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RESEARCH ARTICLE

The time until a random walk exceeds a square root and other barriers

Sheldon M. Ross and Tianchi Zhao

Department of Industrial and Systems Engineering, University of Southern California, Los Angeles, CA, USA
Corresponding author: Tianchi Zhao; Email: tianchiz@usc.edu

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Abstract
This paper investigates the time N until a random walk first exceeds some specified barrier. Letting $X_i, i \geq 1$, be a sequence of independent, identically distributed random variables with a log-concave density or probability mass function, we derive both lower and upper bounds on the probability $P(N > n)$, as well as bounds on the expected value $E[N]$. On barriers of the form $a + b\sqrt{k}$, where a is nonnegative, b is positive, and k is the number of steps, we provide additional bounds on $E[N]$.

1. Introduction

Let $X_i, i \geq 1$, be a sequence of independent and identically distributed random variables with a density or probability mass function $f(x)$ for which $\log(f(x))$ is a concave function. Let $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. For given constants $s_n, n \geq 1$, let

$$N = \min\{k \geq 1 : S_k \geq s_k\}$$

In Section 2 we present bounds on $P(N > n)$. In Section 3, we specialize to the case $s_k = a+b\sqrt{k},\ k \geq 1$, where a is non-negative and b is positive, and present bounds on $E[N]$.

The exploration of random walk behavior with threshold boundaries has a rich history in probability theory. Blackwell and Freedman[1] presented foundational insights into exit times for sums of independent random variables. They considered a simple coin-tossing model where $X_i, i \geq 1$, take values ± 1 with probability $\frac{1}{2}$ each. Let $\tau(N, c)$ be the least $n \geq N$ with $|S_n| > cn^{\frac{1}{2}}$, where c is a constant. Their work demonstrated that $E[\tau(1, 1)]$ is infinite but when $0 < c < 1$, $E[\tau(N, c)]$ is finite for all N.

Building on these concepts, Breiman[2] investigated the asymptotic distribution of first exit times for random walks with a square root boundary, particularly examining both discrete sums of i.i.d. random variables and continuous processes like Brownian motion. Breiman's work established an approximation for the probability $P(N > n)$ as $n \to \infty$, and highlighted that while invariance principles apply for certain distributions, they may not extend to more general cases. This extension provides a framework for understanding the impact of varying boundary functions on exit time distributions.

In more recent work, Hansen[4] examined random walks reflected at general boundaries, focusing on conditions under which the global maximum remains finite almost surely. Specifically, Hansen considered random walks with light-tailed, negatively biased increments, showing that the tail of the distribution for the maximum decays exponentially.

To the best of our knowledge, this is the first paper that examines the bounds of $P(N > n)$ and particularly $E[N]$ for log-concave random walks.

2. Bounds on *P(N > n)*

Proposition 2.1.

$$P(N > n) \geq P(S_1 < s_1) \prod_{k=2}^{n} P(S_k < s_k | S_{k-1} < s_{k-1}).$$

To establish Proposition 2.1, we present some lemmas. The first one is Efron's theorem[3].

Lemma 2.2. Efron's Theorem *If $X_1, \ldots, X_r$ are independent log-concave random variables, then $(X_1, \ldots, X_r) \mid \sum_{t=1}^{r} X_t = x$ is stochastically increasing in x. That is, for any component-wise increasing function $g(x_1, \ldots, x_r)$*

$$E\left[g(X_1, \ldots, X_r) | \sum_{t=1}^{r} X_t = x\right] \text{ is increasing in } x$$

Our next lemma states that $\mathrm{S_k}$ conditional on $S_1 < s_1, \ldots, S_k < s_k$ is likelihood ratio smaller than $\mathrm{S_k}$ conditional on $S_k < s_k$.

