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DUALIZING COMPLEX OF A TORIC FACE RING

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Abstract. A *toric face ring*, which generalizes both Stanley-Reisner rings and affine semigroup rings, is studied by Bruns, Römer and their coauthors recently. In this paper, under the "normality" assumption, we describe a dualizing complex of a toric face ring R in a very concise way. Since R is not a graded ring in general, the proof is not straightforward. We also develop the squarefree module theory over R, and show that the Cohen-Macaulay, Buchsbaum, and Gorenstein* properties of R are topological properties of its associated cell complex.

§1. Introduction

Stanley-Reisner rings and (normal) affine semigroup rings are important subjects of combinatorial commutative algebra. The notion of *toric face rings*, which originated in Stanley [12], generalizes both of them, and has been studied by Bruns, Römer, and their coauthors recently (e.g. [2], [5], [8]). Contrary to Stanley-Reisner rings and affine semigroup rings, a toric face ring does not admit a nice multi-grading in general. So, even if the results can be easily imagined from these classical examples, the proofs sometimes require technical argument.

Now we start the definition of a toric face ring. Let $\mathcal{X}$ be a finite cell complex with $\emptyset \in \mathcal{X}$. Assume that the closure $\overline{\sigma}$ of each i-cell $\sigma \in \mathcal{X}$ is homeomorphic to an i-dimensional ball, and for given two cells $\sigma, \tau \in \mathcal{X}$ there exists $\upsilon \in \mathcal{X}$ with $\overline{\sigma} \cap \overline{\tau} = \overline{\upsilon}$ (we allow the case $\upsilon = \emptyset$). A simplicial complex and the cell complex associated with a polytope are examples of our $\mathcal{X}$.

We assign a pointed polyhedral cone $C_\sigma \subset \mathbb{R}^{d_\sigma}$ to each $\sigma \in \mathcal{X}$ so that the following condition is satisfied. (We say a cone is pointed if it contains no line.)

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(∗) $\dim C_\sigma = \dim \sigma + 1$, and there is a one-to-one correspondence between $\{\text{faces of } C_\sigma\}$ and $\{\tau \in \mathcal{X} \mid \tau \subset \overline{\sigma}\}$. The face of C_σ corresponding to τ is isomorphic to C_τ by a map $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$. These maps satisfy $\iota_{\sigma,\sigma} = \mathrm{id}_{C_\sigma}$ and $\iota_{\sigma,\tau} \circ \iota_{\tau,\upsilon} = \iota_{\sigma,\upsilon}$ for all $\sigma, \tau, \upsilon \in \mathcal{X}$ with $\overline{\sigma} \supset \overline{\tau} \supset \upsilon$.

For example, a pointed fan (i.e., a fan consisting of pointed cones) gives such a structure. Here $\iota_{\sigma,\tau}$'s are inclusion maps, and $\mathcal{X}$ is a "cross-section" of the fan.

Next we define a *monoidal complex* $\mathcal{M}$ supported by $\{C_\sigma\}_{\sigma\in\mathcal{X}}$ as follows.

(∗∗) To each $\sigma \in \mathcal{X}$, we assign a finitely generated additive submonoid $\mathbf{M}_\sigma \subset (\mathbb{Z}^{d_\sigma} \cap C_\sigma) \subset \mathbb{R}^{d_\sigma}$ with $\mathbb{R}_{\geq 0}\mathbf{M}_\sigma = C_\sigma$. For $\sigma, \tau \in \mathcal{X}$ with $\overline{\sigma} \supset \tau$, the map $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ induces an isomorphism $\mathbf{M}_\tau \cong \mathbf{M}_\sigma \cap \iota_{\sigma,\tau}(C_\tau)$ of monoids.

If Σ is a rational pointed fan in $\mathbb{R}^n$, then $\{\mathbb{Z}^n \cap C\}_{C\in\Sigma}$ gives a monoidal complex.

For a monoidal complex $\mathcal{M}$ on a cell complex $\mathcal{X}$, we set $|\mathcal{M}| := \varinjlim_{\sigma\in\mathcal{X}} \mathbf{M}_\sigma$, where the direct limit is taken with respect to $\iota_{\sigma,\tau} : \mathbf{M}_\tau \to \mathbf{M}_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\overline{\sigma} \supset \tau$. If $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$, then $|\mathcal{M}|$ can be identified with $\bigcup_{\sigma\in\mathcal{X}} \mathbf{M}_\sigma \subset \mathbb{Z}^n$. The $\Bbbk$-vector space

$$\Bbbk[\mathcal{M}] := \bigoplus_{a\in|\mathcal{M}|} \Bbbk\, t^a,$$

with the multiplication

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if } a, b \in \mathbf{M}_\sigma \text{ for some } \sigma \in \mathcal{X}; \\ 0 & \text{otherwise,} \end{cases}$$

has a $\Bbbk$-algebra structure. We call $\Bbbk[\mathcal{M}]$ the toric face ring of $\mathcal{M}$. If $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$, then $\Bbbk[\mathcal{M}]$ has a natural $\mathbb{Z}^n$-grading. However, this is not true in general (cf. Example 2.9 below).

Example 1.1. (1) Let Δ be a simplicial complex. Attaching the monoid $\mathbb{N}^{i+1}$ to each i-dimensional face of Δ, we get a monoidal complex $\mathcal{M}$ on Δ. In this case, $\Bbbk[\mathcal{M}]$ coincides with the Stanley-Reisner ring $\Bbbk[\Delta]$. An affine semigroup ring is also a toric face ring corresponding to the case when $\mathcal{X}$ has a unique maximal cell.

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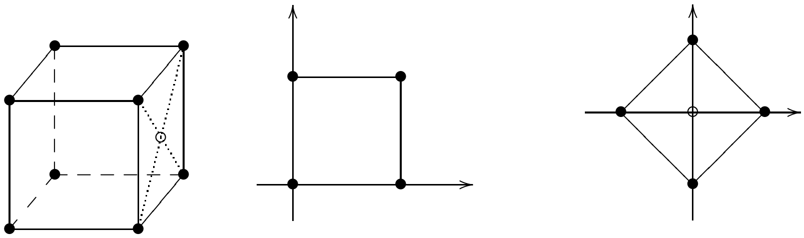
(2) Let $\mathcal{X}$ be a two-dimensional cell complex given by the boundary of a cube. Assigning normal semigroup rings of the form $\Bbbk[x,y,z,w]/(xz-yw)$ to all two-dimensional cells, we get a toric face ring $\Bbbk[\mathcal{M}]$. This $\mathcal{M}$ comes from a fan, and $\Bbbk[\mathcal{M}]$ has a $\mathbb{Z}^3$-grading with $\mathbf{M}_\sigma = \mathbb{Z}^3 \cap C_\sigma$ for all $\sigma \in \mathcal{X}$. (Find such a grading explicitly.) Next, we assign $\Bbbk[x,y,z,w]/(xz-yw)$ to 5 two-dimensional cells and $\Bbbk[x,y,z,w,v]/(xz-v^2, yw-v^2)$ to the 6^{th} one. Then we get a toric face ring $\Bbbk[\mathcal{M}']$, which is observed in [2, pp. 6–7]. While $\Bbbk[\mathcal{M}']$ admits a $\mathbb{Z}^3$-grading and all $\Bbbk[\mathbf{M}'_\sigma]$ is normal, it is impossible to satisfy $\mathbf{M}'_\sigma = \mathbb{Z}^3 \cap C_\sigma$ simultaneously for all σ. A toric face ring without multi-grading is given in Example 2.9.



$\Bbbk[x,y,z,w]/(xz-yw)$ $\quad$ $\Bbbk[x,y,z,w,v]/(xz-v^2, yw-v^2)$

The affine semigroup ring $\Bbbk[\mathbf{M}_\sigma] := \bigoplus_{a \in \mathbf{M}_\sigma} \Bbbk\, t^a$ can be regarded as a quotient ring of a toric face ring $R := \Bbbk[\mathcal{M}]$. In the rest of this section, we assume that $\Bbbk[\mathbf{M}_\sigma]$ is normal for all $\sigma \in \mathcal{X}$, and set $d := \dim R = \dim \mathcal{X} + 1$.

Theorem 1.2. *In the above situation, the cochain complex $I_R^\bullet$ given by*

$$I_R^{-i} := \bigoplus_{\substack{\sigma \in \mathcal{X}, \\ \dim \sigma = i-1}} \Bbbk[\mathbf{M}_\sigma], \quad I_R^\bullet : 0 \longrightarrow I_R^{-d} \longrightarrow I_R^{-d+1} \longrightarrow \cdots \longrightarrow I_R^0 \longrightarrow 0,$$

and

$$\partial : I_R^{-i} \supset \Bbbk[\mathbf{M}_\sigma] \ni 1_\sigma \longmapsto \sum_{\substack{\dim \Bbbk[\tau] = i-1, \\ \tau \subset \overline{\sigma}}} \pm 1_\tau \in \bigoplus_{\substack{\dim \Bbbk[\tau] = i-1, \\ \tau \subset \overline{\sigma}}} \Bbbk[\mathbf{M}_\tau] \subset I_R^{-i+1}$$

is quasi-isomorphic to a normalized dualizing complex $D_R^\bullet$ of R. Here the sign $\pm$ is given by an incidence function of the regular cell complex $\mathcal{X}$.

Clearly, our $I_R^\bullet$ is analogous to the complex constructed in Ishida [9], but, since we assume that all $\Bbbk[\mathbf{M}_\sigma]$ are normal, we do not have to take the (graded) injective hull of $\Bbbk[\mathbf{M}_\sigma]$. If $\mathcal{M}$ comes from a fan in $\mathbb{R}^n$, the above theorem has been obtained in [8, Theorem 5.1] using the $\mathbb{Z}^n$-grading of R.

We also introduce the notion of $\mathbb{Z}\mathcal{M}$-graded R-modules. Since R is not a graded ring, these are not graded modules in the usual sense, but we can consider their "Hilbert functions". In particular, Corollary 6.3, which recaptures a result of [1], gives a formula on the Hilbert function of the local cohomology module $H^i_{\mathfrak{m}}(R)$ at the maximal ideal $\mathfrak{m} := (t^a \mid 0 \neq a \in |\mathcal{M}|)$.

In [14], [16], the second author defined *squarefree modules* M over a normal semigroup ring $\Bbbk[\mathbf{M}_\sigma]$, and gave corresponding constructible sheaves M^+ on the closed ball $\overline{\sigma}$. We can extend this to a toric face ring R, that is, we define squarefree R-modules and associate constructible sheaves on $\mathcal{X}$ with them. In this context, the duality $\mathrm{RHom}_R(-, I_R^\bullet)$ on the derived category of squarefree R-modules corresponds to Poincaré-Verdier duality on the derived category of constructible sheaves on $\mathcal{X}$. For example, the complex $I_R^\bullet$ consists of squarefree modules, and $(I_R^\bullet)^+$ is the Verdier's dualizing complex of the underlying topological space of $\mathcal{X}$.

Corollary 1.3. *The Buchsbaum property, Cohen-Macaulay property and Gorenstein* property are topological properties of the underlying space of $\mathcal{X}$.*

While some parts/cases of Corollary 1.3 have been obtained in existing papers, our argument gives systematic perspective.

§2. Toric face rings

First, we shall recall the definition of a regular cell complex: A *finite regular cell complex* (cf. [4, Section 6.2]) is a topological space X together with a finite set $\mathcal{X}$ of subsets of X such that the following conditions are satisfied:

(1) $\emptyset \in \mathcal{X}$ and $X = \bigcup_{\sigma \in \mathcal{X}} \sigma$;

(2) the subsets $\sigma \in \mathcal{X}$ are pairwise disjoint;

(3) for each $\sigma \in \mathcal{X}$, $\sigma \neq \emptyset$, there exists some $i \in \mathbb{N}$ and a homeomorphism from an i-dimensional ball $\{x \in \mathbb{R}^i \mid \|x\| \leq 1\}$ to the closure $\overline{\sigma}$ of σ which maps $\{x \in \mathbb{R}^i \mid \|x\| < 1\}$ onto σ.

(4) For any $\sigma \in \mathcal{X}$, the closure $\overline{\sigma}$ can be written as the union of some cells in $\mathcal{X}$.

