^{\mu/T}.} \tag{10.17}$$

The thermodynamic pair of variables (T, μ) is often used for describing the properties of \mathcal{F} , instead of the grand-canonical statistical quantities $(\beta$ and $\lambda)$. To do this, it is quite important that appropriate attention be paid to the simple relation

$$\mu = \mu(\lambda, \beta) = \beta^{-1} \ln \lambda. \tag{10.18}$$

Consider, for example, the expression for the energy. With

$$\mathcal{Z}(V, \beta, \lambda) = \tilde{\mathcal{Z}}(V, T, \mu), \tag{10.19}$$

we obtain

$$E = - \frac{d}{d\beta} \ln \mathcal{Z}(V, \beta, \lambda), \tag{10.20}$$

$$= T^2 \frac{d}{dT} \ln \tilde{\mathcal{Z}}(V, T, \mu) + \frac{d}{d\mu} \ln \tilde{\mathcal{Z}}(V, T, \mu) \left. \frac{d\mu(\lambda, \beta)}{d\beta} \right|_{\beta=T^{-1}}. \tag{10.21}$$

The second form is clearly much different from the simple statistical relation Eq. (10.20). However, given Eq. (10.18), we have

$$\left. \frac{d\mu(\lambda, \beta)}{d\beta} \right|_{\beta=T^{-1}} = -T\mu, \tag{10.22}$$

and thus the last term in Eq. (10.21) is $\mu d\mathcal{F}/d\mu$. We hence obtain a form of the the important Gibbs–Duham relation, see Eq. (10.26) below,

$$\boxed{E(V, T, \mu) = \mathcal{F}(V, T, \mu) + TS(V, T, \mu) + \mu b(V, T, \mu),} \tag{10.23}$$

where the baryon number and entropy are

$$\mathcal{F}(V, T, \mu) = -P(T, \mu)V, \tag{10.24a}$$

$$b = -\frac{d}{d\mu}\mathcal{F}(V, T, \mu) = \lambda \frac{d}{d\lambda} \ln \mathcal{Z}(V, \beta, \lambda), \tag{10.24b}$$

$$S = -\frac{d}{dT}\mathcal{F}(V, T, \mu) = \frac{d}{dT}T \ln \tilde{\mathcal{Z}}(V, T, \mu). \tag{10.24c}$$

The expression for the entropy, Eq. (10.24c), takes a much more complex form in terms of statistical variables. Namely, Eq. (10.23) implies that

$$S = \frac{1}{T}(E - \mathcal{F} - \mu b) = \ln \mathcal{Z} - \beta \frac{\partial \ln \mathcal{Z}}{\partial \beta} - (\ln \lambda)\lambda \frac{\partial \ln \mathcal{Z}}{\partial \lambda}. \tag{10.25}$$

In an extensive system, we can greatly simplify Eq. (10.23). We replace \mathcal{F} by $-PV$, Eq. (10.16), and obtain the usual form of the Gibbs–Duham relation:

$$\boxed{P = T\sigma + \mu\nu - \epsilon, \quad \sigma = \frac{S}{V}, \quad \nu = \frac{b}{V}, \quad \epsilon = \frac{E}{V}.} \tag{10.26}$$

For completeness of the discussion, we mention now two more quantities, the enthalpy $H(P, S, b)$ and the Gibbs free energy $G(P, T, b)$. To obtain these two quantities with a new mix of variables, we continue the process of replacement of variables. We recall that, at first, we moved from $E(V, S) \rightarrow F(V, T)$ and subsequently from $E(V, S, b) \rightarrow \mathcal{F}(V, T, \mu)$, i.e., we replaced the extensive variables by the intensive variables. The one extensive variable left is the volume itself. Since we address in this book an isolated system (a fireball) that can expand its volume with entropy and baryon number remaining nearly constant, elimination of V in favor of P would seem a logical step and indeed the statistical (partition-function) analog to $G(P, T, b)$ is the recently proposed generalization of the (grand) canonical partition function to the pressure partition function $\Pi(P, \beta, b)$ [142]. We will not pursue this interesting subject further in this book.

However, we note that, for a given pressure, entropy, and baryon number, it should be convenient to introduce the enthalpy,

$$\boxed{H(P, S, b) \equiv E(V, S, b) + PV.} \quad (10.27)$$

The volume occupied by the system is obtained as the change of H with respect to pressure at fixed S and b :

$$dH = V dP + T dS + \mu db. \quad (10.28)$$

In thermal physics, a quantity often considered in the study of a freely expanding isolated system is the specific enthalpy per particle, the so-called heat function h ,

$$\frac{H}{b} \equiv h = \frac{\epsilon}{\nu} + \frac{P}{\nu}. \quad (10.29)$$

The Gibbs free energy G is introduced to facilitate consideration of evolution, not at a constant entropy, but in a ‘heat bath’, i.e., at a given temperature, though at a fixed baryon number (not in a ‘baryon bath’):

$$\boxed{G(P, T, b) \equiv E + PV - TS} = \mu(P, T, b) b. \quad (10.30)$$

Both the enthalpy $H(P, S, b)$ and the Gibbs free energy $G(P, T, b)$ have not yet been used much in the study of heavy-ion-fireball dynamics.

10.2 Statistical ensembles and fireballs of hadronic matter

We extend the discussion of physical ensembles introduced in chapter 4. The concept of an ensemble consisting of weakly coupled physical systems, $\mathcal{M} = \{M_i, i = 1, \dots, N\}$, was introduced by Gibbs and Boltzmann. It helped to establish a conceptual foundation of statistical physics. A large number, $N \rightarrow \infty$, of such systems is normally considered. The otherwise negligible interactions between individual systems M_i are such that both energy and (conserved) quantum numbers (such as, e.g., the baryon number) can be exchanged between the systems. This establishes a ‘bath’ of energy and baryon number, in which each individual system is immersed, and with which it can equilibrate its properties.

When we examine the microscopic properties in the ensemble, such as energies of individual members, we speak of a *micro-canonical* ensemble. Furthermore, we distinguish between the *canonical* ensemble and the *grand-canonical* ensemble: in both cases, we have adopted a statistical distribution in energy. However, in the canonical ensemble, we still treat discrete quantum numbers (particle number, baryon number, etc.) microscopically, whereas in the grand-canonical ensemble, we have also adapted the statistical-ensemble distribution for the discrete properties

such as baryon number. Of course, when we have more than one conserved discrete quantum number, any of these properties can be treated in ‘canonical’ or ‘grand-canonical’ way.