Lemma 2.3.

$$S_k | (S_1 < s_1, \ldots, S_k < s_k) \ \leq_{lr} \ S_k | S_k < s_k.$$

Proof. We need to show that the ratio of the conditional density of S_k given $S_1 < s_1, \ldots, S_k < s_k$ to the conditional density of S_k given $S_k < s_k$ is decreasing. Now, for $t \leq s_k$

$$f_{S_k|S_1<s_1,\ldots,S_k<s_k}(t) = \frac{f_{S_k}(t) P(S_1 < s_1, \ldots, S_k < s_k | S_k = t)}{P(S_1 < s_1, \ldots, S_k < s_k)}$$

$$f_{S_k|S_k<s_k}(t) = \frac{f_{S_k}(t) P(S_k < s_k | S_k = t)}{P(S_k < s_k)} = \frac{f_{S_k}(t)}{P(S_k < s_k)}$$

Hence we need to show that $P(S_1 < s_1, \ldots, S_k < s_k | S_k = t)$ is a decreasing function of t. However, this follows from Efron's theorem because $g(x_1, \ldots, x_k) = 1 - I\{x_1 < s_1, x_1 + x_2 < s_2, \ldots, x_1 + \ldots + x_k < s_k\}$ is an increasing function of $(x_1, \ldots, x_k)$. □

Lemma 2.4.

$$P(S_k < s_k | S_1 < s_1, \ldots, S_{k-1} < s_{k-1}) \geq P(S_k < s_k | S_{k-1} < s_{k-1})$$

Proof. Because being likelihood ratio smaller implies being stochastically smaller, it follows from Lemma 2.3 that $S_{k-1} | (S_1 < s_1, \ldots, S_{k-1} < s_{k-1})$ is stochastically smaller than $S_{k-1} | S_{k-1} < s_{k-1}$. Now, if $X \leq_{st} Y$ and Z is independent of both X and Y, then $X + Z \leq_{st} Y + Z$. The result thus follows because $S_k = S_{k-1} + X_k$. □

Proof. Proposition 2.1. Proposition 2.1 follows from Lemma 2.4 upon using that

$$\begin{aligned} P(N > n) &= P(S_1 < s_1, \ldots, S_n < s_n) \\ &= P(S_1 < s_1) \prod_{k=2}^{n} P(S_k < s_k | S_1 < s_1, \ldots, S_{k-1} < s_{k-1}) \end{aligned}$$

Proposition 2.1 yields the following lower bound on $E[N]$. □

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Corollary 2.5.

$$\begin{aligned} E[N] &= \sum_{n=0}^{\infty} P(N > n) \\ &\geq 1 + P(S_1 < s_1)\big(1 + \sum_{n=2}^{\infty}\prod_{k=2}^{n} P(S_k < s_k | S_{k-1} < s_{k-1})\big) \end{aligned}$$

Remark. The log concave condition is essential for establishing Proposition 2.1. For a counterexample, suppose that $p_0 = \epsilon,\ p_2 = .2,\ p_5 = .8 - \epsilon$, where $p_j = P(X_i = j)$ and ϵ is a small positive number. Then

$$P(S_3 < 6.5 | S_1 < 1, S_2 < 5.5) = P(X_2 + X_3 < 6.5) \approx .04$$

whereas

$$P(S_3 < 6.5 | S_2 < 5.5) \approx .2$$

The conditional expectation inequality (see [6]) can be used to obtain an upper bound on $P(N > n)$.

Lemma 2.7. The Conditional Expectation Inequality. *For events $B_1, \ldots, B_n$*

$$P(\cup_{i=1}^{n} B_i) \geq \sum_{i=1}^{n} \frac{P(B_i)}{1 + \sum_{j \neq i} P(B_j | B_i)}.$$

With $B_i = A_i^c = \{S_i \geq s_i\}$, the inequality yields that

$$P(N \leq n) \geq \sum_{i=1}^{n} \frac{P^2(B_i)}{P(B_i) + \sum_{j \neq i} P(B_i B_j)}$$

Whereas for many logconcave distributions it is difficult to compute $P(S_k < s_k | S_{k-1} < s_{k-1})$, this is easily accomplished in important special cases such as normal, exponential, binomial, and Poisson. Example 2.8 considers the normal case and Example 2.9 the exponential case.