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An element $\sigma \in \mathcal{X}$ is called a *cell*. We regard $\mathcal{X}$ as a poset with the order $>$ defined as follows; $\sigma \geq \tau$ if $\overline{\sigma} \supset \tau$. If $\overline{\sigma}$ is homeomorphic to an i-dimensional ball, we set $\dim \sigma = i$. Here $\dim \emptyset = -1$. Set $\dim X = \dim \mathcal{X} := \max\{\dim \sigma \mid \sigma \in \mathcal{X}\}$.

Let $\sigma, \tau \in \mathcal{X}$. If $\dim \sigma = i+1$, $\dim \tau = i-1$ and $\tau < \sigma$, then there are exactly two cells $\sigma_1, \sigma_2 \in \mathcal{X}$ between τ and σ. (Here $\dim \sigma_1 = \dim \sigma_2 = i$.) A remarkable property of a regular cell complex is the existence of an *incidence function* ε satisfying the following conditions.

(1) To each pair (σ, τ) of cells, ε assigns a number $\varepsilon(\sigma, \tau) \in \{0, \pm 1\}$.
(2) $\varepsilon(\sigma, \tau) \neq 0$ if and only if $\dim \tau = \dim \sigma - 1$ and $\tau < \sigma$.
(3) If $\dim \sigma = i+1$, $\dim \tau = i-1$ and $\tau < \sigma_1$, $\sigma_2 < \sigma$, $\sigma_1 \neq \sigma_2$, then we have
$$\varepsilon(\sigma, \sigma_1)\, \varepsilon(\sigma_1, \tau) + \varepsilon(\sigma, \sigma_2)\, \varepsilon(\sigma_2, \tau) = 0.$$

We can compute the (co)homology groups of X using the cell decomposition $\mathcal{X}$ and an incidence function ε.

EXAMPLE 2.1. We shall give two typical examples of a finite regular cell complex: one is associated with a simplicial complex Δ on the vertex set $[n] := \{1, \ldots, n\}$, i.e., a subset of the power set $2^{[n]}$ such that, for $F, G \in 2^{[n]}$, $F \subset G$ and $G \in \Delta$ imply $F \in \Delta$. Take its geometric realization $\|\Delta\|$, and let ρ be the map giving the realization (see [4] for the definition of a geometric realization). Then $X := \|\Delta\|$ together with $\{\text{rel-int}(\rho(F)) \mid F \in \Delta\}$ is a regular cell complex, where $\text{rel-int}(\rho(F))$ denotes the relative interior of $\rho(F)$.

The other example is a polytope P. In this case, P itself is the underlying topological space; the cells are the relative interiors of its faces.

DEFINITION 2.2. A *conical complex* consists of the following data.

(1) A finite regular cell complex $\mathcal{X}$ satisfying the intersection property, i.e., for $\sigma, \tau \in \mathcal{X}$, there is a cell $\upsilon \in \mathcal{X}$ such that $\overline{\upsilon} = \overline{\sigma} \cap \overline{\tau}$;
(2) A set Σ of finitely generated cones $C_\sigma \subset \mathbb{R}^{\dim \sigma + 1}$ with $\sigma \in \mathcal{X}$ and $\dim C_\sigma = \dim \sigma + 1$.
(3) An injection $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$ satisfying the following.
    (a) $\iota_{\sigma,\tau}$ can be lifted up to a linear map $\mathbb{R}^{\dim \tau + 1} \to \mathbb{R}^{\dim \sigma + 1}$.

(b) The image $\iota_{\sigma,\tau}(C_\tau)$ is a face of C_σ. Conversely, for a face C' of C_σ, there is a sole cell τ with $\tau \leq \sigma$ such that $\iota_{\sigma,\tau}(C_\tau) = C'$. Thus we have a one-to-one correspondence between {faces of C_σ} and $\{\tau \in \mathcal{X} \mid \tau \leq \sigma\}$.

(c) $\iota_{\sigma,\sigma} = \mathrm{id}_{C_\sigma}$ and $\iota_{\sigma,\tau} \circ \iota_{\tau,\upsilon} = \iota_{\sigma,\upsilon}$ for $\sigma, \tau, \upsilon \in \mathcal{X}$ with $\sigma \geq \tau \geq \upsilon$.

We denote this structure by $(\Sigma, \mathcal{X})$ or Σ simply.

Remark 2.3. (1) We have $\varnothing \in \mathcal{X}$ according to the definition of a regular cell complex, and the corresponding cone $C_\varnothing$ is $\{0\}$. Thus for a conical complex $(\Sigma, \mathcal{X})$, each $C_\sigma \in \Sigma$ is *pointed*, i.e., $\{0\}$ is a face of C_σ.

(2) The concept of conical complexes was first defined by Bruns-Koch-Römer [5] in a slightly different manner, but, under the additional condition that each cone is pointed, their definition is equivalent to ours. That is, our conical complexes are *pointed* conical complexes of [5].

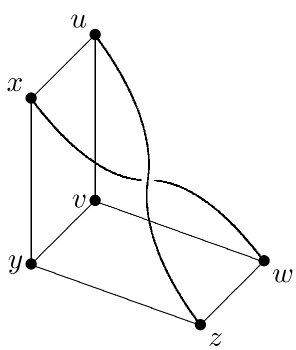
For grasping the image of a conical complex $(\Sigma, \mathcal{X})$, it is helpful to regard the conical complex as the object given by "gluing" each cones along the injections $\iota_{\sigma,\tau}$. A typical example of a conical complex is a pointed fan, i.e., a finite collection Σ of pointed cones in $\mathbb{R}^n$ satisfying the following properties:

(1) for $C' \subset C \in \Sigma$, C' is a face of C if and only if $C' \in \Sigma$;

(2) for $C, C' \in \Sigma$, $C \cap C'$ is a common face of C and C'.

In this case, as an underlying cell complex, we can take $\{\text{rel-int}(C \cap \mathbb{S}^{n-1}) \mid C \in \Sigma\}$, where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, and the injections ι are inclusion maps.

Example 2.4. There exists a conical complex which is not a fan. In fact, consider the Möbius strip as follows.



Regarding each rectangles as the cross-sections of 3-dimensional cones, we have a conical complex that is not a fan (see [3]).

A monoidal complex plays a role similar to the defining semigroup of an affine semigroup ring.

DEFINITION 2.5. ([5]) A *monoidal complex* $\mathcal{M}$ supported by a conical complex $(\Sigma, \mathcal{X})$ is a set of monoids $\{\mathbf{M}_\sigma\}_{\sigma \in \mathcal{X}}$ with the following conditions:

(1) $\mathbf{M}_\sigma \subset \mathbb{Z}^{\dim \sigma + 1}$ for each $\sigma \in \mathcal{X}$, and it is a finitely generated additive submonoid (so $\mathbf{M}_\sigma$ is an affine semigroup);
(2) $\mathbf{M}_\sigma \subset C_\sigma$ and $\mathbb{R}_{\geq 0}\mathbf{M}_\sigma = C_\sigma$ for each $\sigma \in \mathcal{X}$ (hence the cone C_σ is automatically rational);
(3) for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$, the map $\iota_{\sigma,\tau} : C_\tau \to C_\sigma$ induces an isomorphism $\mathbf{M}_\tau \cong \mathbf{M}_\sigma \cap \iota_{\sigma,\tau}(C_\tau)$ of monoids.

For example, let Σ be a rational pointed fan in $\mathbb{R}^n$. Then $\{C \cap \mathbb{Z}^n \mid C \in \Sigma\}$ gives a monoidal complex. More generally, a family of affine semigroups $\{\mathbf{M}_C \subset \mathbb{Z}^n \mid C \in \Sigma\}$ satisfying the following conditions, forms a monoidal complex;

(1) $\mathbb{R}_{\geq 0}\mathbf{M}_C = C$ for each $C \in \Sigma$;
(2) $\mathbf{M}_C \cap C' = \mathbf{M}_{C'}$ for $C, C' \in \Sigma$ with $C' \subset C$.

Remark 2.6. (1) In [2, §2], basic properties of a *rational polyhedral complex*, which gives a conical complex and a monoidal complex in a natural way, are discussed.

(2) Even if a regular cell complex $\mathcal{X}$ satisfies the intersection property, there does not exist a conical complex of the form $(\Sigma, \mathcal{X})$ in general. For example, there is a simplicial complex Δ such that the geometric realization $\|\Delta\|$ is homeomorphic to a 3-dimensional sphere, but Δ is not the boundary complex of any (4-dimensional) polytope. See, for example, [19, Notes of Chap. 8]. Now take a 4-dimensional ball, and let σ be its interior. Triangulating the boundary of the ball, which is a 3-dimensional sphere, according to Δ, we obtain the cell complex $\mathcal{X} := \Delta \cup \{\sigma\}$ such that $\sigma > \tau$ for all $\tau \in \Delta$. If there is a conical complex of the form $(\Sigma, \mathcal{X})$, then the boundary complex of a cross section of the cone $C_\sigma \in \Sigma$ coincides with Δ. This is a contradiction.

On the other hand, for any 2-dimensional regular cell complex $\mathcal{X}$ satisfying the intersection property, there is a conical complex $(\Sigma, \mathcal{X})$ and a monoidal complex $\mathcal{M}$ supported by it as follows.

Let $n \geq 3$ be an integer. It is an easy exercise to construct an affine semigroup $\mathbf{M}_n \subset \mathbb{N}^3$ satisfying the following conditions.

(i) The cone $C := \mathbb{R}_{\geq 0}\mathbf{M}_n \subset \mathbb{R}^3$ has exactly n extremal rays, that is, its cross section is an n-gon.

(ii) For any 2-dimensional face F of C, we have $F \cap \mathbf{M}_n \cong \mathbb{N}^2$ as monoids.

For a 2-dimensional cell $\sigma \in \mathcal{X}$, set $n(\sigma) := \#\{\tau \mid \tau \leq \sigma, \dim \tau = 1\}$. By the intersection property of $\mathcal{X}$, we have $n(\sigma) \geq 3$. The assignment $\mathbf{M}_\sigma := \mathbf{M}_{n(\sigma)}$ for each 2-dimensional cell σ gives a monoidal complex on $\mathcal{X}$.

For a conical complex $(\Sigma, \mathcal{X})$ and a monoidal complex $\mathcal{M}$ supported by Σ, we set

$$|\mathcal{M}| := \varinjlim_{\sigma \in \mathcal{X}} \mathbf{M}_\sigma, \quad |\mathbb{Z}\mathcal{M}| := \varinjlim_{\sigma \in \mathcal{X}} \mathbb{Z}\mathbf{M}_\sigma,$$

where the direct limits are taken with respect to the inclusions $\iota_{\sigma,\tau} : \mathbf{M}_\tau \to \mathbf{M}_\sigma$ and induced map $\mathbb{Z}\mathbf{M}_\tau \to \mathbb{Z}\mathbf{M}_\sigma$ respectively, for $\sigma, \tau \in \mathcal{X}$ with $\sigma \geq \tau$.

Let $a, b \in |\mathbb{Z}\mathcal{M}|$. If there is some $\sigma \in \mathcal{X}$ with $a, b \in \mathbb{Z}\mathbf{M}_\sigma$, by the intersection property of $\mathcal{X}$, there is a unique minimal cell among these σ's. Hence we can define $a \pm b \in |\mathbb{Z}\mathcal{M}|$.

Definition 2.7. ([5]) Let $(\Sigma, \mathcal{X})$ be a conical complex, $\mathcal{M}$ a monoidal complex supported by Σ, and $\Bbbk$ a field. Then the $\Bbbk$-vector space

$$\Bbbk[\mathcal{M}] := \bigoplus_{a \in |\mathcal{M}|} \Bbbk\, t^a,$$

where t is a variable, equipped with the following multiplication

$$t^a \cdot t^b = \begin{cases} t^{a+b} & \text{if } a, b \in \mathbf{M}_\sigma \text{ for some } \sigma \in \mathcal{X}; \\ 0 & \text{otherwise,} \end{cases}$$

has a $\Bbbk$-algebra structure. We call $\Bbbk[\mathcal{M}]$ the *toric face ring* of $\mathcal{M}$ over $\Bbbk$.