A colloquial way to explain the difference between the ensembles is to say that, in the micro-canonical ensemble, we consider each individual system M_i as being decoupled from the others. In the canonical ensemble, we allow only for coupling of energy, and in the grand-canonical ensemble, we allow for exchange of energy and quantum number (particle number). Practically, we maintain the picture of an ensemble of many weakly coupled systems \mathcal{M} in place, but, for describing physical properties within micro-canonical, canonical, or grand-canonical ensembles, we use different physical variables.

Said differently, in a theoretical description, it is our choice how we characterize the properties of the system as long as there is a precise mathematical transformation we can use to make a transition between the different descriptions of the same physical situation, and the description of choice is what is most convenient. There is an exception to this ‘convenience principle’: the color-confining nature of strong interactions imposes color neutrality on all ‘drops’ of QGP we consider, thus, in principle, we may not use a color-grand-canonical ensemble, and, if it is used, the question to consider is that of whether results obtained in this way make good physical sense.

What is, in our context, the individual ensemble element M_i ? Can we view it as a single hadron, or do we have to take the entire drop of highly excited hadronic matter formed in the nuclear collision as being the element in the Gibbs ensemble? In chapter 4, we have wondered if a single particle can be seen as the element of the ensemble. This is motivated by the fact that, in our physical environment, the number of particles is not fixed, and their variable number is an expression of the sharing of the total energy and baryon number (or other conserved number). In support of this point of view, we will next show in section 10.3 that, allowing for a change in numbers of particles, the state of maximum entropy at fixed energy and baryon number is the conventional statistical-equilibrium distribution.

We know that there are physical processes of particle production that allow conversion of energy into particles, such that their yields reach (chemical) equilibrium. The well-known Boltzmann collision dynamics assures that the momentum distributions are equilibrated in (binary) collisions. Microscopic processes of particle production and interaction can establish, in a particle ensemble M_i , a distribution that is normally associated with ensemble elements consisting of larger drops of matter.

Each high-energy heavy-ion reaction forms a many-body system, a fireball, which evolves into a final state with thousands of particles. The

study of the average rather than individual microscopic properties of such a large system makes sense, if the distribution of individual properties of the sub-components has a ‘peaked’ shape. What this means is that most individual objects considered should be found near to their common average, just a few may be far from it. On intuitive grounds, it seems that otherwise many-body systems equilibrate exceedingly slowly, if at all. Since there are many examples of dynamic systems that do not satisfy this criterion, we see that in general there is *a priori* no guarantee that strongly interacting confined matter will ever equilibrate. However, experimental results suggest that strongly interacting particle systems of practically any size approach statistical equilibrium very rapidly. Why this is the case remains today an open issue, see section 5.6.

In the following, we develop further the physical properties of ideal relativistic gases introduced in section 4.4. We assemble useful formulas, including in the discussion chemically nonequilibrated gases, which have been treated only sparingly before.

10.3 The ideal gas revisited

The additivity of different gas fractions (i.e., flavors) ‘f’ originates from the additive property of the logarithm of the partition functions Eq. (4.19):

$$\mathcal{Z} = \prod_{\text{f}} \mathcal{Z}_{\text{f}}, \quad \ln \mathcal{Z} = \sum_{\text{f}} \ln \mathcal{Z}_{\text{f}}. \quad (10.31)$$

For an ideal Fermi gas, such as a quark gas in the deconfined phase, we have for each flavor, as seen in Eqs. (4.38) and (4.39),

$$\ln \mathcal{Z}_{\text{F}} = g_{\text{F}} V \int \frac{d^3 p}{(2\pi)^3} [\ln(1 + \gamma \lambda e^{-\beta \varepsilon}) + \ln(1 + \gamma \lambda^{-1} e^{-\beta \varepsilon})], \quad (10.32)$$

where the degeneracy factor is, e.g., $g_{\text{F}} = g_{\text{s}} g_{\text{c}}$ and comprises $g_{\text{s}} = 2$ for spin- $\frac{1}{2}$ degeneracy and $g_{\text{c}} = 3$ for color. For bosons with degeneracy g_{B} , in principle, we must allow for the possibility of condensation (macroscopic occupancy) in the lowest energy state ε_0 :

$$\begin{aligned} \ln \mathcal{Z}_{\text{B}} = & -g_{\text{B}} V \int \frac{d^3 p}{(2\pi)^3} [\ln(1 - \gamma \lambda e^{-\beta \varepsilon}) + \ln(1 - \gamma \lambda^{-1} e^{-\beta \varepsilon})] \\ & - g_{\text{B}} [\ln(1 - \gamma \lambda e^{-\beta \varepsilon_0}) + \ln(1 - \gamma \lambda^{-1} e^{-\beta \varepsilon_0})]. \end{aligned} \quad (10.33)$$

We will not address further in this book the condensation phenomena and will not pursue further the last term in Eq. (10.33).

Differentiating with respect to the energy of the particle, see Eq. (4.41), we obtain the single-particle distribution functions. For the fermions and

antifermions seen in Eq. (4.42), respectively, we have

$$f_{\text{F}}(\varepsilon, \mu) = \frac{1}{\gamma^{-1}e^{\beta(\varepsilon-\mu)} + 1}, \quad (10.34a)$$

$$\bar{f}_{\text{F}}(\varepsilon, \mu) = \frac{1}{\gamma^{-1}e^{\beta(\varepsilon+\mu)} + 1}, \quad (10.34b)$$

and similarly for bosons and antibosons:

$$f_{\text{B}}(\varepsilon, \mu) = \frac{1}{\gamma^{-1}e^{\beta(\varepsilon-\mu)} - 1}, \quad (10.35a)$$

$$\bar{f}_{\text{B}}(\varepsilon, \mu) = \frac{1}{\gamma^{-1}e^{\beta(\varepsilon+\mu)} - 1}. \quad (10.35b)$$

We will also use the short-hand notation

$$f_{\text{F,B}}^{\pm} = f_{\text{F,B}} \pm \bar{f}_{\text{F,B}}, \quad (10.36)$$

since these combinations occur in evaluations of statistical properties of gases.