Example 2.8. Suppose the X_i are normal random variables with mean μ and variance 1. Let Z be a standard normal whose distribution function is Φ; let U be uniform on $(0, 1)$; and let $c_{n-1} = \frac{s_{n-1} - (n-1)\mu}{\sqrt{n-1}}$. Because S_{n-1} is normal with mean $(n-1)\mu$ and variance $n-1$, it follows that

$$\begin{aligned} S_{n-1} | S_{n-1} < s_{n-1} &=_{st} (n-1)\mu + \sqrt{n-1}\, Z |\ Z < c_{n-1} \\ &=_{st} (n-1)\mu + \sqrt{n-1}\, \Phi^{-1}(U) \mid \Phi^{-1}(U) < c_{n-1} \\ &=_{st} (n-1)\mu + \sqrt{n-1}\, \Phi^{-1}(U) \mid U < \Phi(c_{n-1}) \\ &=_{st} (n-1)\mu + \sqrt{n-1}\, \Phi^{-1}(U\Phi(c_{n-1})) \end{aligned}$$

Hence

$$\begin{aligned} P(S_n < s_n | S_{n-1} < s_{n-1}) &= E[E[I\{S_n < s_n\} | S_{n-1}] | S_{n-1} < s_{n-1}] \\ &= E[\Phi(s_n - \mu - S_{n-1}) | S_{n-1} < s_{n-1}] \\ &= E[\Phi(s_n - n\mu - \sqrt{n-1}\Phi^{-1}(U\Phi(c_{n-1}))] \\ &= \int_0^1 g(x) dx, \end{aligned}$$

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where $g(x) = \Phi\left(s_n - n\mu - \sqrt{n-1}\,\Phi^{-1}(x\Phi(c_{n-1})\right)$. Because Φ and Φ^{-1} are both increasing functions, it follows that $g(x)$ is a decreasing function of x. Since $\int_0^1 g(x)dx = \sum_{i=1}^r \int_{(i-1)/r}^{i/r} g(x)dx$, this shows that, for any r,

$$\frac{1}{r}\sum_{i=1}^{r} g(\frac{i-1}{r}) \geq \int_0^1 g(x)dx \geq \frac{1}{r}\sum_{i=1}^{r} g(\frac{i}{r})$$

To utilize the conditional expectation inequality we need to compute $P(S_i > s_i, S_j > s_j), i \neq j$. To do so, suppose that $i < j$, and let $c_i = \frac{s_i - i\mu}{\sqrt{i}}$. Then, with $\bar{\Phi} = 1 - \Phi$, arguing as before yields

$$\begin{aligned} S_i|S_i > s_i \quad &=_{st} \quad i\mu + \sqrt{i}\,\Phi^{-1}(U) \mid U > \Phi(c_i) \\ &=_{st} \quad i\mu + \sqrt{i}\,\Phi^{-1}\left(\Phi(c_i) + \bar{\Phi}(c_i)U\right) \end{aligned}$$

Using that $S_j|S_i$ is normal with mean $S_i + (j-i)\mu$ and variance $(j-i)^2$ yields that

$$\begin{aligned} P(S_j > s_j|S_i > s_i) &= E\big[E[I\{S_j > s_j\}|S_i]|S_i > s_i\big] \\ &= E\big[\bar{\Phi}(\frac{s_j - S_i - (j-i)\mu}{\sqrt{j-i}})|S_i > s_i\big] \\ &= E\big[\bar{\Phi}\left(\frac{s_j - j\mu - \sqrt{i}\,\Phi^{-1}\left(\Phi(c_i) + \bar{\Phi}(c_i)U\right)}{\sqrt{j-i}}\right)\big] \\ &= \int_0^1 h_{i,j}(x)dx, \end{aligned}$$