It is easy to see that $\dim R = \dim \mathcal{X} + 1$. When Σ is a rational pointed fan, $\Bbbk[\mathcal{M}]$ coincides with a toric face ring of Ichim-Römer's sense ([8]). Moreover, if we choose $C_\sigma \cap \mathbb{Z}^n$ as $\mathbf{M}_\sigma$ for each σ, $\Bbbk[\mathcal{M}]$ is just an earlier version due to Stanley ([12]). Henceforth we refer a toric face ring of $\mathcal{M}$ supported by a fan as an *embedded* toric face ring. Every Stanley-Reisner ring and every affine semigroup ring (associated with a positive affine semigroup) can be established as embedded toric face rings (see Example 1.1). The most difference between an embedded toric face ring and a non-embedded one, is whether it has a nice $\mathbb{Z}^n$-grading or not; an embedded toric face ring always has the natural $\mathbb{Z}^n$-grading such that the dimension, as a $\Bbbk$-vector space, of each homogeneous component is less than or equal to 1. However a non-embedded one does not have such a grading.

Toric face rings can be expressed as a quotient ring of a polynomial ring. Let $\mathcal{M}$ be a monoidal complex supported by a conical complex $(\Sigma, \mathcal{X})$, and $\{a_e\}_{e\in E}$ a family of elements of $|\mathcal{M}|$ generating $\Bbbk[\mathcal{M}]$ as a $\Bbbk$-algebra, or equivalently, $\{a_e\}_{e\in E} \cap \mathbf{M}_\sigma$ generates $\mathbf{M}_\sigma$ for each $\sigma \in \mathcal{X}$. Then the polynomial ring $S := \Bbbk[X_e \mid e \in E]$ surjects on $\Bbbk[\mathcal{M}]$. We denote, by $I_{\mathcal{M}}$, its kernel. Similarly we have the surjection $S_\sigma := \Bbbk[X_e \mid a_e \in \mathcal{M}_\sigma,\ e \in E] \twoheadrightarrow \Bbbk[\mathbf{M}_\sigma]$, where $\Bbbk[\mathbf{M}_\sigma]$ denotes the affine semigroup ring of $\mathbf{M}_\sigma$, and denote its kernel by $I_{\mathbf{M}_\sigma}$.

PROPOSITION 2.8. ([5, Proposition 2.6]) *With the above notation, we have*

$$I_{\mathcal{M}} = A_{\mathcal{M}} + \sum_{i=1}^{n} S I_{\mathbf{M}_{\sigma_i}},$$

where $\sigma_1, \dots, \sigma_n$ are the maximal cells of $\mathcal{X}$, and $A_{\mathcal{M}}$ is the ideal of S generated by the squarefree monomials $\prod_{h\in H} X_h$ for which $\{a_h \mid h \in H\}$ is not contained in $\mathbf{M}_\sigma$ for any $\sigma \in \mathcal{X}$.

EXAMPLE 2.9. ([5, Example 4.6]) Consider the conical complex given in Example 2.4, and choose each rectangles to be a unit square. In this case, we can construct a monoidal complex $\mathcal{M}$ such that $\mathbf{M}_\sigma = C_\sigma \cap \mathbb{Z}^{\dim C_\sigma}$ for all σ, and then u, v, w, x, y, z are generators of $\mathcal{M}$. We set $S := \Bbbk[X_u, X_v, X_w, X_x, X_y, X_z]$, where $X_u, \dots, X_z$ are variables. Clearly, $\Bbbk[\mathbf{M}_\sigma]$ is a polynomial ring if $\dim \sigma \le 1$, and one of the following

$$\Bbbk[X_u, X_v, X_x, X_y]/(X_x X_v - X_u X_y),$$
$$\Bbbk[X_v, X_w, X_y, X_z]/(X_v X_z - X_y X_w),$$
$$\Bbbk[X_u, X_w, X_x, X_z]/(X_x X_z - X_u X_w),$$

if $\dim\sigma = 2$. Therefore we conclude that

$$I_{\mathcal{M}} = (X_xX_v - X_uX_y,\, X_vX_z - X_yX_w,\, X_xX_z - X_uX_w,\\ X_uX_vX_w,\, X_uX_vX_z) \subset S.$$

We leave the reader to verify that the other squarefree monomials in $A_{\mathcal{M}}$, e.g. $X_xX_yX_z$, are indeed contained in the above ideal.

In this paper, we often assume that $\Bbbk[\mathcal{M}]$ satisfies the following condition.

Definition 2.10. We say a toric face ring $\Bbbk[\mathcal{M}]$ (or a monoidal complex $\mathcal{M}$) is *cone-wise normal*, if the affine semigroup ring $\Bbbk[\mathbf{M}_\sigma]$ is normal for all $\sigma \in \mathcal{X}$.

If $\Bbbk[\mathcal{M}]$ is cone-wise normal, then $\Bbbk[\mathbf{M}_\sigma]$ is Cohen-Macaulay for all $\sigma \in \mathcal{X}$. Clearly, the toric face rings given in Examples 1.1 and 2.9 are cone-wise normal.

Remark 2.11. The notion of a cone-wise normal monoidal complex $\mathcal{M}$ is equivalent to that of the lattice points $\mathcal{W}F(\Pi_{\mathrm{rat}})$ of a *weak fan* $\mathcal{W}F$ introduced by Bruns and Gubeladze in [2, Definition 2.6]. In this case, our ring $\Bbbk[\mathcal{M}]$ is the same thing as the ring $\Bbbk[\mathcal{W}F]$ of [2].

An affine semigroup ring $A = \Bbbk[\mathbf{M}_\sigma]$ has a graded ring structure $A = \bigoplus_{i\in\mathbb{N}} A_i$ with $A_0 = \Bbbk$. The toric face ring given in Example 2.9 also has an $\mathbb{N}$-grading given by $\deg X_u = \cdots = \deg X_z = 1$. This is not true in general; there is a monoidal complex whose toric face ring does not have an $\mathbb{N}$-grading. See [2, Example 2.7].

For a commutative ring A, let $\operatorname{Mod} A$ (resp. $\operatorname{mod} A$) denote the category of (resp. finitely generated) A-modules.

Definition 2.12. Let $R := \Bbbk[\mathcal{M}]$ be a toric face ring of a monoidal complex $\mathcal{M}$ supported by a conical complex $(\Sigma, \mathcal{X})$.

(1) $M \in \operatorname{Mod} R$ is said to be $\mathbb{Z}\mathcal{M}$-graded if the following conditions are satisfied;

 (a) $M = \bigoplus_{a\in|\mathbb{Z}\mathcal{M}|} M_a$ as $\Bbbk$-vector spaces;

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(b) $t^a \cdot M_b \subset M_{a+b}$ if $a \in \mathbf{M}_\sigma$ and $b \in \mathbb{Z}\mathbf{M}_\sigma$ for some $\sigma \in \mathcal{X}$, and $t^a \cdot M_b = 0$ otherwise.

(2) $M \in \operatorname{Mod} R$ is said to be $\mathcal{M}$-graded if it is $\mathbb{Z}\mathcal{M}$-graded and $M_a = 0$ for $a \notin |\mathcal{M}|$.

Of course, setting $R_a := \Bbbk\, t^a$ for each $a \in |\mathcal{M}|$, we see that R itself is $|\mathcal{M}|$-graded. Any *monomial ideal*, i.e., an ideal generated by elements of the form t^a for some $a \in |\mathcal{M}|$, is $\mathcal{M}$-graded, and hence $\mathbb{Z}\mathcal{M}$-graded. Conversely, every $\mathbb{Z}\mathcal{M}$-graded ideal is a monomial ideal.

Let $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ (resp. $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$) denote the subcategory of $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) whose objects are $\mathbb{Z}\mathcal{M}$-graded R-modules and morphisms are degree preserving maps, i.e., R-homomorphisms $f : M \to N$ such that $f(M_a) \subset N_a$ for $a \in |\mathbb{Z}\mathcal{M}|$. It is clear that $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ and $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ are abelian.

For each $\sigma \in \mathcal{X}$, the ideal $\mathfrak{p}_\sigma := (t^a \mid a \notin \mathbf{M}_\sigma) \subset R$ is a $\mathbb{Z}\mathcal{M}$-graded prime ideal since $R/\mathfrak{p}_\sigma \cong \Bbbk[\mathbf{M}_\sigma]$. Conversely, every $\mathbb{Z}\mathcal{M}$-graded prime ideals are of this form.

LEMMA 2.13. *There is a one-to-one correspondence between the cells in $\mathcal{X}$ and the $\mathbb{Z}\mathcal{M}$-graded prime ideals of R.*

$$\begin{array}{ccc} \mathcal{X} & \longleftrightarrow & \left\{ \begin{array}{c} \text{all the } \mathbb{Z}\mathcal{M}\text{-graded} \\ \text{prime ideals of } R \end{array} \right\} \\ \Cup & & \Cup \\ \sigma & \longleftrightarrow & \mathfrak{p}_\sigma \end{array}$$

Proof. The proof is quite the same as [8, Lemma 2.1]. □

For an ideal I of R, we denote, by I^*, the ideal of R generated by all the monomials belonging to I. As in the case of a usual grading, we have the following:

LEMMA 2.14. *For a prime ideal $\mathfrak{p}$ of R, $\mathfrak{p}^*$ is also prime, and hence is a $\mathbb{Z}\mathcal{M}$-graded prime ideal.*

Proof. Since the ideal 0 can be decomposed as follows

$$\bigcap_{\substack{\sigma\in\mathcal{X} \\ \sigma:\text{maximal}}} \mathfrak{p}_\sigma = 0,$$

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$\{\mathfrak{p}_\sigma \mid \sigma$ is a maximal cell of $\mathcal{X}\}$ is the set of minimal primes of R. Hence $\mathfrak{p}$ must contain $\mathfrak{p}_\sigma$ for some $\sigma \in \mathcal{X}$. It follows that $\mathfrak{p}^* \supset \mathfrak{p}_\sigma$. Consider the images $\rho(\mathfrak{p})$ and $\rho(\mathfrak{p}^*)$ by the surjection $\rho : R \twoheadrightarrow \Bbbk[\mathbf{M}_\sigma]$. Then $\rho(\mathfrak{p})$ is prime and $\rho(\mathfrak{p}^*)$ is the ideal generated by the monomials contained in $\rho(\mathfrak{p})$, whence is prime. Therefore we conclude that $\mathfrak{p}^*$ is also prime. □

Corollary 2.15. *Let $\mathfrak{a}$ be a $\mathbb{Z}\mathcal{M}$-graded ideal of R. Then its radical ideal $\sqrt{\mathfrak{a}}$ is also $\mathbb{Z}\mathcal{M}$-graded.*

Proof. Since $\mathfrak{a} \subset \mathfrak{p}^*$ holds for a prime ideal $\mathfrak{p}$ with $\mathfrak{a} \subset \mathfrak{p}$, we have

$$\bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}^* \subset \bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p} = \sqrt{\mathfrak{a}} \subset \bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}^*,$$

and therefore $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p}\supset\mathfrak{a}} \mathfrak{p}^*$. □

§3. Cěch complexes and local cohomologies

Let $(\Sigma, \mathcal{X})$ be a conical complex, and $\mathcal{M}$ a monoidal complex. For $\sigma \in \mathcal{X}$, set $T_\sigma := \{t^a \mid a \in \mathbf{M}_\sigma\} \subset R := \Bbbk[\mathcal{M}]$. Then T_σ forms a multiplicatively closed subset consisting of monomials. Moreover, a multiplicatively closet subset T consisting of monomials is contained in some T_σ, unless $T \ni 0$.