The particle densities are

$$\rho_{\text{F}} \equiv \frac{N_{\text{F}}}{V} = \frac{1}{V} \lambda \frac{d}{d\lambda} \ln \mathcal{Z}_{\text{F}} = g_{\text{F}} \int \frac{d^3p}{(2\pi)^3} f_{\text{F}}^-, \quad (10.37a)$$

$$\rho_{\text{B}} \equiv \frac{N_{\text{B}}}{V} = \frac{1}{V} \lambda \frac{d}{d\lambda} \ln \mathcal{Z}_{\text{B}} = g_{\text{B}} \int \frac{d^3p}{(2\pi)^3} f_{\text{B}}^-. \quad (10.37b)$$

These distributions determine the local equilibrium particle densities, for example, the local density of quarks and antiquarks given by the integral of the Fermi distribution, Eqs. (10.34a) and (10.34b):

$$n_{\text{q}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{1 + \gamma_i^{-1} \lambda_i^{-1} e^{\varepsilon(p)/T}} \rightarrow \gamma_i \lambda_i \int \frac{d^3p}{(2\pi)^3} e^{\varepsilon(p)/T}, \quad (10.38a)$$

$$n_{\bar{\text{q}}} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{1 + \gamma_i^{-1} \lambda_i e^{\varepsilon(p)/T}} \rightarrow \gamma_i \lambda_i^{-1} \int \frac{d^3p}{(2\pi)^3} e^{\varepsilon(p)/T}. \quad (10.38b)$$

The Boltzmann limit, which is applicable when the phase-space cells have small overall occupancy, is also indicated in Eqs. (10.38a) and (10.38b). In this limit, the chemical-abundance factors enter as coefficients of the distributions. We note that, while the chemical potential enhances the abundance of particles, it suppresses the abundance of antiparticles.

In order to obtain other statistical properties, such as, e.g., the energy content of the system, one can also apply the rule that the occupation functions Eqs. (10.34a)–(10.35b) and Eq. (4.46) can be folded with

the quantity of interest. To obtain the energy density, we fold with the single-particle energy ε the sum of particle and antiparticle spectra. The correctness of this prescription is seen on evaluating the derivative of $\ln \mathcal{Z}$ with respect to β , Eq. (10.20):

$$\epsilon_B = g_B \int \frac{d^3p}{(2\pi)^3} \varepsilon f_B^+, \tag{10.39a}$$

$$\epsilon_F = g_F \int \frac{d^3p}{(2\pi)^3} \varepsilon f_F^+, \tag{10.39b}$$

$$\epsilon_g = g_g \int \frac{d^3p}{(2\pi)^3} \varepsilon f_g. \tag{10.39c}$$

The gluon distribution f_g is seen in Eq. (4.46). The total energy density is the sum of all contributing terms:

$$\epsilon = \sum_i \epsilon_i. \tag{10.40}$$

10.4 The relativistic phase-space integral

To evaluate the properties of ideal relativistic gases, we need to evaluate the relativistic momentum integral, which appears in all phase-space integrals in a similar form. To do this we consider the definition of the Bessel function K_ν ,

$$K_\nu(z) = \frac{\sqrt{\pi}(z/2)^\nu}{\Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zt} (t^2 - 1)^{\nu - \frac{1}{2}} dt, \quad \Re \nu > -\frac{1}{2}, \tag{10.41}$$

valid for $|\arg z| < \pi/2$. We used before the case $\nu = 1$, Eq. (8.7), which arises on substituting in Eq. (10.41) $t \rightarrow \cosh t$. The connection to the class of integrals which we now require is obtained by recognizing that $z = \beta m$ and substituting into Eq. (10.41):

$$t \rightarrow \sqrt{p^2 + m^2}/m. \tag{10.42}$$

With $\varepsilon = \sqrt{p^2 + m^2}$, we obtain

$$K_\nu(\beta m) = \frac{\sqrt{\pi}}{\Gamma(\nu + \frac{1}{2})} \left(\frac{\beta}{2m}\right)^\nu \int_0^\infty \frac{p^{2\nu}}{\varepsilon} e^{-\beta\varepsilon} dp. \tag{10.43}$$

On integrating by parts with the relation

$$\frac{\partial}{\partial p} e^{-\beta\varepsilon} = -\beta \frac{p}{\varepsilon} e^{-\beta\varepsilon},$$

we obtain ($\nu > \frac{1}{2}$)

$$K_\nu(\beta m) = \frac{\sqrt{\pi}}{\Gamma(\nu - 1/2)} \frac{1}{m} \left(\frac{\beta}{2m}\right)^{\nu-1} \int_0^\infty p^{2\nu-2} e^{-\beta\varepsilon} dp. \tag{10.44}$$

We recall that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}; \quad \Gamma\left(\frac{3}{2}\right) = \sqrt{\pi}/2; \quad \Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right); \quad \dots$$

Two interesting limits arise from the well known series expansion of the Bessel function Eq. (10.41).

- The non-relativistic limit in which we use p/m as the small parameter:

$$K_\nu(z) \rightarrow \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)}{2!(8z)^2} + \frac{(4\nu^2 - 1)(4\nu^2 - 9)(4\nu^2 - 25)}{3!(8z)^3} + \dots \right). \tag{10.45}$$

We note that this expansion is rather slowly convergent. The special case of interest to us is

$$\frac{K_1(z)}{K_2(z)} = 1 - \frac{3}{2} \frac{1}{z} + \frac{15}{8} \frac{1}{z^2} - \frac{15}{8} \frac{1}{z^3} + \frac{135}{128} \frac{1}{z^4} + \mathcal{O}(z^{-5}). \tag{10.46}$$

- The relativistic limit, in which the mass is negligible relative to the typical energies, and thus effectively $m \simeq 0$. For the relevant two cases, we have, for $z \rightarrow 0$,

$$K_1(z) = \frac{1}{z} + \left[\ln\left(\frac{z}{2}\right) + \gamma_E \right] \frac{z}{2} + \left[\ln\left(\frac{z}{2}\right) + \gamma_E - \frac{5}{4} \right] \frac{z^3}{16} + \dots,$$

$$K_2(z) = \frac{2}{z^2} - \frac{1}{2} - \left[\ln\left(\frac{z}{2}\right) + \gamma_E - \frac{3}{4} \right] \frac{z^2}{8} - \left[\ln\left(\frac{z}{2}\right) + \gamma_E - \frac{17}{12} \right] \frac{z^4}{96} + \dots,$$

leading to

$$\frac{K_1(z)}{K_2(z)} = \frac{z}{2} + \left[\ln\left(\frac{z}{2}\right) + \gamma_E \right] \frac{z^3}{4} + \dots. \tag{10.47}$$

We recall that

$$\gamma_E = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{k} - \ln n = 0.577\ 215\ 664\ 9 \dots$$

is the Euler constant.