where $h_{i,j}(x) = \bar{\Phi}\left(\frac{s_j - j\mu - \sqrt{i}\,\Phi^{-1}(\Phi(c_i) + \bar{\Phi}(c_i)x)}{\sqrt{j-i}}\right)$. Because $h_{i,j}(x)$ is an increasing function of x, this gives

$$\frac{1}{r}\sum_{i=1}^{r} h_{i,j}(\frac{i}{r}) \geq \int_0^1 h(x)dx \geq \frac{1}{r}\sum_{i=1}^{r} h_{i,j}(\frac{i-1}{r}).$$

Example 2.9. Suppose the X_i are exponential random variables with rate λ. Let $N(t)$ be the number of events by time t of the Poisson process that has X_i as its i^{th} interarrival time, $i \geq 1$. Now, if $s_{n-1} \leq s_n$ then

$$\begin{aligned} P(S_n < s_n, S_{n-1} < s_{n-1}) &= P(N(s_n) \geq n, N(s_{n-1}) \geq n-1) \\ &= \sum_{i=n-1}^{\infty} P(N(s_n) \geq n|N(s_{n-1}) = i)P(N(s_{n-1}) = i) \\ &= (1 - e^{-\lambda(s_n - s_{n-1})})P(N(s_{n-1}) = n-1) + \sum_{i=n}^{\infty} P(N(s_{n-1}) = i) \\ &= P(N(s_{n-1}) \geq n-1) - e^{-\lambda(s_n - s_{n-1})}P(N(s_{n-1}) = n-1) \end{aligned}$$

giving that

$$P(S_n < s_n|S_{n-1} < s_{n-1}) = 1 - \frac{e^{-\lambda(s_n - s_{n-1})}P(N(s_{n-1}) = n-1)}{P(N(s_{n-1}) \geq n-1)}, \quad s_{n-1} \leq s_n$$

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Table 1. *Probability bounds and Monte Carlo estimates for $P(N > n)$ with $X_i \sim N(1,1)$ and $s_k = 2 + 2\sqrt{k}$ for n = 2 to 20.*

n	Lower bound	$P(N > n)$	Upper bound
2	0.977	0.977	0.977
3	0.918	0.918	0.925
4	0.825	0.826	0.851
5	0.714	0.718	0.767
6	0.596	0.609	0.682
7	0.483	0.502	0.600
8	0.381	0.409	0.524
9	0.294	0.325	0.455
10	0.222	0.254	0.394
11	0.164	0.203	0.340
12	0.120	0.153	0.294
13	0.086	0.117	0.254
14	0.061	0.088	0.219
15	0.042	0.067	0.190
16	0.029	0.050	0.165
17	0.020	0.037	0.144
18	0.014	0.028	0.126
19	0.009	0.020	0.111
20	0.006	0.015	0.098

If $s_{n-1} \geq s_n$, then $P(S_n < s_n, S_{n-1} < s_{n-1}) = P(N(s_n) \geq n)$, giving that

$$P(S_n < s_n | S_{n-1} < s_{n-1}) = \frac{P(N(s_n) \geq n)}{P(N(s_{n-1}) \geq n-1)}$$

To compute $P(S_i > s_i, S_j > s_j) = P(N(s_i) < i, N(s_j) < j)$ suppose that $i < j$. If $s_j > s_i$, conditioning on $N(s_i)$ yields

$$P(S_i > s_i, S_j > s_j) = \sum_{r=0}^{i-1} P(N(s_i) = r) P(N(s_j - s_i) < j - r)$$

If $s_i > s_j$, then

$$P(S_i > s_i, S_j > s_j) = P(S_i > s_i) = P(N(s_i) < i).$$

In Tables 1–2 and Figure 1, we present two numerical results for the probability bounds of $P(N > n)$. One assumes that X_i follows a normal distribution with mean 1 and variance 1, denoted $X_i \sim N(1,1)$. The other assumes that X_i follows an exponential distribution with rate parameter 1, denoted $X_i \sim Exp(1)$. The boundary $s_k = 2 + 2\sqrt{k}$, $k \geq 1$. For values of n ranging from 2 to 20, we calculate the lower and upper bounds for $P(N > n)$ using the analytical method described alongside Monte Carlo simulation estimates.