Lemma 3.1. *Let $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$, and let T be a multiplicatively closed subset of R consisting of monomials. Then $T^{-1}M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$.*

Proof. Take any $x/t^a \in T^{-1}M$ with $a \in |\mathcal{M}|$, $b \in |\mathbb{Z}\mathcal{M}|$, and $x \in M_b$. If there is no $\sigma \in \mathcal{X}$ with $a, b \in \mathbb{Z}\mathbf{M}_\sigma$, then $x/t^a = (xt^a)/t^{2a} = 0$; otherwise, $b - a$ is well-defined and in $|\mathbb{Z}\mathcal{M}|$. Now for $\lambda \in |\mathbb{Z}\mathcal{M}|$, set

$$(T^{-1}M)_\lambda := \sum_{x\in M_b,\, b-a=\lambda} \Bbbk \cdot \frac{x}{t^a}$$

Then we have $T^{-1}M = \bigoplus_{\lambda\in|\mathbb{Z}\mathcal{M}|}(T^{-1}M)_\lambda$ as $\Bbbk$-vector spaces, which gives $T^{-1}M$ a $|\mathbb{Z}\mathcal{M}|$-grading. □

Well, set

$$L_R^i := \bigoplus_{\substack{\sigma\in\mathcal{X} \\ \dim\sigma=i-1}} T_\sigma^{-1}R$$

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and define $\partial : L_R^i \to L_R^{i+1}$ by

$$\partial(x) = \sum_{\substack{\tau \geq \sigma \\ \dim \tau = i}} \varepsilon(\tau, \sigma) \cdot f_{\tau,\sigma}(x)$$

for $x \in T_\sigma^{-1}R \subset L_R^i$, where ε is an incidence function on $\mathcal{X}$ and $f_{\tau,\sigma}$ is the natural map $T_\sigma^{-1}R \to T_\tau^{-1}R$ for $\sigma \leq \tau$. Then $(L_R^\bullet, \partial)$ forms a complex in $\mathrm{Mod}_{\mathbb{Z}\mathcal{M}}\, R$:

$$L_R^\bullet : 0 \longrightarrow L_R^0 \xrightarrow{\partial} L_R^1 \xrightarrow{\partial} \cdots \xrightarrow{\partial} L_R^d \longrightarrow 0,$$

where $d = \dim R = \dim \mathcal{X} + 1$. We set $\mathfrak{m} := (t^a \mid 0 \neq a \in |\mathcal{M}|)$. This is a maximal ideal of R.

PROPOSITION 3.2. (cf. [8, Theorem 4.2]) *For any R-module M,*

$$H_\mathfrak{m}^i(M) \cong H^i(L_R^\bullet \otimes_R M),$$

for all i.

Proof. It suffices to show the following:

(1) $H^0(L_R^\bullet \otimes_R M) \cong H_\mathfrak{m}^0(M)$;

(2) for a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathrm{Mod}\, R$, the induced one $0 \to L_R^\bullet \otimes_R M_1 \to L_R^\bullet \otimes_R M_2 \to L_R^\bullet \otimes_R M_3 \to 0$ is also exact;

(3) for any injective R-module I, $H^i(L_R^\bullet \otimes_R I) = 0$ for all $i \geq 1$.

Let $\mathfrak{a}$ be the ideal generated by elements t^a with $0 \neq a \in C_\sigma$ for some 1-dimensional cone C_σ. Since $\mathrm{Ker}(L_R^0 \otimes_R M \to L_R^1 \otimes_R M) = H_\mathfrak{a}^0(M)$, to prove (1), we only have to show that $\sqrt{\mathfrak{a}} = \mathfrak{m}$. Let $\mathfrak{p}$ be a prime containing $\mathfrak{a}$. Since $\mathfrak{a}$ is graded, we have $\mathfrak{p}^* \supset \mathfrak{a}$. Thus there exists $\tau \in \mathcal{X}$ such that $\mathfrak{p}_\tau \supset \mathfrak{a}$, but then C_τ contains no 1-dimensional face. Therefore we conclude that $\mathfrak{p}_\tau = \mathfrak{p}_\varnothing = \mathfrak{m}$, which implies $\sqrt{\mathfrak{a}} = \mathfrak{m}$.

The condition (2) follows easily from the flatness of the localization. For (3), we can apply the same argument of Ichim and Römer [8] for embedded toric face rings (but we need to use Lemma 2.14). ∎

Let $\mathbf{R}\Gamma_\mathfrak{m} : D^b(\mathrm{Mod}\, R) \to D^b(\mathrm{Mod}\, R)$ be the right derived functor of $\Gamma_\mathfrak{m} := \varinjlim_n \mathrm{Hom}(R/\mathfrak{m}^n, -)$, where $D^b(\mathrm{Mod}\, R)$ is the bounded derived category of $\mathrm{Mod}\, R$. Recall that $H^i(\mathbf{R}\Gamma_\mathfrak{m}(M)) = H_\mathfrak{m}^i(M)$ for all i and

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$M \in \operatorname{Mod} R$. The usual spectral sequence argument of double complexes tells us that $L_R^\bullet$ is a flat resolution of $\mathrm{R}\Gamma_{\mathfrak{m}}(R)$, and therefore we have the following.

Corollary 3.3. *For a bounded complex $M^\bullet$ of R-modules,* $\mathrm{R}\Gamma_{\mathfrak{m}}(M^\bullet)$ *and $L_R^\bullet \otimes_R M^\bullet$ are isomorphic in $D^b(\operatorname{Mod} R)$.*

When M is $\mathbb{Z}\mathcal{M}$-graded, by Lemma 3.1, $T_\sigma^{-1}R \otimes_R M$ is also $\mathbb{Z}\mathcal{M}$-graded, and moreover the differentials of $L_R^\bullet \otimes_R M$ are in $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$. Thus if $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$, $H^i(L_R^\bullet \otimes_R M)$ has a $\mathbb{Z}\mathcal{M}$-grading induced by $L_R^\bullet \otimes M$. Hence we have the following.

Corollary 3.4. $H_{\mathfrak{m}}^i(M) \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ *for* $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$.

§4. Squarefree Modules

In this section, we assume that all the toric face rings are cone-wise normal. Let $(\Sigma, \mathcal{X})$ be a conical complex, $\mathcal{M}$ a monoidal complex, and R the toric face ring of $\mathcal{M}$. For $a \in |\mathcal{M}|$, there exists a unique cell $\sigma \in \mathcal{X}$ such that $\text{rel-int}(C_\sigma) \ni a$. We denote this σ by $\operatorname{supp}(a)$.

Definition 4.1. An R-module $M \in \operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ is said to be *squarefree* if it is $\mathcal{M}$-graded and the multiplication map $M_a \ni x \mapsto t^b x \in M_{a+b}$ is an isomorphism of $\Bbbk$-vector spaces for all $a, b \in |\mathcal{M}|$ with $\operatorname{supp}(a+b) = \operatorname{supp}(a)$.

For a monomial ideal I of R, it is a squarefree R-module, if and only if so is R/I, if and only if $I = \sqrt{I}$. In particular, $\mathfrak{p}_\sigma$ and $R/\mathfrak{p}_\sigma$ are squarefree. We denote, by $\operatorname{Sq} R$, the full subcategory of $\operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$ consisting of squarefree R-modules. As in the case of affine semigroup rings or Stanley-Reisner rings (see [14], [15]), $\operatorname{Sq} R$ has nice properties. Since their proofs are also quite similar to these cases, we omit some of them.

Lemma 4.2. (cf. [14], [15]) *Let $M \in \operatorname{Sq} R$. Then for $a, b \in |\mathcal{M}|$ with* $\operatorname{supp}(a) \geq \operatorname{supp}(b)$, *there exists a $\Bbbk$-linear map $\varphi_{a,b}^M : M_b \to M_a$ satisfying the following properties*:

(1) $\varphi_{a,b}^M$ *is bijective if* $\operatorname{supp}(a) = \operatorname{supp}(b)$;
(2) $\varphi_{a,a}^M = \mathrm{id}$ *and* $\varphi_{a,b}^M \circ \varphi_{b,c}^M = \varphi_{a,c}^M$ *for $a, b, c \in |\mathcal{M}|$ with* $\operatorname{supp}(c) \leq \operatorname{supp}(b) \leq \operatorname{supp}(a)$;

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(3) *For* $a, a', b, b' \in |\mathcal{M}|$ *with* $\operatorname{supp}(a) \leq \operatorname{supp}(a')$ *and* $\operatorname{supp}(a+b) \leq \operatorname{supp}(a'+b')$, *the following diagram*

$$\begin{array}{ccc} M_a & \xrightarrow{t^b} & M_{a+b} \\ {\scriptstyle \varphi^M_{a',a}} \downarrow & & \downarrow {\scriptstyle \varphi^M_{a'+b',a+b}} \\ M_{a'} & \xrightarrow[t^{b'}]{} & M_{a'+b'} \end{array}$$

commutes.

Let Λ denote the incidence algebra of the regular cell complex $\mathcal{X}$ over $\Bbbk$ (regarding $\mathcal{X}$ as a poset by its order $>$). That is, Λ is a finite dimensional associative $\Bbbk$-algebra with basis $\{e_{\sigma,\tau} \mid \sigma, \tau \in \mathcal{X} \text{ with } \sigma \geq \tau\}$, and its multiplication is defined by

$$e_{\sigma,\tau} \cdot e_{\tau',\upsilon} = \begin{cases} e_{\sigma,\upsilon} & \text{if } \tau = \tau'; \\ 0 & \text{otherwise.} \end{cases}$$

We write $e_\sigma := e_{\sigma,\sigma}$ for $\sigma \in \mathcal{X}$. Each e_σ is idempotent, and moreover Λe_σ is indecomposable as a left Λ-module. It is easy to verify that $e_\sigma \cdot e_\tau = 0$ if $\sigma \neq \tau$ and that $1 = \sum_{\sigma \in \mathcal{X}} e_\sigma$. Hence Λ, as a left Λ-module, can be decomposed as $\Lambda = \bigoplus_{\sigma \in \mathcal{X}} \Lambda e_\sigma$.

Let $\operatorname{mod} \Lambda$ denote the category of finitely generated left Λ-modules. As a $\Bbbk$-vector space, any $M \in \operatorname{mod} \Lambda$ has the decomposition $M = \bigoplus_{\sigma \in \mathcal{X}} e_\sigma M$. Henceforth we set $M_\sigma := e_\sigma M$.

For each $\sigma \in \mathcal{X}$, we can construct an indecomposable injective object in $\operatorname{mod} \Lambda$ as follows; set

$$\bar{E}(\sigma) := \bigoplus_{\tau \in \mathcal{X}, \tau \leq \sigma} \Bbbk\, \bar{e}_\tau,$$

where $\bar{e}_\tau$'s are basis elements. The multiplication on $\bar{E}(\sigma)$ from the left defined by

$$e_{\upsilon,\omega} \cdot \bar{e}_\tau = \begin{cases} \bar{e}_\upsilon & \text{if } \tau = \omega \text{ and } \upsilon \leq \sigma; \\ 0 & \text{otherwise,} \end{cases}$$

bring $\bar{E}(\sigma)$ a left Λ-module structure. The following is well known.

PROPOSITION 4.3. *The category* $\operatorname{mod} \Lambda$ *is abelian and enough injectives, and any indecomposable injective object is isomorphic to* $\bar{E}(\sigma)$ *for some* $\sigma \in \mathcal{X}$.

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As in the case of affine semigroup rings and Stanley-Reisner rings, we have

Proposition 4.4. (cf. [14], [15]) *There is an equivalence between* $\operatorname{Sq} R$ *and* $\operatorname{mod} \Lambda$. *Hence* $\operatorname{Sq} R$ *is abelian, and enough injectives. Any indecomposable injective object in* $\operatorname{Sq} R$ *is isomorphic to* $R/\mathfrak{p}_\sigma$ *for some* $\sigma \in \mathcal{X}$.

Proof. First, we will show the category equivalence. The object $M \in \operatorname{Sq} R$ corresponding to $N \in \operatorname{mod} \Lambda$ is given as follows. Set $M_a := N_{\operatorname{supp}(a)}$ for each $a \in |\mathcal{M}|$. For $a, b \in |\mathcal{M}|$ such that $a+b$ exists, define the multiplication $M_a \ni x \mapsto t^b \cdot x \in M_{a+b}$ by

$$M_a = N_{\operatorname{supp}(a)} \ni x \longmapsto e_{\operatorname{supp}(a+b),\, \operatorname{supp}(a)} \cdot x \in N_{\operatorname{supp}(a+b)} = M_{a+b}.$$

Then M becomes a squarefree module. See [14], [15] for details (though *right* Λ-modules are treated in [14], [15], there is no essential difference).