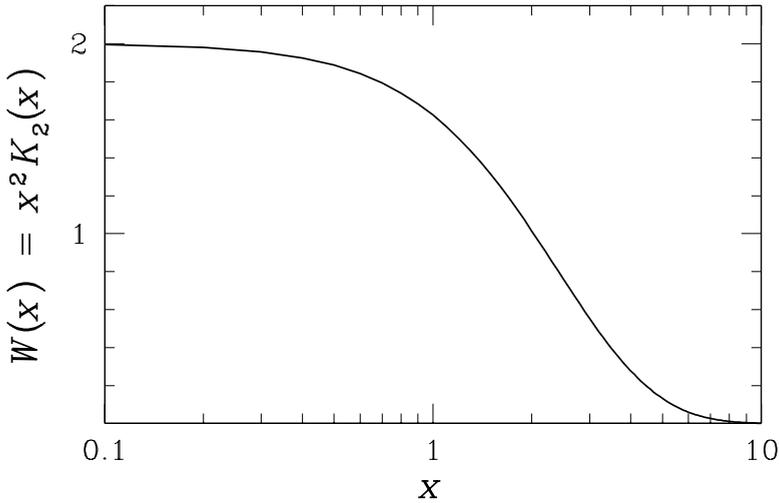


Fig. 10.1. The relativistic distribution function $W(x) = x^2 K_2$.

With this preparation, we are now in a position to study properties of relativistic gases. Let us first look at the classical limit, the first, ‘classical’ term in the expansion of the partition function Eq. (4.51). For $n = 1$, we obtain, from Eq. (4.53),

$$\ln Z_{\text{cl}} = Z^{(1)} = \sum_f \gamma_f (\lambda_f + \lambda_f^{-1}) Z_f^{(1)}, \tag{10.48}$$

with

$$Z_f^{(1)} = g_f V \int \frac{d^3 p}{(2\pi)^3} e^{-\beta \varepsilon(\mathbf{p})} \equiv g_f \frac{\beta^{-3} V}{2\pi^2} W(\beta m_f). \tag{10.49}$$

The sum, in Eq. (10.48), includes all Fermi and Bose particles.

We encounter in Eq. (10.49) the function Eq. (10.44), with $\nu = 2$,

$$W(\beta m) \equiv \beta^3 \int e^{-\beta \varepsilon} p^2 dp = (\beta m)^2 K_2(\beta m), \tag{10.50a}$$

$$\rightarrow 2, \quad \text{for } m \rightarrow 0, \tag{10.50b}$$

$$\rightarrow \sqrt{\frac{\pi m^3}{2T^3}} e^{-m/T}, \quad \text{for } m \gg T, \tag{10.50c}$$

where, in the last limit, we exploited the large-argument limit shown in Eq. (10.45). As can be seen from Fig. 10.1, the change between the two asymptotic limits occurs near $\beta m = 1$.

In the classical (Boltzmann) limit indicated by superscript ‘cl’ the number of particles of each species, Eq. (4.56), is given by

$$\boxed{N|^{cl} = \lambda \frac{\partial}{\partial \lambda} \ln \mathcal{Z}_{cl} = \lambda Z^{(1)} = g\lambda \frac{\beta^{-3}V}{2\pi^2} W(\beta m),} \tag{10.51}$$

where we have combined the factors $\gamma^{\pm 1} \lambda \rightarrow \lambda$, since, when only one particle species is considered, one fugacity suffices. The most useful and often quoted property of a relativistic gas is the average energy per particle:

$$\left. \frac{E}{N} \right|^{cl} = \frac{-(\partial/\partial\beta) \ln \mathcal{Z}_{cl}}{\lambda(\partial/\partial\lambda)\partial \ln \mathcal{Z}_{cl}} = 3T + m \frac{K_1(\beta m)}{K_2(\beta m)}. \tag{10.52}$$

The fugacity coefficients λ for the particles cancel out. To obtain Eq. (10.52), we exploited the property of the function $W = x^2 K_2$,

$$\frac{d}{dx} W(x) = -x^2 K_1(x), \tag{10.53}$$

arising from the recursion relation of the K -functions,

$$\frac{d}{dx} K_\nu(x) = -K_{\nu-1}(x) - \frac{\nu}{x} K_\nu(x), \tag{10.54a}$$

written in the form

$$\frac{d}{dx} (x^\nu K_\nu(x)) = -x^\nu K_{\nu-1}(x). \tag{10.54b}$$

- In the relativistic limit $\beta m \rightarrow 0$, we can use Eq. (10.47) and obtain

$$\boxed{\left. \frac{E}{N} \right|_{m=0}^{cl} = 3T.} \tag{10.55}$$

Equation (10.55) can be improved; see Eq. (10.68). However, further refinement in the limit $m/T \rightarrow 0$ requires that quantum statistics be considered.

- In the non-relativistic limit $\beta m \gg 1$, the ratio appearing in Eq. (10.52) is as given in Eq. (10.46), and we obtain

$$\left. \frac{E}{N} \right|_{nr}^{cl} = m + \frac{3}{2}T \left(1 + \frac{5}{4} \frac{T}{m} - \frac{5}{4} \frac{T^2}{m^2} + \frac{45}{64} \frac{T^3}{m^3} \dots \right), \quad \frac{m}{T} > 1. \tag{10.56}$$

Note that, to obtain the correct first $3T/2$ term in the non-relativistic limit, the next-to-leading term in Eq. (10.46) needs to be considered. The slow convergence of the series, Eq. (10.56), is also to be remembered, i.e., the non-relativistic limit requires a truly a non-relativistic $m \gg T$ condition. For $m \simeq T$, the relativistic limit offers a better approximation.

We continue with a more thorough discussion of the energy per baryon in the next subsection, addressing there, in particular, the differences arising for bosons and fermions.

