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ON A THEOREM OF HEILBRONN CONCERNING THE FRACTIONAL PART OF θn^2

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1. In 1948 Heilbronn [**4**] proved the following theorem.

THEOREM H. *For every real θ and every positive integer N, there is an integer n satisfying*

$$(1.1) \qquad 1 \leqq n \leqq N, \qquad \|\theta n^2\| < C(\epsilon)N^{-1/2+\epsilon},$$

where ϵ is an arbitrarily small number, $C(\epsilon)$ depends only on ϵ, and $\|t\|$ means the distance from t to the nearest integer.

The interest of the result (1.1) is that the inequality is uniform in θ, and is therefore analogous to the classical inequality of Dirichlet for the fractional part of θn. In this paper we shall prove the following theorem.

THEOREM. *For every real θ and every positive integer N, there is an integer n satisfying*

$$(1.2) \qquad 1 \leqq n \leqq N, \qquad \|\theta n^2\| < AN^{-1/2+\epsilon(N)},$$

where A is an absolute constant and $\epsilon(N) = 1/\log\log N$. Furthermore, there is a positive integer N_1 such that for each $N \geqq N_1$, (1.2) is true for $A = 1$.

2. In what follows, we always assume that N is a sufficiently large positive integer, say $N \geqq N_0$, such that all the subsequent asymptotic approximations and inequalities are satisfied. Thus it is difficult to define N_0 at the beginning or at any particular point. We use the following notation: $x \ll y$ means $x < Ay$, where A is a positive absolute constant. $[t]$ is the integral part of t. $\epsilon(N)$ means $1/\log\log N$ and for real α, we write $e(\alpha) = \exp\{2\pi\alpha i\}$.

We need several lemmas.

LEMMA 1. *Let $d(n)$ be the number of divisors of an integer n, including 1 and n. Then there exists some positive integer n_0 such that for all $n \geqq n_0$ we have*

$$(2.1) \qquad d(n) < n^{(3/4)\epsilon(n)}.$$

Proof. Lemma 1 follows if in [3, p. 262, Theorem 317] we choose $\epsilon > 0$ such that $2^{1+\epsilon} \leqq e^{3/4}$.

It is remarked that for $n > e^e$, $n^{(3/4)\epsilon(n)}$ is an increasing sequence tending to infinity and $\log n = o(n^{a\epsilon(n)})$ for any positive constant a.

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LEMMA 2. *Suppose that Δ is a number satisfying $0 < \Delta < \frac{1}{2}$ and r is a positive integer. Then there exists a real function $\psi(x)$, periodic with period 1, which satisfies*

$$\psi(x) = 0 \quad \textit{if } \|x\| \geqq \Delta, \tag{2.2}$$

and

$$\psi(x) = \sum_{k=-\infty}^{\infty} c_k e(kx),$$

where c_k are real and

$$c_0 = \Delta, \qquad |c_k| \ll \min\left(\Delta, \left(\frac{r}{\pi}\right)^r \Delta^{-r}|k|^{-r-1}\right) \tag{2.3}$$

for $k \neq 0$.

Proof. This is a particular case of [**6**, p. 32, Lemma 12] with $\beta = -\alpha = \frac{1}{2}\Delta$.

LEMMA 3. *Let $S = \sum_{n=1}^{N} e(\theta n^2)$. Then*

$$|S|^2 \ll \left(N + N^{(3/4)\epsilon(N)} \sum_{m=1}^{2N} \min\left(N, \frac{1}{\|\theta m\|}\right)\right). \tag{2.4}$$

Proof. Replacing ϵ in [**1**, p. 229, Theorem 5.7] by $\frac{3}{4}\epsilon(N)$ and using our Lemma 1 we can prove Lemma 3 in exactly the same way as [**1**, Theorem 5.7].

LEMMA 4. *Let*

$$q > 0, \qquad \left|\theta - \frac{a}{q}\right| < \frac{1}{q^2}, \qquad (a, q) = 1.$$

Then

$$\sum_{j=p+1}^{p+q} \min\left(N, \frac{1}{\|\theta j\|}\right) \ll (N + q \log q),$$

where p is some positive integer.