Whereas the bounds on $P(N > n)$ also yield bounds on $E[N]$, additional bounds for a square root barrier are given in the next section.

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Table 2. *Probability bounds and Monte Carlo estimates for* $P(N > n)$ *with* $X_i \sim Exp(1)$ *and* $s_k = 2{+}2\sqrt{k}$ *for* n = 2 *to* 20

n	Lower bound	$P(N > n)$	Upper bound
2	0.950	0.950	0.954
3	0.900	0.903	0.916
4	0.834	0.838	0.866
5	0.752	0.759	0.806
6	0.659	0.670	0.738
7	0.561	0.577	0.665
8	0.464	0.483	0.591
9	0.372	0.394	0.518
10	0.290	0.312	0.450
11	0.220	0.242	0.387
12	0.162	0.181	0.331
13	0.117	0.134	0.281
14	0.082	0.098	0.239
15	0.056	0.067	0.203
16	0.037	0.046	0.173
17	0.024	0.031	0.147
18	0.015	0.020	0.126
19	0.010	0.014	0.109
20	0.006	0.008	0.095

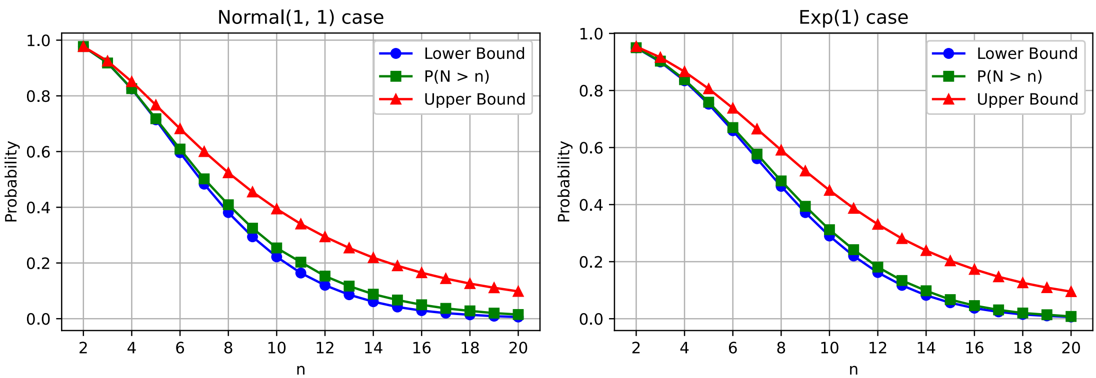

Figure: Probability bounds and Monte Carlo estimates for $P(N > n)$. $s_k = 2 + 2\sqrt{k}$, Left: $Xi \sim N(1, 1)$. Right: $Xi \sim \text{Exp}(1)$.

Figure 1. *Probability bounds and Monte Carlo estimates for* $P(N > n).s_k = 2{+}2\sqrt{k}$, *Left:* $X_i \sim N(1, 1)$. *Right:* $X_i \sim \text{Exp}(1)$.