10.5 Quark and gluon quantum gases

In the deconfined QGP phase, we have to consider the quantum nature of the effectively massless, relativistic quark and gluon gases. Many of the results for quantum gases that we require arise in terms of a series expansion of which the Boltzmann approximation is the first term. We are, in particular, interested in the properties of the equation of state, i.e., the relation between the energy density ϵ , Eq. (10.40), and the pressure P , Eq. (10.11).

Integrating by parts Eqs. (10.32) and (10.33),

$$\begin{aligned} \pm \int \frac{d^3p}{(2\pi)^3} \ln(1 \pm \gamma \lambda e^{-\beta\epsilon}) &= \frac{\beta}{3} \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \frac{\partial \epsilon}{\partial |\vec{p}|} f_{F,B} \\ &= \frac{\beta}{3} \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{\epsilon} f_{F,B}, \end{aligned} \tag{10.57}$$

where the factor $\frac{1}{3}$ arises from the $p^2 dp$ momentum integral. We have used the (relativistic-dispersion) relation Eq. (4.31) in the last equality. We obtain for the pressure Eq. (10.11), noting Eqs. (10.32), (10.33), and (10.36), and using Eq. (4.31) to eliminate the momentum,

$$3P = g_F \int \frac{d^3p}{(2\pi)^3} \left(\epsilon - \frac{m^2}{\epsilon} \right) f_F^+ + g_B \int \frac{d^3p}{(2\pi)^3} \left(\epsilon - \frac{m^2}{\epsilon} \right) f_B^+ \leq \epsilon. \tag{10.58}$$

Since the particle-occupation probabilities $f_{B,F}^+$ are always positive, the terms proportional to m^2 in Eq. (10.58) always reduce the pressure. For this reason the maximum absolute value of the ideal-gas pressure, for given thermal parameters, is subject to the relativistic bound

$$\epsilon - 3P = g_F m \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{\epsilon} \right) f_F^+ + g_B m \int \frac{d^3p}{(2\pi)^3} \left(\frac{m}{\epsilon} \right) f_B^+ \geq 0. \tag{10.59}$$

The right-hand side of Eq. (10.59) is cast into the form which is natural considering the trace of the energy-momentum tensor of quantum fields. It can be evaluated using Eq. (10.43), when the series expansion of quantum distributions exists. The leading (Boltzmann) term is

$$\epsilon - 3P = \frac{gT^4}{2\pi^2} x^3 K_1(x), \quad x = m/T. \tag{10.60}$$

For high temperatures relative to (vanishingly small) mass, we find the relativistic equation of state,

$$3P \rightarrow \epsilon, \quad \text{for } \beta m \rightarrow 0, \tag{10.61}$$

corresponding to the maximum mobility of particles. Massive particles move slowly relative to the velocity of light and are far away from this

limit. In fact, the pressure in the normal world around us is vanishingly small, seen on the scale of energy density comprising the rest mass. This is expressed in the power of 30 arising in Eq. (1.2), which separates the pressure on Earth from that in the QGP.

We turn now to consider the energy per particle in relativistic quantum gases in more detail. We employ the series expansion appearing in Eq. (4.51), and obtain, by expanding Eqs. (10.32) and (10.33) for each particle species,

$$\ln \mathcal{Z} = \frac{\beta^{-3}V}{2\pi^2} \sum_{n=1}^{\infty} g_n \frac{\lambda^n}{n^4} (n\beta m)^2 K_2(n\beta m). \tag{10.62}$$

We have combined the factor $(-)^{n+1}$ for fermions with the degeneracy g to form the factor g_n . This expansion Eq. (10.62) can not be used if the condition $m - \mu < 0$ arises. This happens, in particular, for massless quarks at finite baryon density. We will be able to deal with this interesting case exactly for $m \rightarrow 0$; see Eq. (10.74). In the HG phase, for a very narrow parameter range, allowing in particular condensation of kaons, the expression Eq. (10.62) is also not valid. Apart from these exceptions, Eq. (10.62) can be used as the basis for the evaluation of the properties of hot, strongly interacting matter.

Using the series expansion Eq. (10.62), the quantum generalization of the classical particle number, Eq. (10.51), is

$$N = \lambda \frac{\partial}{\partial \lambda} \ln \mathcal{Z} = \frac{\beta^{-3}V}{2\pi^2} \sum_{n=1}^{\infty} g_n \frac{\lambda^n}{n^3} (n\beta m)^2 K_2(n\beta m). \tag{10.63}$$

As noted, the masses and fugacities are such that $m - \mu > 0$, so that the series expansion exists. The relativistic limit $m \rightarrow 0$ is now

$$\frac{N}{V} \Big|_{m=0}^B = \frac{gT^3}{\pi^2} \zeta(3), \tag{10.64}$$

$$\frac{N}{V} \Big|_{m=0}^F = \frac{gT^3}{\pi^2} \eta(3). \tag{10.65}$$

We have introduced the Riemann zeta function

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}. \tag{10.66a}$$

We note that

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(3) \simeq 1.202, \quad \zeta(4) = \frac{\pi^4}{90}. \tag{10.66b}$$

For a Fermi occupation function, the signs of the terms in the sums in Eq. (10.63) are alternating, which leads to the eta function

$$\eta(k) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^k} = (1 - 2^{1-k})\zeta(k), \tag{10.67a}$$

and thus

$$\eta(3) = \frac{3}{4}\zeta(3) = 0.9015, \quad \eta(4) = \frac{7}{8}\zeta(4) = \frac{7}{720}\pi^4. \tag{10.67b}$$

The generalization of the energy per particle, Eq. (10.52), to quantum statistics yields

$$\boxed{\frac{E}{N} = 3T \frac{\sum_{n=1}^{\infty} g_n \frac{\lambda^n}{n^4} \left((nx)^2 K_2(nx) + \frac{1}{3}(nx)^3 K_1(nx) \right)}{\sum_{n=1}^{\infty} g_n \frac{\lambda^n}{n^3} (nx)^2 K_2(nx)},} \tag{10.68}$$

where $x = m\beta$. For the non-relativistic limit, the Boltzmann approximation Eq. (10.56) is quite appropriate, resulting in Eq. (10.56). The ultra-relativistic limit with $m \rightarrow 0$ yields

$$\left. \frac{E}{N} \right|_{m=0}^B = 3T \frac{\zeta(4)}{\zeta(3)} = 2.70 T, \tag{10.69}$$

$$\left. \frac{E}{N} \right|_{m=0}^F = 3T \frac{\eta(4)}{\eta(3)} = 3.15 T. \tag{10.70}$$

The factor $\frac{7}{8}$ seen in Eq. (10.67b) enters Eq. (10.70) and is the source of the reduction of the number of fermionic degrees of freedom in a Fermi gas compared with that in a Bose gas. However, we have to allow for the presence both of quarks and of antiquarks; thus the radiation term in the quark gas is actually $\frac{7}{4}$ times as large as that in the gluon gas, apart from the other statistical flavor and color multiplication factors.