Proof. Lemma 4 is well known. See, for example, [**5**, p. 23, Lemma 3.5].

3. The proof of the theorem is essentially a refinement of Davenport's method [2]. We suppose that

$$\|\theta n^2\| \geqq N^{-1/2+\epsilon(N)} \tag{3.1}$$

for $n = 1, 2, \ldots, N$. Putting $\Delta = N^{-1/2+\epsilon(N)}$ in Lemma 2 we have

$$\sum_{n=1}^{N} \psi(\theta n^2) = \sum_{n=1}^{N} \sum_{k=-\infty}^{\infty} c_k e(k\theta n^2) = c_0 N + \sum_{k=1}^{\infty} c_k S_k + \sum_{k=1}^{\infty} c_{-k} S_{-k} = 0,$$

where $S_k = \sum_{n=1}^{N} e(k\theta n^2)$. Hence

$$\Delta N \leqq \sum_{k=1}^{\infty} |c_k S_k| + \sum_{k=1}^{\infty} |c_{-k} S_{-k}| = T_1 + T_2, \quad \text{say}. \tag{3.2}$$

We first estimate the value of T_1.

$$T_1 = \sum_{k=1}^{\infty} |c_k S_k| = \left(\sum_{k=1}^{M} + \sum_{M+1}^{\infty}\right) |c_k S_k| = T_{11} + T_{12}, \quad \text{say},$$

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where $M = [N^{1/2-(31/32)\epsilon(N)}]$. By (2.3) we have

$$T_{12} = \sum_{k=M+1}^{\infty} |c_k S_k| \tag{3.3}$$
$$\ll N\left(\frac{r}{\pi}\right)^r \Delta^{-r} \sum_{k=M+1}^{\infty} k^{-r-1}$$
$$\ll N r^{r-1} \Delta^{-r} M^{-r}$$
$$\ll N\Delta(r^r \Delta^{-r-1} M^{-r}).$$

$$T_{11} = \sum_{k=1}^{M} |c_k S_k| \tag{3.4}$$
$$\ll \Delta \sum_{k=1}^{M} |S_k|.$$

Since $S_{-k} = \bar{S}_k$, we have the same estimate of the value of T_2. It follows from (3.2), (3.3), and (3.4) that

$$N(1 - Ar^r \Delta^{-r-1} M^{-r}) \ll \sum_{k=1}^{M} |S_k|, \tag{3.5}$$

where A is some absolute constant. Putting $r = [32/\epsilon(N)] = [32 \log\log N]$, we see that

$$r^r \Delta^{-r-1} M^{-r} \ll r^r N^{1/2-(1/32) r \epsilon(N) - \epsilon(N)}$$
$$\ll (32 \log\log N)^{(32 \log\log N)} N^{-1/2}$$
$$= o(1),$$

as $N \to \infty$. It follows from (3.5) that $N \ll \sum_{k=1}^{M} |S_k|$. Using Hölder's inequality we have

$$M^{-1}N^2 \ll \sum_{k=1}^{M} |S_k|^2.$$

By Lemma 3 we see that

$$M^{-1}N^2 \ll \sum_{k=1}^{M} \left(N + N^{(3/4)\epsilon(N)} \sum_{m=1}^{2N} \min\left(N, \frac{1}{||\theta km||}\right)\right)$$
$$\ll MN + N^{(3/4)\epsilon(N)} \sum_{k=1}^{M} \sum_{m=1}^{2N} \min\left(N, \frac{1}{||\theta km||}\right).$$

Since $M^2 N^{-1} \leqq N^{-\epsilon(N)} = o(1)$ as $N \to \infty$, we have

$$M^{-1} N^{2-(3/4)\epsilon(N)} \ll \sum_{k=1}^{M} \sum_{m=1}^{2N} \min\left(N, \frac{1}{||\theta km||}\right). \tag{3.6}$$