Since $R/\mathfrak{p}_\sigma$ corresponds to $\bar{E}(\sigma)$ in this equivalence, the other statements follow from Proposition 4.3. □

Let $D^b(\operatorname{Sq} R)$ be the bounded derived category of $\operatorname{Sq} R$. We shall define the functor $\mathbb{D} : D^b(\operatorname{Sq} R) \to D^b(\operatorname{Sq} R)^{\mathsf{op}}$. This functor will play an important role in the next section. First, we choose elements $a(\sigma) \in |\mathcal{M}|$ with $\operatorname{supp}(a(\sigma)) = \sigma$ for each $\sigma \in \mathcal{X}$, and set $\varphi^M_{\sigma,\tau} := \varphi^M_{a(\sigma),\, a(\tau)}$ for $M \in \operatorname{Sq} R$ and $\sigma, \tau \in \mathcal{X}$ with $\tau \le \sigma$, where $\varphi^M_{a(\sigma),\, a(\tau)}$ is the map given in Lemma 4.2. To a bounded complex $M^\bullet$ of squarefree R-modules, we assign the complex $\mathbb{D}(M^\bullet)$ defined as follows: the component of cohomological degree p is

$$\mathbb{D}(M^\bullet)^p := \bigoplus_{i+\dim C_\sigma = -p} (M^i_{a(\sigma)})^* \otimes_{\Bbbk} R/\mathfrak{p}_\sigma,$$

where $(-)^*$ denotes the $\Bbbk$-dual, but the "degree" of $(M^i_{a(\sigma)})^*$ is $0 \in |\mathbb{Z}\mathcal{M}|$. Define $d' : \mathbb{D}(M^\bullet)^p \to \mathbb{D}(M^\bullet)^{p+1}$ and $d'' : \mathbb{D}(M^\bullet)^p \to \mathbb{D}(M^\bullet)^{p+1}$ by

$$d'(y \otimes r) = \sum_{\substack{\tau \le \sigma, \\ \dim \tau = \dim \sigma - 1}} \varepsilon(\sigma, \tau) \cdot (\varphi^{M^i}_{\sigma,\tau})^*(y) \otimes g_{\tau,\sigma}(r),$$

$$d''(y \otimes r) = (-1)^p \cdot (\partial^i_{M^\bullet})^*(y) \otimes r$$

for $y \in M^i_{a(\sigma)}$ with $i + \dim C_\sigma = -p$ and $r \in R/\mathfrak{p}_\sigma$. Here $\varepsilon(\sigma, \tau)$ is an incidence function on $\mathcal{X}$ and $g_{\tau,\sigma} : R/\mathfrak{p}_\sigma \to R/\mathfrak{p}_\tau$ is the surjection induced by the inclusion $\mathfrak{p}_\sigma \subset \mathfrak{p}_\tau$. Clearly, $(\mathbb{D}(M^\bullet), d' + d'')$ forms a bounded complex in $\operatorname{Sq} R$, and Lemma 4.2 guarantees the independence of $\mathbb{D}(M^\bullet)$ from the choice of $a(\sigma)$'s.

Let $K^b(\operatorname{Sq} R)$ be the bounded homotopy category of $\operatorname{Sq} R$. Since the above assignment preserves mapping cones, it gives a triangulated functor of $K^b(\operatorname{Sq} R) \to K^b(\operatorname{Sq} R)^{\mathsf{op}}$, and an usual argument using spectral sequences indicates that it preserves quasi-isomorphisms. Hence it induces the functor $D^b(\operatorname{Sq} R) \to D^b(\operatorname{Sq} R)^{\mathsf{op}}$, which is denoted by $\mathbb{D}$ again.

Up to translation, the functor $\mathbb{D}$ coincides with the functor $\mathbf{D}: D^b(\operatorname{mod} \Lambda) \to D^b(\operatorname{mod} \Lambda)^{\mathsf{op}}$ defined in [17], through the equivalence $\operatorname{Sq} R \cong \operatorname{mod} \Lambda$ in Proposition 4.4. Hence by [17, Theorem 3.4 (1)], we have the following.

PROPOSITION 4.5. *The functor* $\mathbb{D} : D^b(\operatorname{Sq} R) \to D^b(\operatorname{Sq})^{\mathsf{op}}$ *satisfies* $\mathbb{D} \circ \mathbb{D} \cong \mathrm{id}$.

§5. Dualizing complexes

We first recall the following useful result due to Sharp ([11]).

THEOREM 5.1. (Sharp) *Let A and B be commutative noetherian rings, and $f : A \to B$ a ring homomorphism. Assume that A has a dualizing complex $D_A^\bullet$ and B, regarded as an A-module by f, is finitely generated. Then* $\operatorname{Hom}_A(B, D_A^\bullet)$ *is a dualizing complex of B.*

For a commutative ring A, we denote, by $E_A(-)$, the injective hull in $\operatorname{Mod} A$. Let $(\Sigma, \mathcal{X})$ be a conical complex, $\mathcal{M}$ a cone-wise normal monoidal complex supported by Σ, and $R := \Bbbk[\mathcal{M}]$ its toric face ring. Since R is a finitely generated $\Bbbk$-algebra, we can take a polynomial ring which surjects onto R. Thus, Proposition 5.1 implies that R has a normalized dualizing complex

$$D_R^\bullet : 0 \longrightarrow \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = d}} E_R(R/\mathfrak{p}) \longrightarrow \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = d-1}} E_R(R/\mathfrak{p}) \longrightarrow \cdots$$
$$\cdots \longrightarrow \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = 0}} E_R(R/\mathfrak{p}) \longrightarrow 0,$$

where $d := \dim R = \dim \mathcal{X} + 1$ and cohomological degrees are given by

$$D_R^i := \bigoplus_{\substack{\mathfrak{p} \in \operatorname{Spec} R, \\ \dim R/\mathfrak{p} = -i}} E_R(R/\mathfrak{p}).$$

On the other hand, set

$$I_R^i := \bigoplus_{\substack{\sigma \in \mathcal{X} \\ \dim R/\mathfrak{p}_\sigma = -i}} R/\mathfrak{p}_\sigma$$

for $i = 0, \dots, d$, and define $I_R^{-i} \to I_R^{-i+1}$ by

$$x \longmapsto \sum_{\substack{\dim \Bbbk[\tau] = i-1 \\ \tau \le \sigma}} \varepsilon(\sigma, \tau) \cdot g_{\tau,\sigma}(x)$$

for $x \in R/\mathfrak{p}_\sigma \subset I_R^{-i}$, where $\varepsilon(\sigma, \tau)$ denotes an incidence function of $\mathcal{X}$, and $g_{\tau,\sigma}$ is the surjection $R/\mathfrak{p}_\sigma \to R/\mathfrak{p}_\tau$. Then

$$I_R^\bullet : 0 \longrightarrow I_R^{-d} \longrightarrow I_R^{-d+1} \longrightarrow \cdots \longrightarrow I_R^0 \longrightarrow 0$$

is a complex.

THEOREM 5.2. *With the above situation (in particular, R is cone-wise normal), $I_R^\bullet$ is quasi-isomorphic to the normalized dualizing complex $D_R^\bullet$ of R.*

For the embedded case, Theorem 5.2 was already shown by Ichim and Römer [8], using the natural $\mathbb{Z}^n$-graded structure. However, in the general case, we cannot apply the same argument.

PROPOSITION 5.3. *With the hypothesis in Theorem 5.2, $I_R^\bullet$ is a sub-complex of $D_R^\bullet$.*

Proof. We shall go through some steps.

Step 1. Some observations.

For $\sigma \in \mathcal{X}$, we set $\Bbbk[\sigma] := R/\mathfrak{p}_\sigma \cong \Bbbk[\mathbf{M}_\sigma]$ and $d_\sigma := \dim C_\sigma = \dim \Bbbk[\sigma] = \dim \sigma + 1$. Note that

$$D_\sigma^\bullet := \operatorname{Hom}_R(\Bbbk[\sigma], D_R^\bullet)$$

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is a normalized dualizing complex of $\Bbbk[\sigma]$ by Proposition 5.1. Since $\Bbbk[\sigma]$ is $\mathbb{Z}^{d_\sigma}$-graded, we also have the $\mathbb{Z}^{d_\sigma}$-graded version of a normalized dualizing complex

$$
{}^*D_\sigma^\bullet : 0 \longrightarrow \bigoplus_{\substack{\tau\le\sigma,\\ \dim \Bbbk[\tau]=d_\sigma}} {}^*E_{\Bbbk[\sigma]}(\Bbbk[\tau]) \longrightarrow \bigoplus_{\substack{\tau\le\sigma,\\ \dim \Bbbk[\tau]=d_\sigma-1}} {}^*E_{\Bbbk[\sigma]}(\Bbbk[\tau]) \longrightarrow \cdots
$$
$$
\cdots \longrightarrow {}^*E_{\Bbbk[\sigma]}(\Bbbk) \longrightarrow 0,
$$

where ${}^*E_{\Bbbk[\sigma]}(-)$ denotes the injective hull in the category of $\mathbb{Z}^{d_\sigma}$-graded $\Bbbk[\sigma]$-modules, and cohomological degrees are given by the same way as $D_R^\bullet$.

It is easy to see that the *positive part*

$$
\bigoplus_{a\in\mathbf{M}_\sigma} [{}^*E_{\Bbbk[\sigma]}(\Bbbk[\tau])]_a
$$

of ${}^*E_{\Bbbk[\sigma]}(\Bbbk[\tau])$ is isomorphic to $\Bbbk[\tau]$. Set

$$
I_\sigma^\bullet := \bigoplus_{a\in\mathbf{M}_\sigma} [{}^*D_\sigma^\bullet]_a \subset {}^*D_\sigma^\bullet. \tag{5.1}
$$

Clearly, $I_\sigma^\bullet$ is a complex with

$$
I_\sigma^i := \bigoplus_{\substack{\tau\le\sigma,\\ \dim \Bbbk[\tau]=-i}} \Bbbk[\tau]. \tag{5.2}
$$

As is well-known, $D_\sigma^\bullet$ is an injective resolution of ${}^*D_\sigma^\bullet$ in the category $\operatorname{Mod}(\Bbbk[\sigma])$, and the latter can be seen as a subcomplex of the former in a non-canonical way. By the construction, $I_\sigma^\bullet$ is a subcomplex of ${}^*D_\sigma^\bullet$, and $D_\sigma^\bullet$ is a subcomplex of $D_R^\bullet$. Combining them, we have an embedding $I_\sigma^\bullet \hookrightarrow D_R^\bullet$. Thus the problem is the compatibility of the embeddings $I_\sigma^\bullet \hookrightarrow D_R^\bullet$ and $I_\tau^\bullet \hookrightarrow D_R^\bullet$ for $\sigma, \tau \in \Sigma$.

Step 2. Canonical (up to scalar multiplication) embedding $\Bbbk[\sigma] \hookrightarrow D_R^{-d_\sigma}$.