In Fig. 10.2, we show the variation of the energy per particle in units of m , as a function of T/m . The solid line depicts the Boltzmann limit; the long-dashed line, fermions with $\lambda = 1$; and the short-dashed line, bosons. Asymptotic conditions are indicated by dotted lines corresponding to the limits Eqs. (10.55), (10.69), and (10.70). For finite mass, we see at $T \rightarrow 0$ the non-relativistic linear rise common to all three cases, Eq. (10.56). The transition from non-relativistic to relativistic behavior occurs within the entire temperature domain shown in Fig. 10.2.

Our discussion of the properties of gases of relativistic particles cannot be complete without a review of the particularly interesting case of a free

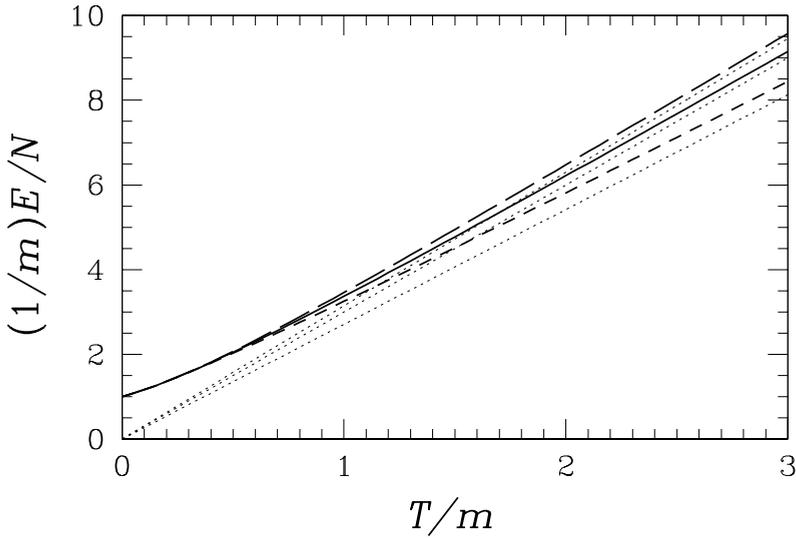


Fig. 10.2. The energy per particle in units of m , as a function of T/m . Solid line: Boltzmann limit; long-dashed line: fermions; short-dashed lines: bosons. Asymptotic conditions are indicated by dotted lines. Quantum gases are evaluated with fugacities $\gamma, \lambda = 1$.

gas of massless quarks. In particular, at finite baryon density, for which we cannot expand the Fermi distribution function in the presence of strong quantum degeneracy, this analytically soluble case offers the only practical method for studying the behavior of an ideal gas of quarks. To see this, let us assume that we are at a finite positive chemical potential, which means that there is a net number of quarks present. Fermi distributions as functions of ε/T both for particles, Eq. (10.34a), and for antiparticles (dashed), Eq. (10.34b), are shown in Fig. 10.3, for a typical situation of $\mu/T = 0.5$ (that is, $\lambda = 1.65$).

Let us restate the mathematical problem more precisely. The grand partition function of the Fermi system, Eq. (10.32), can be written, using Eq. (10.57), in the form

$$3 \frac{T}{V} \ln Z_F = g_F \int \frac{d^3 p}{(2\pi)^3} \frac{\vec{p}^2}{\varepsilon} \left(\frac{1}{e^{\beta(\varepsilon-\mu)} + 1} + \frac{1}{e^{\beta(\varepsilon+\mu)} + 1} \right). \quad (10.71)$$

Our usual series expansion would work for the momentum range such that $\varepsilon > \mu$. For the massless case, we would have to split the integral into two, and expand in a slightly different fashion. On approaching this problem in this way, one actually discovers that, for $m = 0$, the partition function can be exactly integrated.

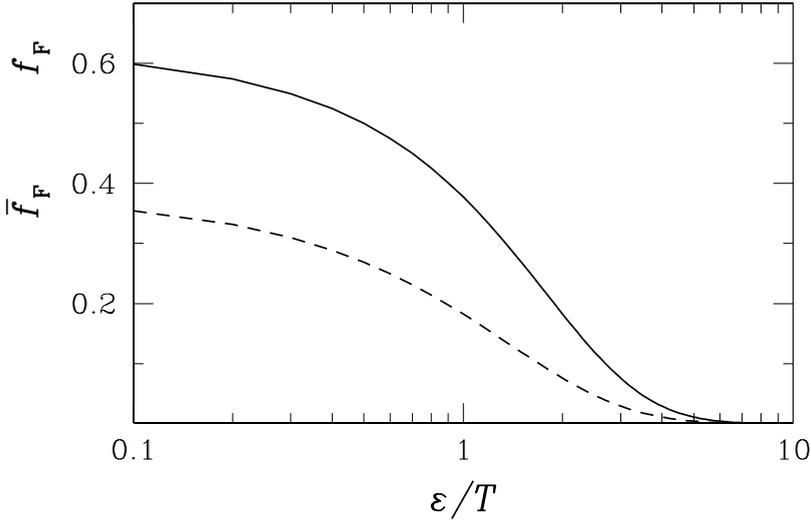


Fig. 10.3. A comparison of particle (solid) and antiparticle (dashed) Fermi distribution functions, as functions of ϵ/T for $\mu/T = 0.5$.