Let $j = km$ ($k = 1, 2, \ldots, M$; $m = 1, 2, \ldots, 2N$). Since, by Lemma 1,

$$d(j) \ll (2MN)^{(3/4)\epsilon(2MN)}$$
$$\ll N^{(9/8)\epsilon(N)},$$

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we have

$$M^{-1}N^{2-(15/8)\epsilon(N)} \ll \sum_{j=1}^{2MN} \min\left(N, \frac{1}{\|\theta j\|}\right). \tag{3.7}$$

Suppose that a/q is any irreducible fraction such that

$$\left|\theta - \frac{a}{q}\right| < 1/q^2. \tag{3.8}$$

We divide the sum on the right of (3.7) into blocks of q terms. The number of blocks is at most $q^{-1}2MN + 1$. By Lemma 4 we see that

$$M^{-1}N^{2-(15/8)\epsilon(N)} \ll (q^{-1}MN + 1)(N + q \log q).$$

Let

$$q \leqq M^{-1}N^{2-2\epsilon(N)}. \tag{3.9}$$

We see that

$$\begin{aligned} N &\ll M^{-1}N^{2-(15/8)\epsilon(N)}N^{-1/2} = o(M^{-1}N^{2-(15/8)\epsilon(N)}); \\ MN \log q &\ll M^{-1}N^{2-(15/8)\epsilon(N)}(N^{-(1/16)\epsilon(N)} \log N) \\ &= o(M^{-1}N^{2-(15/8)\epsilon(N)}); \\ q \log q &\ll M^{-1}N^{2-(15/8)\epsilon(N)}(N^{-(1/8)\epsilon(N)} \log N) \\ &= o(M^{-1}N^{2-(15/8)\epsilon(N)}), \end{aligned}$$

as $N \to \infty$. Thus

$$M^{-1}N^{2-(15/8)\epsilon(N)} \ll q^{-1}MN^2,$$

or

$$\begin{aligned} q &\ll M^2N^{(15/8)\epsilon(N)} \\ &\ll N^{1-(1/16)\epsilon(N)} \\ &\leqq N. \end{aligned} \tag{3.10}$$

Then the consequence of the assumption (3.1) made at the beginning of this section is that if a/q satisfies (3.8) and (3.9), then it necessarily satisfies (3.10). By Dirichlet's theorem, there exists a/q such that $q \leqq M^{-1}N^{2-2\epsilon(N)}$ and

$$\left|\theta - \frac{a}{q}\right| < q^{-1}MN^{-2+2\epsilon(N)}.$$

This q must also satisfy (3.10). Hence

$$\begin{aligned} \|\theta q^2\| &< |\theta q^2 - aq| \\ &< qMN^{-2+2\epsilon(N)} \\ &\ll N^{-1/2+(31/32)\epsilon(N)} \\ &< AN^{-(1/32)\epsilon(N)}N^{-1/2+\epsilon(N)}. \end{aligned}$$

Put $q = n$ and define $N_1 > N_0$ such that $AN_1^{-(1/32)\epsilon(N_1)} < 1$. This proves the theorem.

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References

1. R. Ayoub, *An introduction to the analytic theory of numbers*, Mathematical Surveys, No. 10 (Amer. Math. Soc., Providence, R.I., 1963).
2. H. Davenport, *On a theorem of Heilbronn*, Quart. J. Math. Oxford Ser. *18* (1967), 339–344.
3. G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 3rd ed. (Oxford, at the Clarendon Press, 1954).
4. H. Heilbronn, *On the distribution of the sequence* $n^2\theta$ (mod 1), Quart. J. Math. Oxford Ser. *19* (1948), 249–256.
5. L. K. Hua, *Additive theory of prime numbers*, Translations of Mathematical Monographs, Vol. 13 (Amer. Math. Soc., Providence, R.I., 1965).
6. I. M. Vinogradov, *The method of trigonometric sums in the theory of numbers* (Interscience, New York, 1954).

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