For $\sigma \in \mathcal{X}$, let $\omega_{\Bbbk[\sigma]}$ be the canonical module of $\Bbbk[\sigma]$. By our hypothesis that $\mathcal{M}$ is cone-wise normal, we see that $\omega_{\Bbbk[\sigma]}$ is just the ideal generated by $\{t^a \in \Bbbk[\sigma] \mid a \in \text{rel-int}(C_\sigma)\cap \mathbf{M}_\sigma\}$ (cf. [4, Theorem 6.3.5]). Whence we have the exact sequence:

$$
0 \longrightarrow \omega_{\Bbbk[\sigma]} \longrightarrow \Bbbk[\sigma] \longrightarrow \Bbbk[\sigma]/\omega_{\Bbbk[\sigma]} \longrightarrow 0.
$$

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Since $\mathrm{Hom}_R(\Bbbk[\sigma]/\omega_{\Bbbk[\sigma]}, E_R(\Bbbk[\sigma])) = 0$, applying $\mathrm{Hom}_R(-, E_R(\Bbbk[\sigma]))$ to the above exact sequence yields the canonical isomorphism

$$\mathrm{Hom}_R(\Bbbk[\sigma], E_R(\Bbbk[\sigma])) \cong \mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, E_R(\Bbbk[\sigma])),$$

and thus the canonical embedding

$$\mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, E_R(\Bbbk[\sigma])) \cong \{x \in E_R(\Bbbk[\sigma]) \mid \mathfrak{p}_\sigma x = 0\} \subset E_R(\Bbbk[\sigma]). \tag{5.3}$$

Since we have

$$\begin{aligned}\mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, D_R^{-d_\sigma}) &= \bigoplus_{\substack{\mathfrak{p}\in \mathrm{Spec}\, R,\\ \dim R/\mathfrak{p}=d_\sigma}} \mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, E_R(R/\mathfrak{p}))\\ &= \mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, E_R(\Bbbk[\sigma])),\end{aligned}$$

in conjunction with (5.3), we obtain the canonical embedding

$$\mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, D_R^{-d_\sigma}) \subset E_R(\Bbbk[\sigma]) \subset D_R^{-d_\sigma}.$$

Since $\mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, D_R^{-d_\sigma-1}) = 0$, it follows that

$$\begin{aligned}\mathrm{Ext}_R^{-d_\sigma}(\omega_{\Bbbk[\sigma]}, D_R^\bullet) &= \mathrm{Ker}(\mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, D_R^{-d_\sigma}) \to \mathrm{Hom}_R(\omega_{\Bbbk[\sigma]}, D_R^{-d_\sigma+1}))\\ &= \{x \in D_R^{-d_\sigma} \mid \mathfrak{p}_\sigma x = 0 \text{ and } \partial(J_\sigma x) = 0\},\end{aligned}$$

where $J_\sigma := \{t^a \mid a \in \text{rel-int}(C_\sigma) \cap \mathbf{M}_\sigma\}$ and $\partial : D^{-d_\sigma} \to D^{-d_\sigma+1}$ is the differential map. Consequently, we have

$$\Bbbk[\sigma] \cong \mathrm{Ext}_R^{-d_\sigma}(\omega_{\Bbbk[\sigma]}, D_R^\bullet) \subset D_R^{-d_\sigma} \tag{5.4}$$

canonically.

Using this, we have a canonical injection

$$I_R^i = \bigoplus_{\substack{\sigma\in\mathcal{X}\\ \dim \Bbbk[\sigma]=-i}} \Bbbk[\sigma] \hookrightarrow D_R^i \tag{5.5}$$

for each i.

Step 3. Compatibility.

For $\sigma, \tau \in \mathcal{X}$ with $\tau \le \sigma$, set

$$\underline{\mathrm{Ext}}^i_{\Bbbk[\sigma]}(\omega_{\Bbbk[\tau]}, {}^*D_\sigma^\bullet) := H^i(\mathrm{Hom}^\bullet_{\Bbbk[\sigma]}(\omega_{\Bbbk[\tau]}, {}^*D_\sigma^\bullet)).$$

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This module has a $\mathbb{Z}^{d_\sigma}$-grading, since so does $\omega_{\Bbbk[\tau]}$. Applying the same argument as in Step 2 (replacing R by $\Bbbk[\sigma]$ and $D_R^\bullet$ by ${}^*D_\sigma^\bullet$), we have a canonical embedding which is the first injection of the sequence

$$\Bbbk[\tau] \cong \underline{\operatorname{Ext}}_{\Bbbk[\sigma]}^{-d_\tau}(\omega_{\Bbbk[\tau]}, {}^*D_\sigma^\bullet) \longrightarrow {}^*D_\sigma^{-d_\tau} \longrightarrow D_R^{-d_\tau}. \tag{5.6}$$

Here the last injection is not canonical. Since the inclusions ${}^*D_\sigma^\bullet \hookrightarrow D_\sigma^\bullet \hookrightarrow D_R^\bullet$ give the isomorphisms

$$\underline{\operatorname{Ext}}_{\Bbbk[\sigma]}^{-d_\tau}(\omega_{\Bbbk[\tau]}, {}^*D_\sigma^\bullet) \cong \operatorname{Ext}_{\Bbbk[\sigma]}^{-d_\tau}(\omega_{\Bbbk[\tau]}, D_\sigma^\bullet) \cong \operatorname{Ext}_R^{-d_\tau}(\omega_{\Bbbk[\tau]}, D_R^\bullet),$$

the embedding $\Bbbk[\tau] \hookrightarrow D_R^{-d_\tau}$ given in (5.6) coincides with the one given in Step 2. (So the image of (5.6) does not depend on the choice of an injection ${}^*D_\sigma^{-d_\tau} \hookrightarrow D_R^{-d_\tau}$.)

It is easy to see that the inclusion (5.1) (see also (5.2)) is same as the one given by (5.6). Therefore, through any ${}^*D_\sigma^\bullet \hookrightarrow D_R^\bullet$, the embeddings of (5.1) and (5.5) are compatible. So under this embedding, we have $I_\sigma^i \subset I_R^i \subset D_R^i$. Since $I_\sigma^\bullet$ is a subcomplex of $D_R^\bullet$ for all $\sigma \in \mathcal{X}$, $\bigoplus_{i\in\mathbb{Z}} I_R^i$ forms a subcomplex of $D_R^\bullet$.

We can take a generator $1_\sigma \in \Bbbk[\sigma] \subset I_R^{-d_\sigma} \subset D_R^{-d_\sigma}$ for each $\sigma \in \mathcal{X}$ satisfying

$$\partial_{D_R^\bullet}(1_\sigma) = \sum \varepsilon'(\sigma, \tau) \cdot 1_\tau$$

for some incidence function ε' on $\mathcal{X}$. Recall that we have fixed an incidence function ε to define the differential of $I_R^\bullet$. While ε and ε' do not coincide in general, their difference is well-regulated (cf. [4, p. 265]). So, after a suitable change of $\{1_\sigma\}_{\sigma\in\mathcal{X}}$, we have

$$\partial_{D_R^\bullet}(1_\sigma) = \sum \varepsilon(\sigma, \tau) \cdot 1_\tau.$$

Therefore we conclude that $I_R^\bullet$ is a subcomplex of $D_R^\bullet$ as is desired. ∎

When R is a normal semigroup ring, the second author showed in [18, Lemma 3.8] that there is a natural isomorphism between $\mathbb{D}$ and $\operatorname{RHom}(-, D_R^\bullet)$. The next result generalizes this to toric face rings.

PROPOSITION 5.4. *There is the following commutative diagram;*

$$\begin{array}{ccc} D^b(\operatorname{Sq} R) & \xrightarrow{\ \mathbb{U}\ } & D^b(\operatorname{Mod} R) \\ {\scriptstyle\mathbb{D}}\downarrow & & \downarrow{\scriptstyle \operatorname{RHom}(-, D_R^\bullet)} \\ D^b(\operatorname{Sq} R)^{\mathsf{op}} & \xrightarrow[\ \mathbb{U}\ ]{} & D^b(\operatorname{Mod} R)^{\mathsf{op}}, \end{array}$$

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where $\mathbb{U}$ *is the functor induced by the forgetful functor* $\operatorname{Sq} R \to \operatorname{Mod} R$. *In particular, we have* $\mathbb{D}(M^\bullet) \cong \operatorname{RHom}_R(M^\bullet, D_R^\bullet)$ *in* $D^b(\operatorname{Mod} R)$ *for any* $M^\bullet \in D^b(\operatorname{Sq} R)$, *and hence* $\operatorname{Ext}^i_R(M^\bullet, D_R^\bullet)$ *has a* $\mathbb{Z}\mathcal{M}$*-grading induced by* $\mathbb{D}(M^\bullet)$.

Proof. Let Inj-Sq be the full subcategory of $\operatorname{Sq} R$ consisting of all injective objects, that is, finite direct sums of $\Bbbk[\sigma]$ for various $\sigma \in \mathcal{X}$. As is well-known (cf. [7, Proposition 4.7]), the bounded homotopy category $K^b(\text{Inj-Sq})$ is equivalent to $D^b(\operatorname{Sq} R)$. It is easy to see that $\mathbb{D}(\Bbbk[\sigma]) = \operatorname{Hom}^\bullet_R(\Bbbk[\sigma], I^\bullet_R)$. Moreover, $\mathbb{D}(J^\bullet) = \operatorname{Hom}^\bullet_R(J^\bullet, I^\bullet_R)$ for all $J^\bullet \in K^b(\text{Inj-Sq})$. Since $I^\bullet_R$ is a subcomplex of $D^\bullet_R$ as shown in Proposition 5.3, we have a chain map $\operatorname{Hom}^\bullet_R(J^\bullet, I^\bullet_R) \to \operatorname{Hom}^\bullet_R(J^\bullet, D^\bullet_R)$. This map induces a natural transformation $\Psi : \mathbb{U} \circ \mathbb{D} \to \operatorname{RHom}_R(-, D^\bullet_R) \circ \mathbb{U}$. If $M \in \operatorname{Sq} R$ is a $\Bbbk[\sigma]$-module, then $\mathbb{D}(M) \cong \operatorname{RHom}_{\Bbbk[\sigma]}(M, D^\bullet_\sigma) \cong \operatorname{RHom}_R(M, D^\bullet_R)$ by [18, Lemma 3.8]. In particular, $\Psi(\Bbbk[\sigma])$ is isomorphism for all $\sigma \in \mathcal{X}$. Hence applying [7, Proposition 7.1], we see that $\Psi(M^\bullet)$ is an isomorphism for all $M^\bullet \in D^b(\operatorname{Sq} R)$. □

The most part of the proof of Theorem 5.2 has done now.

Proof of Theorem 5.2. Since $R \in \operatorname{Sq} R$, we have

$$I_R = \mathbb{D}(R) \cong \operatorname{RHom}_R(R, D^\bullet_R) \cong D^\bullet_R$$

by Proposition 5.4. □

Let $M \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$. We can construct the *graded Matlis dual* $M^\vee \in \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ of M as follows: For each $a \in |\mathbb{Z}\mathcal{M}|$, $(M^\vee)_a$ is the $\Bbbk$-dual space of M_{-a}. For $a, b \in |\mathbb{Z}\mathcal{M}|$ such that $a+b$ exists (that is, $a, b, a+b \in \mathbf{M}_\sigma$ for some $\sigma \in \mathcal{X}$), the multiplication map $(M^\vee)_a \ni x \mapsto t^b x \in (M^\vee)_{a+b}$ is the $\Bbbk$-dual of $M_{-a-b} \ni y \mapsto t^b y \in M_{-a}$. Otherwise, $t^b x = 0$ for all $x \in (M^\vee)_a$.

It is obvious that $M^\vee$ is actually a $\mathbb{Z}\mathcal{M}$-graded R-module. If $\dim_\Bbbk M_a < \infty$ for all $a \in |\mathbb{Z}\mathcal{M}|$ (e.g. $M \in \operatorname{mod}_{\mathbb{Z}\mathcal{M}} R$), then $M^{\vee\vee} \cong M$. Clearly, $(-)^\vee$ defines an exact contravariant functor from $\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$ to itself. We can extend this functor to the functors $K^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R) \to K^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathsf{op}}$ and $D^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R) \to D^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)^{\mathsf{op}}$. We simply denote them by $(-)^\vee$.

PROPOSITION 5.5. *As functors from* $D^b(\operatorname{Sq} R)$ *to* $D^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)$, *we have* $\mathrm{R}\Gamma_{\mathfrak{m}} \cong (-)^\vee \circ \mathbb{U} \circ \mathbb{D}$, *where* $\mathbb{U} : D^b(\operatorname{Sq} R) \to D^b(\operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R)$ *is induced by the forgetful functor* $\operatorname{Sq} R \to \operatorname{Mod}_{\mathbb{Z}\mathcal{M}} R$. *In particular, if* $M \in \operatorname{Sq} R$, *then* $H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{-i}_R(M, D^\bullet_R)^\vee$ *as* $\mathbb{Z}\mathcal{M}$*-graded modules for all* i.