We substitute the arguments of f_F and \bar{f}_F with $x = \beta(\epsilon \pm \mu)$:

$$3 \frac{T}{V} \ln Z_F = \frac{g_F}{2\pi^2} T^4 \left(\int_{\beta(m-\mu)}^{\infty} dx \frac{[(x + \mu/T)^2 - (m/T)^2]^{3/2}}{e^x + 1} + \int_{\beta(m+\mu)}^{\infty} dx \frac{[(x - (\mu/T))^2 - (m/T)^2]^{3/2}}{e^x + 1} \right). \quad (10.72)$$

A systematic expansion in m/T was carried out in [113]. We consider the leading term for $m = 0$. For what follows it is important to note that, in Eq. (10.72), one of the factors under the integral is, for $m \rightarrow 0$,

$$[(x \pm \mu/T)^2 - (m/T)^2]^{3/2} \rightarrow (|x \pm \beta\mu|)^3.$$

The range of the integrals is now split to be from $\pm\beta\mu \rightarrow 0$ and from $0 \rightarrow \infty$. The final-range integrals can be recombined to give an elementary polynomial integral,

$$\begin{aligned} \int_{-\beta\mu}^0 dx \frac{|x + \beta\mu|^3}{1 + e^x} - \int_0^{\beta\mu} dx \frac{(x - \beta\mu)^3}{1 + e^x} \\ = \int_0^{\beta\mu} dx \frac{(\beta\mu - x)^3}{1 + e^{-x}} + \int_0^{\beta\mu} dx \frac{(\beta\mu - x)^3}{1 + e^x}, \\ = \int_0^{\beta\mu} dx (\beta\mu - x)^3 = \frac{(\beta\mu)^4}{4}, \end{aligned} \quad (10.73)$$

where we have changed variable from x to $-x$ in the first integrand. This term is the usual Fermi-integral contribution remaining in the limit $T \rightarrow 0$. The remaining infinite-range integral is evaluated by expansion in power series along the lines of the method shown in section 10.5, and it leads, in a straightforward way, to the first two terms in large parentheses in the following final result:

$$\ln \mathcal{Z}_F|_{m=0} = \frac{g_F V \beta^{-3}}{6\pi^2} \left(\frac{7\pi^4}{60} + \frac{\pi^2}{2} \ln^2 \lambda + \frac{1}{4} \ln^4 \lambda \right). \quad (10.74)$$

The net quark density follows immediately from Eq. (10.74):

$$\rho_q \equiv n_q - n_{\bar{q}} = 3\rho_q \frac{1}{V} \lambda \frac{\partial}{\partial \lambda} \ln \mathcal{Z}_F|_{m=0},$$

$$\rho_q = \frac{g_F \beta^{-3}}{6\pi^2} (\pi^2 \ln \lambda + \ln^3 \lambda) = \frac{g_F}{6} \left(\mu T^2 + \frac{\mu^3}{\pi^2} \right). \quad (10.75)$$

At zero temperature, the second term is the well-known expression for the degenerate Fermi gas. However, already at a modestly high temperature $T > \mu/\pi$, the first term dominates. In the range of parameters of interest to us, when $\lambda_q \simeq 1.2$ – 2.5 and $T > 140$ MeV this is always the case. The resulting proportionality of the quark (i.e., baryon) density to the chemical potential, and the accompanying quadratic temperature dependence, offer a very counterintuitive environment for a reader used to working with cold Fermi gases.

We refer to section 4.6 for the energy and pressure of the quark and gluon gases, and a more thorough discussion the properties of a QGP is given in chapter 16.

10.6 Entropy of classical and quantum gases

We consider next the single-particle entropy associated with hadrons. We recall the expressions for entropy presented in section 7.1, Eq. (7.1) for a Fermi–Bose gases and Eq. (7.2) for a Boltzmann gas. We use the Gibbs–Duhem relation Eq. (10.26) as well as the statistical-physics analog, Eq. (10.25) and obtain

$$\frac{S}{N} = \frac{PV + E}{TN} - \frac{\mu}{T} = \frac{\ln \mathcal{Z} - \beta \frac{\partial}{\partial \beta} \ln \mathcal{Z}}{\lambda \frac{\partial}{\partial \lambda} \ln \mathcal{Z}} - \ln \lambda. \quad (10.76)$$

We will consider several cases of physical interest and note that, for pions, even at a temperature $m/T \simeq 1$, the relativistic Boltzmann limit

is of interest, while the non-relativistic limit is of interest primarily for understanding the entropy of the baryon contribution.

- The classical-gas case: the partition function is, in the Boltzmann approximation, proportional to the fugacity; see, e.g., Eq. (4.40). Thus Eq. (10.76) simplifies to

$$\left. \frac{S}{N} \right|^{cl} = 1 + \frac{E}{TN} - \frac{\mu}{T} = 1 - \frac{\beta \frac{\partial}{\partial \beta} \ln \mathcal{Z}}{\lambda \frac{\partial}{\partial \lambda} \ln \mathcal{Z}} - \ln \lambda. \tag{10.77}$$

Here, $\ln \lambda = \mu/T$, with the understanding that λ , in the present context, is synonymous with the abundance fugacity γ , and in what follows for the pion gas $\mu \equiv T \ln \gamma$.

By inserting Eq. (10.52) into Eq. (10.77), we obtain

$$\left. \frac{S}{N} \right|^{cl} = 4 + \beta m \frac{K_1}{K_2} - \frac{\mu}{T}. \tag{10.78}$$

We consider first the limit $m/T \rightarrow 0$, Eq. (10.47):

$$\left. \frac{S}{N} \right|^{cl} \simeq 4 + \frac{m^2}{2T^2} + \left[\ln \left(\frac{m}{2T} \right) + \gamma_E \right] \frac{m^4}{4T^4} - \frac{\mu}{T} + \dots, \quad \frac{m}{T} \rightarrow 0. \tag{10.79}$$

An expansion suitable for the non-relativistic case, $m/T \gg 1$, can also be obtained using Eq. (10.46):

$$\left. \frac{S}{N} \right|^{cl} \simeq \frac{5}{2} + \frac{m}{T} + \frac{15}{8} \frac{T}{m} - \frac{15}{8} \frac{T^2}{m^2} + \frac{135}{128} \frac{T^3}{m^3} - \frac{\mu}{T} + \dots, \quad \frac{m}{T} \gg 1. \tag{10.80}$$

Numerical calculation shows that the Boltzmann specific entropy is monotonically falling, as shown in Fig. 10.4 by the solid line, toward the asymptotic value $S/N = 4$ (dashed line), Eq. (10.78). The two approximants, Eqs. (10.79) and (10.80), are depicted as dotted lines. Both fourth-order approximants describe the exact result well in general, except near to the physically interesting case $m/T \simeq 1$, for which the entropy per particle is $S/N = 4.4$, in the absence of a chemical potential, i.e., for a chemically equilibrated classical (Boltzmann) gas.