3. Additional bounds on $E[N]$ for a square root barrier

Suppose that $s_k = a + b\sqrt{k}, k \geq 1$, where $a \geq 0, b > 0$. Also, suppose the log concave random variables X_i have a positive mean. Now, conditional on N and S_{N-1}, the random variable S_N is distributed as $a + b\sqrt{N}$ plus the amount by which X, a random variable having density f exceeds the positive value $a + b\sqrt{N} - S_{N-1}$ given that it does exceed that value. But a log concave random variable X conditioned to be positive has an increasing failure rate (see Shaked and Shanthikumar[5]) implying that

Table 3. *Comparison of Simulated* $E[N]$ *with Lower and Upper Bounds for Different Values of* a *and* b.

a	b	$\hat{E[N]}$	LB	UB_1	UB_2
1	0.5	2.574	2.522	3.556	3.179
1	1	3.566	3.447	5.031	4.381
1	2	6.699	6.316	9.408	7.914
1	5	27.453	25.648	34.539	29.397
1	10	102.596	97.698	116.299	104.525
2	0.5	3.763	3.682	5.016	4.328
2	1	4.952	4.791	6.660	5.668
2	2	8.337	7.967	11.272	9.429
2	5	29.456	27.677	36.495	31.229
2	10	104.430	99.755	118.214	106.474
5	0.5	7.159	7.026	9.002	7.673
5	1	8.721	8.510	11.021	9.344
5	2	12.818	12.383	16.211	13.687
5	5	34.994	33.313	42.023	36.492
5	10	110.295	105.786	123.847	112.223
10	0.5	12.609	12.418	15.168	13.097
10	1	14.624	14.345	17.615	15.184
10	2	19.566	19.068	23.544	20.298
10	5	43.490	41.866	50.500	44.727
10	10	119.846	115.452	132.887	121.527

$S_N - (a + b\sqrt{N})$ is stochastically smaller than $X|X > 0$. As this is true no matter what the values of N and S_{N-1}, it follows that

$$E[S_N] \leq a + bE[\sqrt{N}] + E[X|X > 0]$$

Using Wald's equation and Jensen's inequality the preceding implies that

$$\mu E[N] \leq a + b\sqrt{E[N]} + E[X|X > 0]$$

With $d = a + E[X|X > 0]$, the preceding can be written as

$$\mu E[N] - d \leq b\sqrt{E[N]}$$

If $d \leq \mu E[N]$, which can be checked using Corollary 2.5, the preceding yields that

$$\mu^2 E^2[N] + d^2 - (2d\mu + b^2)E[N] \leq 0$$

Because the function $g(x) = \mu^2 x^2 - (2d\mu + b^2)x + d^2$ is convex with $g(0) > 0, \lim_{x\to\infty} g(x) = \infty$, it follows that $g(x) < 0$ in the region between the two roots of $g(x) = 0$. Thus, $E[N]$ lies between these two roots.

Remarks. 1. If f is the normal density with mean $\mu > 0$ and variance 1, then $E[X|X > 0] = \mu + \frac{e^{-\mu^2/2}}{\sqrt{2\pi}\Phi(\mu)}$, where Φ is the standard normal distribution function.

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2. Whereas the condition $d \leq \mu E[N]$ involves the unknown $E[N]$, it can often be verified by showing that $d \leq \mu\, LB,$ where LB is the lower bound for $E[N]$ given by Corollary 2.5. (Of course, it is possible that $d \leq \mu E[N]$ but $d > \mu\, LB$).

In Table 3, we give the numerical results of the lower and upper bounds of $E[N]$ and compare them with the Monte Carlo estimate of $E[N]$. In this case, $X_i \sim N(1, 1)$ and $s_k = a + b\sqrt{k}, k \geq 1$.

Let $E[\hat{N}]$ be the Monte Carlo estimate of $E[N]$, and let LB denote the lower bound in Corollary 2.5. Since the smaller root of $\mu^2 x^2 - (2d\mu + b^2)x + d^2 = 0$ does not yield a good result, it is excluded. UB_1 is the upper bound by conditional expectation inequality, and UB_2 is the larger root. The results are as follows.

Remark. From the numerical results across all cases shown in Table 3, UB_2 is consistently smaller than UB_1.

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