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Proof. We use the notation of the proofs of the above results. If $M \in \mathrm{Mod}_{\mathbb{Z}\mathcal{M}} R$, then the $|\mathcal{M}|$*-graded part* $\bigoplus_{a \in |\mathcal{M}|} M_a$ of M is clearly an R-submodule. For $\tau \in \Sigma$, recall that $T_\tau = \{t^a \mid a \in \mathbf{M}_\tau\}$ is a multiplicatively closed set. It is easy to see that, for $\sigma, \tau \in \Sigma$, the localization $T_\tau^{-1}\Bbbk[\sigma]$ is non-zero if and only if $\tau \leq \sigma$. When $\tau \leq \sigma$, the $|\mathcal{M}|$-graded part of $(T_\tau^{-1}\Bbbk[\sigma])^\vee$ is isomorphic to $\Bbbk[\tau]$.

Let $L_R^\bullet$ be the Cěch complex of R defined in Section 3. It is easy to see that the $|\mathcal{M}|$-graded part of $(L_R^\bullet \otimes_R \Bbbk[\sigma])^\vee$ is isomorphic to $\mathbb{D}(\Bbbk[\sigma])$. Moreover, if $J^\bullet \in K^b(\text{Inj-Sq})$, then the $|\mathcal{M}|$-graded part of $(L_R^\bullet \otimes_R J^\bullet)^\vee$ is isomorphic to $\mathbb{D}(J^\bullet)$. Thus $\mathbb{D}(J^\bullet)$ is a subcomplex of $(L_R^\bullet \otimes_R J^\bullet)^\vee$, and there is a chain map $L_R^\bullet \otimes_R J^\bullet \to \mathbb{D}(J^\bullet)^\vee$. Recall that $L_R^\bullet \otimes_R J^\bullet$ is quasi-isomorphic to $\mathrm{R}\Gamma_\mathfrak{m}(J^\bullet)$ by Corollary 3.3. Hence we have a natural transformation $\Phi : \mathrm{R}\Gamma_\mathfrak{m} \to (-)^\vee \circ \mathbb{U} \circ \mathbb{D}$, where we regard $\mathrm{R}\Gamma_\mathfrak{m}$ and $(-)^\vee \circ \mathbb{U} \circ \mathbb{D}$ as functors from $K^b(\text{Inj-Sq})$ ($\cong D^b(\mathrm{Sq}\, R)$) to $D^b(\mathrm{Mod}_{\mathbb{Z}\mathcal{M}} R)$. Since $\Phi(\Bbbk[\sigma])$ is an isomorphism for all $\sigma \in \mathcal{X}$, Φ is a natural isomorphism by [7, Proposition 7.1]. □

§6. Sheaves associated with squarefree modules

Throughout this section, $\mathcal{M}$ is a cone-wise normal monoidal complex supported by a conical complex $(\Sigma, \mathcal{X})$. Recall that $X = \bigcup_{\sigma \in \mathcal{X}} \sigma$ is the underlying topological space of the cell complex $\mathcal{X}$. As in the previous section, let Λ be the incidence algebra of the poset $\mathcal{X}$ over $\Bbbk$, and $\mathrm{mod}\,\Lambda$ the category of finitely generated left Λ-modules.

Let $\mathrm{Sh}(X)$ be the category of sheaves of finite dimensional $\Bbbk$-vector spaces on X. We say $\mathcal{F} \in \mathrm{Sh}(X)$ is *constructible* with respect to the cell decomposition $\mathcal{X}$, if the restriction $\mathcal{F}|_\sigma$ is a constant sheaf for all $\emptyset \neq \sigma \in \mathcal{X}$.

In [17], the second author constructed the functor $(-)^\dagger : \mathrm{mod}\,\Lambda \to \mathrm{Sh}(X)$. (Under the convention that $\emptyset \notin \mathcal{X}$, this functor has been well-known to specialists.) Here we give a precise construction for the reader's convenience.

For $M \in \mathrm{mod}\,\Lambda$, set

$$\mathrm{Sp\acute{e}}(M) := \bigcup_{\emptyset \neq \sigma \in \mathcal{X}} \sigma \times M_\sigma.$$

Let $\pi : \mathrm{Sp\acute{e}}(M) \to X$ be the projection map which sends $(p, m) \in \sigma \times M_\sigma \subset \mathrm{Sp\acute{e}}(M)$ to $p \in \sigma \subset X$. For an open subset $U \subset X$ and a map $s : U \to \mathrm{Sp\acute{e}}(M)$, we will consider the following conditions:

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(∗) $\pi \circ s = \mathrm{id}_U$ and $s_p = e_{\sigma,\tau} \cdot s_q$ for all $p \in \sigma \cap U$, $q \in \tau \cap U$ with $\sigma \geq \tau$. Here s_p (resp. s_q) is the element of M_σ (resp. M_τ) with $s(p) = (p, s_p)$ (resp. $s(q) = (q, s_q)$).

(∗∗) There is an open covering $U = \bigcup_{i \in I} U_i$ such that the restriction of s to U_i satisfies (∗) for all $i \in I$.

Now we define a sheaf $M^\dagger \in \mathrm{Sh}(X)$ from M as follows. For an open set $U \subset X$, set

$$M^\dagger(U) := \{s \mid s : U \to \mathrm{Spé}(M) \text{ is a map satisfying } (**)\}$$

and the restriction map $M^\dagger(U) \to M^\dagger(V)$ is the natural one. It is easy to see that $M^\dagger$ is a constructible sheaf with respect to the cell decomposition $\mathcal{X}$. For $\sigma \in \mathcal{X}$, let $U_\sigma := \bigcup_{\tau \geq \sigma} \tau$ be an open set of X. Then we have $M^\dagger(U_\sigma) \cong M_\sigma$. Moreover, if $\sigma \leq \tau$, then we have $U_\sigma \supset U_\tau$ and the restriction map $M^\dagger(U_\sigma) \to M^\dagger(U_\tau)$ corresponds to the multiplication map $M_\sigma \ni x \mapsto e_{\tau,\sigma} x \in M_\tau$. For a point $p \in \sigma$, the stalk $(M^\dagger)_p$ of $M^\dagger$ at p is isomorphic to M_σ. This construction gives the exact functor $(-)^\dagger : \mathrm{mod}\,\Lambda \to \mathrm{Sh}(X)$. We also remark that $M_\emptyset$ is irrelevant to $M^\dagger$.

As in the previous sections, let $R = \Bbbk[\mathcal{M}]$ be the toric face ring, and $\mathrm{Sq}\,R$ the category of squarefree R-modules. Through the equivalence $\mathrm{Sq}\,R \cong \mathrm{mod}\,\Lambda$, $(-)^\dagger : \mathrm{mod}\,\Lambda \to \mathrm{Sh}(X)$ gives the exact functor

$$(-)^+ : \mathrm{Sq}\,R \longrightarrow \mathrm{Sh}(X).$$

Recall that X admits Verdier's dualizing complex $\mathcal{D}_X^\bullet \in D^b(\mathrm{Sh}(X))$ with coefficients in $\Bbbk$ (see [10, V. Section 2]). In [17], the second author considered the duality functor $\mathbf{D} : D^b(\mathrm{mod}\,\Lambda) \to D^b(\mathrm{mod}\,\Lambda)$. Through the functor $(-)^\dagger : \mathrm{mod}\,\Lambda \to \mathrm{Sh}(X)$, $\mathbf{D}$ corresponds to Poincaré-Verdier duality on $D^b(\mathrm{Sh}(X))$. More precisely, [17, Theorem 3.2] states that, for $M^\bullet \in D^b(\mathrm{mod}\,\Lambda)$, we have

$$\mathbf{D}(M^\bullet)^\dagger \cong \mathrm{R}\mathcal{H}\mathrm{om}((M^\bullet)^\dagger, \mathcal{D}_X^\bullet)$$

in $D^b(\mathrm{Sh}(X))$. On the other hand, through the equivalence $\mathrm{mod}\,\Lambda \cong \mathrm{Sq}\,R$, the duality $\mathbf{D}$ on $D^b(\mathrm{mod}\,\Lambda)$ corresponds to our duality $\mathbb{D}$ on $D^b(\mathrm{Sq}\,R)$ up to translation. More precisely, $\mathbb{D}(-)[-1]$ corresponds to $\mathbf{D}(-)$, where the complex $M^\bullet[-1]$ of a complex $M^\bullet$ denotes the degree shifting of $M^\bullet$ with $M^\bullet[-1]^i = M^{i-1}$. So we have the following.

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THEOREM 6.1. *For $M^\bullet \in D^b(\operatorname{Sq} R)$, we have*

$$\mathbb{D}(M^\bullet)^+[-1] \cong \mathrm{R}\mathcal{H}\mathrm{om}((M^\bullet)^+, \mathcal{D}_X^\bullet)$$

in $D^b(\mathrm{Sh}(X))$. In particular, $(I_R^\bullet)^+[-1] \cong \mathcal{D}_X^\bullet$, where $I_R^\bullet$ is the complex constructed in the previous section.

By Proposition 5.5, if $M \in \operatorname{Sq} R$, then we have

$$H_{\mathfrak{m}}^i(M)^\vee \cong \operatorname{Ext}_R^{-i}(M, D_R^\bullet) \in \operatorname{Sq} R.$$

Hence $H_{\mathfrak{m}}^i(M)$ is $-|\mathcal{M}|$-graded and the next result determines the "Hilbert function" of $H_{\mathfrak{m}}^i(M)$.

THEOREM 6.2. *If $M \in \operatorname{Sq} R$, we have the following.*

(*a*) *There is an isomorphism*

$$H^i(X, M^+) \cong [H_{\mathfrak{m}}^{i+1}(M)]_0 \quad \textit{for all } i \geq 1,$$

and an exact sequence

$$0 \longrightarrow [H_{\mathfrak{m}}^0(M)]_0 \longrightarrow M_0 \longrightarrow H^0(X, M^+) \longrightarrow [H_{\mathfrak{m}}^1(M)]_0 \longrightarrow 0.$$

(*b*) *If $0 \neq a \in |\mathcal{M}|$ with $\sigma = \operatorname{supp}(a)$, then*

$$[H_{\mathfrak{m}}^i(M)]_{-a} \cong H_c^{i-1}(U_\sigma, M^+|_{U_\sigma})$$

for all $i \geq 0$. Here $U_\sigma = \bigcup_{\tau \geq \sigma} \tau$ is an open set of X, and $H_c^\bullet(-)$ stands for the cohomology with compact support.

Proof. (a) We have $H^i(\mathbb{D}(M)) \cong \operatorname{Ext}_R^i(M, D_R^\bullet) \cong H_{\mathfrak{m}}^{-i}(M)^\vee$ by Proposition 5.5. On the other hand, via the equivalence $\operatorname{Sq} R \cong \operatorname{mod} \Lambda$, $\mathbb{D}(-)[-1]$ corresponds to the duality $\mathbf{D}(-) = \mathrm{RHom}_\Lambda(-, \omega^\bullet)$ of $D^b(\operatorname{mod} \Lambda)$ introduced in [17]. So the assertion follows from [17, Corollary 3.5, Theorem 2.2].

(b) Similarly, it follows from [17, Lemma 5.1]. ∎

In the sequel, $\tilde{H}^i(X; \Bbbk)$ denotes the i^{th} reduced cohomology of X with coefficients in $\Bbbk$. That is, $\tilde{H}^i(X; \Bbbk) \cong H^i(X; \Bbbk)$ for all $i \geq 1$, and $\tilde{H}^0(X; \Bbbk) \oplus \Bbbk \cong H^0(X; \Bbbk)$. Here $H^i(X; \Bbbk)$ is the usual cohomology of X with coefficients in $\Bbbk$.

Corollary 6.3. (cf. Brun et al. [1, Theorem 1.3]) *With the above notation, we have* $[H^i_{\mathfrak{m}}(R)]_0 \cong \tilde{H}^{i-1}(X;\Bbbk)$ *and* $[H^i_{\mathfrak{m}}(R)]_{-a} \cong H^{i-1}_c(U_\sigma, \underline{\Bbbk}_{U_\sigma})$ *for all* $i \geq 0$ *and all* $0 \neq a \in |\mathcal{M}|$. *Here* $\sigma = \mathrm{supp}(a)$, *and* $\underline{\Bbbk}_{U_\sigma}$ *is the* $\Bbbk$-*constant sheaf on* U_σ.