- The low-density nucleon gas case: we consider the non-relativistic expansion, $m/T > 1$, but we need to retain in our consideration the baryon number fugacity. Moreover, it is the entropy per baryon rather than the entropy per particle which is of interest. Recalling that in the Boltzmann approximation $\ln \mathcal{Z}_{cl} \propto (\lambda + \lambda^{-1}) = 2 \cosh(\mu/T)$, and $b = \lambda(d/d\lambda) \ln \mathcal{Z}_{cl} \propto (\lambda - \lambda^{-1}) = 2 \sinh(\mu/T)$, Eq. (10.77) now implies that

$$\left. \frac{S}{b} \right|^{cl}_N = \left(4 + \beta m \frac{K_1(\beta m)}{K_2(\beta m)} \right) \coth \left(\frac{\mu}{T} \right) - \frac{\mu}{T}. \tag{10.81}$$

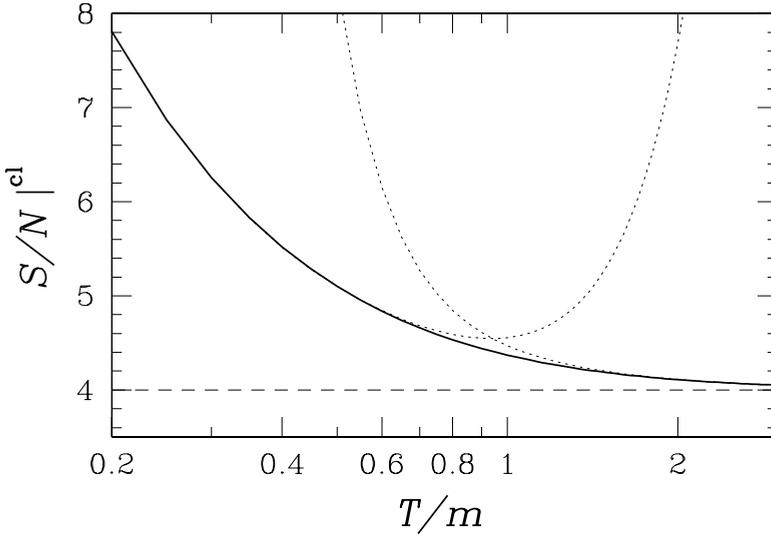


Fig. 10.4. The entropy per particle of a classical (Boltzmann) gas. Dashed line, asymptotic value $S/N = 4$; dotted line, fourth-order approximants, Eqs. (10.79) and (10.80).

Using the asymptotic expansion Eq. (10.46), we obtain

$$\left. \frac{S}{b} \right|_N^{\text{cl}} = \frac{5}{2} \left(1 + \frac{3T}{4m} - \frac{3T^2}{4m^2} + \frac{27T^3}{64m^3} + \dots \right) \coth\left(\frac{\mu}{T}\right) - \frac{\mu - m \coth(\mu/T)}{T}. \quad (10.82)$$

• We find the entropy for quantum quark and gluon gases using the Gibbs–Duhem relation in the form Eq. (10.76) and the relativistic equation of state Eq. (10.61). The entropy is

$$S|_{m=0} = \frac{4PV}{T} - \sum_f \frac{\mu_f}{T} N_f = \frac{4E}{3T} - \sum_f \frac{\mu_f}{T} N_f, \quad (10.83)$$

where the sum over different kinds of component f is implied, $E = \sum_f E_f$, etc. For each component, we obtain in the relativistic limit $m/T \ll 1$, and, for $\mu = 0$, dividing by N , and using Eqs. (10.69) and (10.70),

$$\left. \frac{S}{N} \right|_{m=0}^{\text{B}} = 4 \frac{\zeta(4)}{\zeta(3)} = 3.61, \quad (10.84)$$

$$\left. \frac{S}{N} \right|_{m=0}^{\text{F}} = 4 \frac{\eta(4)}{\eta(3)} = 4.20. \quad (10.85)$$

To obtain the complete dependence on m/T , we use Eq. (10.68). For each particle species, subject to the existence of the series representation of the integral, as addressed earlier, the result is

$$\frac{S}{N} = 4 \frac{\sum_{n=1}^{\infty} \frac{(u\lambda)^n}{n^4} \left((nx)^2 K_2(nx) + \frac{1}{4} (nx)^3 K_1(nx) \right)}{\sum_{n=1}^{\infty} \frac{(u\lambda)^n}{n^3} (nx)^2 K_2(nx)} - \ln \lambda, \quad (10.86)$$

where $x = m\beta$. $u = -1$ for fermions and $u = 1$ otherwise. For the non-relativistic limit $x > 1$, one can use Eq. (10.46) in Eq. (10.86) to obtain

$$\frac{S}{N} = \frac{\sum_{n=1}^{\infty} \frac{(u\lambda)^n}{n^4} (nx)^2 K_2(nx) I(nx)}{\sum_{n=1}^{\infty} \frac{(u\lambda)^n}{n^3} (nx)^2 K_2(nx)} - \ln \lambda, \quad (10.87)$$

$$I(nx) = \frac{5}{2} + nx + \frac{15}{8} \frac{1}{nx} - \frac{15}{8} \frac{1}{(nx)^2} + \frac{135}{128} \frac{1}{(nx)^3} + \dots, \quad (10.88)$$

which, for $n = 1$, yields the result Eq. (10.82), once we rearrange terms of two components to include particles and antiparticles and divide by the baryon number (particle–antiparticle difference).

For the case of a vanishing chemical potential, the non-relativistic Boltzmann approximation, Eq. (10.80), is quite appropriate. In Fig. 10.5, we compare the entropy per particle, evaluated at zero chemical potential ($\lambda = 1$), for the Fermi (long-dashed line), Bose (short-dashed line) and Boltzmann (solid line, see Fig. 10.4) particles.

11 Hadronic gas

11.1 Pressure and energy density in a hadronic resonance gas

We now consider the physical properties of a hadronic, confined phase, such as energy density, pressure, and abundances of various particles, assuming that we have a locally thermally and chemically equilibrated phase. Although full chemical equilibrium is most certainly not attainable in the short time of the nuclear-collision interaction, see chapter 5, this study provides very useful guidance and a reference point for understanding the properties of hadronic matter out of chemical equilibrium.

There are two ways to look at a hadronic gas: the first is that we can study its properties using the known hadronic states. This approach will