Proof. The second isomorphism is a direct consequence of Theorem 6.2 (b) and the fact that $R^+ \cong \underline{\Bbbk}_X$. So it suffices to show the first. By the isomorphism of Theorem 6.2 (a), $[H^i_{\mathfrak{m}}(R)]_0 \cong H^{i-1}(X, R^+) \cong H^{i-1}(X, \underline{\Bbbk}_X) \cong H^{i-1}(X;\Bbbk) \cong \tilde{H}^{i-1}(X;\Bbbk)$ for all $i \geq 2$. Similarly, by the exact sequence of the theorem and that $H^0_{\mathfrak{m}}(R) = 0$, we have $0 \to R_0 \to H^0(X;\Bbbk) \to [H^1_{\mathfrak{m}}(R)]_0 \to 0$. Since $R_0 = \Bbbk$, we have $[H^1_{\mathfrak{m}}(R)]_0 \cong \tilde{H}^0(X;\Bbbk)$. ☐

We say R is a *Buchsbaum ring*, if $R_{\mathfrak{m}'}$ is a Buchsbaum local ring for all maximal ideal $\mathfrak{m}'$. See [13] for further information.

Theorem 6.4. *Set* $\dim X = d$ (*equivalently,* $\dim R = d+1$). *Then* R *is Buchsbaum if and only if* $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ *for all* $i \neq -d$. *In particular, the Buchsbaum property of* R *is a topological property of* X (*while it might depend on* $\mathrm{char}(\Bbbk)$).

Proof. Assume that $\mathcal{H}^i(\mathcal{D}^\bullet_X) \neq 0$ for some $i \neq -d$ (equivalently, $-d+1 \leq i \leq 0$). Then $[H^{i-1}(I^\bullet_R)]_a \neq 0$ for some $0 \neq a \in |\mathcal{M}|$ by Theorem 6.1. Since $H^{i-1}(I^\bullet_R)$ is squarefree, we have $\dim_\Bbbk(H^{i-1}(I^\bullet_R) \otimes_R R_{\mathfrak{m}}) = \infty$. Since $H^{i-1}(I^\bullet_R) \otimes_R R_{\mathfrak{m}}$ is the Matlis dual of $H^{1-i}_{\mathfrak{m}}(R_{\mathfrak{m}})$ over the local ring $R_{\mathfrak{m}}$, we have $\dim_\Bbbk H^{1-i}_{\mathfrak{m}}(R_{\mathfrak{m}}) = \infty$ and $R_{\mathfrak{m}}$ is not Buchsbaum.

Conversely, assume that $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ for all $i \neq -d$. Then $H^i(I^\bullet_R) = [H^i(I^\bullet_R)]_0$ for all $i \neq -d-1$, and they are $\Bbbk$-vector spaces (that is, $R/\mathfrak{m}$-modules). Hence $H^i(I^\bullet_R) \otimes_R R_{\mathfrak{m}'} = 0$ for all $i \neq -d-1$ and all $\mathfrak{m}'$ with $\mathfrak{m}' \neq \mathfrak{m}$. Thus $R_{\mathfrak{m}'}$ is Cohen-Macaulay (in particular, Buchsbaum). It remains to show that $R_{\mathfrak{m}}$ is Buchsbaum. Set $T^\bullet := \tau_{-d-1} I^\bullet_R$. Here, for a complex $M^\bullet$ and an integer r, $\tau_{-r} M^\bullet$ denotes the truncated complex

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Im}(M^{-r} \to M^{-r+1}) \longrightarrow M^{-r+1} \longrightarrow M^{-r+2} \longrightarrow \cdots .$$

By the assumption, we have $H^i(T^\bullet) = [H^i(T^\bullet)]_0$ for all i. Since $T^\bullet$ is a complex of $\mathcal{M}$-graded modules, $U^\bullet := \bigoplus_{0 \neq a \in |\mathcal{M}|} (T^\bullet)_a$ is a subcomplex of $T^\bullet$, and a natural map $T^\bullet \to (T^\bullet / U^\bullet)$ is a quasi-isomorphism by the above observation. Since $T^\bullet / U^\bullet$ is a complex of $\Bbbk$-vector spaces, $R_{\mathfrak{m}}$ is Buchsbaum by [13, II.Theorem 4.1]. ☐

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If $\dim X = d$ and R is Buchsbaum, we set $or_X := \mathcal{H}^{-d}(\mathcal{D}^\bullet_X) \in \mathrm{Sh}(X)$. The next fact follows from [10, IX, (4.1)].

PROPOSITION 6.5. (Poincaré duality) *With the above situation, we have* $H^i(X;\Bbbk) \cong H^{d-i}(X, or_X)$ *for all* i.

If X is a d-dimensional manifold (with or without boundary), then R is Buchsbaum and or_X is the usual *orientation sheaf* of X with coefficients in $\Bbbk$ (see, for example, [10, III, §8]). When X is an orientable manifold, then $or_X \cong \underline{\Bbbk}_X$. In this case, Proposition 6.5 is nothing other than the classical Poincaré duality.

Assume that $\dim X = d$, equivalently, $\dim R = d+1$. If R is Buchsbaum, we call $\omega_R := H^{-d-1}(I^\bullet_R) \in \mathrm{Sq}\, R$ the *canonical module* of R. Clearly, $(\omega_R)^+ \cong or_X$.

EXAMPLE 6.6. Recall the toric face ring R given in Example 2.9, whose underlying topological space X is the Möbius strip. Clearly, X is a manifold with boundary and R is Buchsbaum. It is easy to see that $\tilde{H}^2(X;\Bbbk) = 0$ and $or_X \cong i_!\, \underline{\Bbbk}_{X\setminus\partial X}$, where $\underline{\Bbbk}_{X\setminus\partial X}$ is the $\Bbbk$-constant sheaf on $X\setminus\partial X$ (∂X denotes the boundary of X), and $i : X\setminus\partial X \hookrightarrow X$ is the embedding map. Hence the canonical module ω_R is isomorphic to the monomial ideal I with $I^+ \cong i_!\underline{\Bbbk}_{X\setminus\partial X}$. So we have $\omega_R \cong (X_xX_u, X_zX_w, X_vX_y, X_xX_z, X_yX_w, X_xX_v)$, where the right side is an ideal of R.

We say R is *Gorenstein**, if it is Cohen-Macaulay and $\omega_R \cong R$ as $\mathbb{Z}\mathcal{M}$-graded modules.

THEOREM 6.7. *Set* $d := \dim X$.

(a) (*Caijun,* [6]) *R is Cohen-Macaulay if and only if* $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ *for all* $i \neq -d$, *and* $\tilde{H}^i(X;\Bbbk) = 0$ *for all* $i \neq d$.

(b) *Assume that* $d \geq 1$ *and R is Cohen-Macaulay. Then R is Gorenstein*, if and only if* $or_X \cong \underline{\Bbbk}_X$, *if and only if* $(or_X)_p \cong \Bbbk$ *for all* $p \in X$ *and* $H^d(X;\Bbbk) \neq 0$. *Here* $\underline{\Bbbk}_X$ *denotes the $\Bbbk$-constant sheaf on X and* $(or_X)_p$ *is the stalk of the sheaf* or_X *at* p.

Proof. (a) Since $\dim R = d+1$, R is Cohen-Macaulay if and only if $H^i(I^\bullet_R)$ $(= \mathrm{Ext}^i_R(R, D^\bullet_R)) = 0$ for all $i \neq -d-1$. By Theorem 6.1, the above conditions are also equivalent to that $\mathcal{H}^i(\mathcal{D}^\bullet_X) = 0$ for all $i \neq -d$

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and $[H^i(I_R^\bullet)]_0 = 0$ for all $i \neq -d-1$. Since $[H^i(I_R^\bullet)]_0 \cong ([H_\mathfrak{m}^{-i}(R)]_0)^* \cong \tilde{H}^{-i-1}(X;\Bbbk)^*$, we are done.

(b) We show the first equivalence. If R is Gorenstein*, then $or_X \cong (\omega_R)^+ \cong R^+ \cong \underline{\Bbbk}_X$. So we get the necessity. Next assume that or_X $(= (\omega_R)^+) \cong \underline{\Bbbk}_X$. Then we have that

$$[\omega_R]_a = \Bbbk \quad \text{for all } 0 \neq a \in |\mathcal{M}|. \tag{6.1}$$

On the other hand, by Proposition 6.5, we have $[\omega_R]_0^\vee \cong [H_\mathfrak{m}^{d+1}(R)]_0 \cong H^d(X;\Bbbk) \cong H^0(X, or_X) \cong H^0(X;\Bbbk) \cong \Bbbk$ (since R is Cohen-Macaulay and $d \geq 1$, $\tilde{H}^0(X;\Bbbk) = 0$ and X is connected). Take a non-zero element $x \in [\omega_R]_0$. Since ω_R is a squarefree R-module, $M := Rx$ is a squarefree submodule of ω_R. Set

$$\begin{aligned} \Upsilon &:= \{\operatorname{supp}(a) \mid a \in |\mathcal{M}|, M_a = [\omega_R]_a\} \\ &= \{\operatorname{supp}(a) \mid a \in |\mathcal{M}|, M_a \neq 0\} \subset \mathcal{X}. \end{aligned}$$

Here the second equality follows from the condition (6.1). It is easy to see that $\sigma \leq \tau \in \Upsilon$ implies $\sigma \in \Upsilon$. So we have a direct sum decomposition $\omega_R = M \oplus (\bigoplus_{\operatorname{supp}(a) \in |\mathcal{M}| \setminus \Upsilon} [\omega_R]_a)$ as an R-module. On the other hand, ω_R is indecomposable. Hence $\omega_R = M \cong R$ as $\mathbb{Z}\mathcal{M}$-graded modules. So we get the sufficiency.

For the second equivalence, it is enough to prove the sufficiency. Since $[\omega_R]_0 \cong H^d(X;\Bbbk) \neq 0$, we can take $0 \neq x \in [\omega_R]_0$. By argument similar to the above, $(Rx)^+$ is a direct summand of or_X. Note that X is connected and $\underline{\Bbbk}_X$ is indecomposable. Since $\underline{\Bbbk}_X \cong \mathcal{E}xt^{-d}(or_X, \mathcal{D}_X^\bullet)$, or_X is also indecomposable. Hence $or_X \cong (Rx)^+ \cong \underline{\Bbbk}_X$. We are done. ∎

Corollary 6.8. *The Cohen-Macaulay property and Gorenstein* property of R are topological properties of X (while it may depend on* char$(\Bbbk)$).

Proof. Most of the statement is a direct consequence of Theorems 6.7. It remains to consider the Gorenstein* property in the case $\dim R = 0$. Then R is Gorenstein* if and only if X consists of exactly two points. So the assertion is clear. ∎

Remark 6.9. The main result of Caijun [6] is much more general than our Theorems 6.7 (a). However, since he worked in a wider context, his argument does not give precise information of local cohomologies and canonical modules.

Recall that $\mathcal{M}$ admits a finite subset $\{a_e\}_{e\in E}$ of $|\mathcal{M}|$ generating $\Bbbk[\mathcal{M}]$ as a $\Bbbk$-algebra. Then the polynomial ring $S := \Bbbk[X_e \mid e \in E]$ surjects on $\Bbbk[\mathcal{M}]$. Let $I_{\mathcal{M}}$ be its kernel (i.e., $\Bbbk[\mathcal{M}] = S/I_{\mathcal{M}}$). A remarkable result [5, Theorem 3.8] of Bruns et al. shows that (if $\mathcal{M}$ is cone-wise normal) there is a generating set $\{a_e\}_{e\in E}$ and a term order $\succ$ on S such that the initial ideal $\mathrm{in}_{\succ}(I_{\mathcal{M}})$ is a radical monomial ideal. In this case, $\mathrm{in}_{\succ}(I_{\mathcal{M}})$ equals to the Stanley-Reisner ring I_Δ of a simplicial complex Δ which gives a triangulation of X. Hence, by a basic fact on Gröbner bases, the sufficiency of Theorems 6.4 and 6.7 (b) follow from their result, at least under the additional assumption that R admits an $\mathbb{N}$-grading with $R_0 = \Bbbk$.

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