EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR AN ELLIPTIC EQUATION WITH $p(x)$-GROWTH CONDITIONS

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Abstract. We study a partial differential equation on a bounded domain $\Omega \subset \mathbb{R}^N$ with a $p(x)$-growth condition in the divergence operator and we establish the existence of at least two nontrivial weak solutions in the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. Such equations have been derived as models of several physical phenomena. Our proofs rely essentially on critical point theory combined with corresponding variational techniques.

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1. Introduction. The study of partial differential equations with $p(x)$-growth conditions has received more and more interest in recent decades. The specific attention accorded to such problems is due to their applications in mathematical physics. More precisely, such equations are used to model phenomena which arise in elastic mechanics or electrorheological fluids. For a general account of the underlying physics and for some technical applications we refer to [**1**, **4**, **14**, **16**, **20**, **21**, **24**, **25**] and the references therein.

A typical model of an elliptic equation with $p(x)$-growth conditions is

$$-\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = g(x, u).$$

The operator $\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplace operator and it is a natural generalization of the p-Laplace operator, in which $p(x) = p > 1$ is a constant. For this reason the equations studied in the case in which the $p(x)$-Laplace operator is involved are, in general, extensions of p-Laplacian problems. (See for example [**3**, **9**, **11**].) However, we point out that such generalizations are not trivial since the $p(x)$-Laplace operator possesses more complicated nonlinearity; for example it is inhomogeneous. We just remember the fact that defining

$$\lambda_1 := \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\, dx}{\displaystyle\int_\Omega |u|^{p(x)}\, dx}$$

we often have $\lambda_1 = 0$ for general $p(x)$, and only under some special conditions $\lambda_1 > 0$. (See [**12**].)

In this paper we study the existence of solutions for a boundary value problem of the type

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u) & (\text{for} \quad x \in \Omega), \\ u = 0, & (\text{for} \quad x \in \partial\Omega), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary, $1 < p(x)$ and $p(x) \in C(\overline{\Omega})$.

Equation (1) will be studied in the framework of the variable Lebesgue and Sobolev spaces $L^{p(x)}$ and $W^{1,p(x)}$ which will be briefly described in the following section.

2. Preliminary results. Variable exponent Lebesgue spaces $L^{p(x)}$, where $p(x)$ is a real-valued function, appeared in the literature for the first time already in a 1931 article by W. Orlicz [19]. In the years 1950 this study was carried on by Nakano [18] who made the first systematic study of spaces with variable exponent. Later, the Polish mathematicians investigated the modular function spaces. (See, for example, Musielak [17].) Variable exponent Lebesgue spaces on the real line have been independently developed by Russian researchers. In this context we refer to the work of Tsenov [23], Sharapudinov [22] and Zhikov [25, 26].

Throughout this paper we assume that $p(x) > 1$, $p(x) \in C(\overline{\Omega})$.

Set

$$C_+(\overline{\Omega}) = \{h; \; h \in C(\overline{\Omega}), \; h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x \in \Omega} h(x) \quad \text{and} \quad h^- = \inf_{x \in \Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \{u; \; u \text{ is a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\}.$$

We define a norm, the so-called *Luxemburg norm*, on this space by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0; \; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)}\,dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces [15, Theorem 2.5], the Hölder inequality holds [15, Theorem 2.1], they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ [15, Corollary 2.7] and continuous functions are dense if $p^+ < \infty$ [15, Theorem 2.11]. The inclusion between Lebesgue spaces also generalizes naturally [15, Theorem 2.8]: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponent, so that $p_1(x) \leq p_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$, whose norm does not exceed $|\Omega| + 1$.

We denote by $L^{q(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ the Hölder type inequality

$$\left| \int_\Omega uv \, dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{q^-} \right) |u|_{p(x)} |v|_{q(x)} \tag{2}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space. This is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)} \, dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$, then the following relations holds true

$$|u|_{p(x)} > 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^+}, \tag{3}$$

$$|u|_{p(x)} < 1 \quad \Rightarrow \quad |u|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq |u|_{p(x)}^{p^-}, \tag{4}$$

$$|u|_{p(x)} = 1 \quad \Rightarrow \quad \rho_{p(x)}(u) = 1. \tag{5}$$

Spaces with $p^+ = \infty$ have been studied by Edmunds, Lang and Nekvinda [5].

Next, we define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\| = |\nabla u|_{p(x)}.$$

The space $(W_0^{1,p(x)}(\Omega), \| \cdot \|)$ is a separable and reflexive Banach space. We note that if $q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$, for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous. Here $p^\star(x) = \frac{Np(x)}{N-p(x)}$ if $p(x) < N$ or $p^\star(x) = +\infty$ if $p(x) \geq N$. We refer to [6, 7, 10, 13, 15] for further properties of variable exponent Lebesgue-Sobolev spaces.

3. The main result. In this paper we study problem (1) in the particular case

$$f(x, t) = A|t|^{a-2}t + B|t|^{b-2}t$$

with $1 < a < p^- < p^+ < b < \min\{N, \frac{Np^-}{N-p^-}\}$ and $A, B > 0$. More precisely, we consider the degenerate boundary value problem

$$\begin{cases} -\text{div}(|\nabla u|^{p(x)-2}\nabla u) = A|u|^{a-2}u + B|u|^{b-2}u & \text{(for } x \in \Omega), \\ u = 0 & \text{(for } x \in \partial\Omega). \end{cases} \tag{6}$$

We seek solutions for problem (6) belonging to the space $W_0^{1,p(x)}(\Omega)$ in the sense below.

DEFINITION 1. *We say that* $u \in W_0^{1,p(x)}(\Omega)$ *is a weak solution for problem* (1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx - A \int_\Omega |u|^{a-2}uv \, dx - B \int_\Omega |u|^{b-2}uv \, dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$.

We prove the following result.

THEOREM 1. *There exists $\lambda > 0$ such that, for any $A \in (0, \lambda)$ and any $B \in (0, \lambda)$, problem (6) has at least two distinct nontrivial weak solutions.*

REMARK. In the case where in equation (6) we assume that $A = B = \mu$ we deal with an eigenvalue problem, namely

$$
\begin{cases}
-\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) = \mu(|u|^{a-2}u + |u|^{b-2}u) & \text{(for } x \in \Omega), \\
u = 0 & \text{(for } x \in \partial\Omega).
\end{cases}
\tag{7}
$$

Theorem 1 ensures that problem (7) has a continuous family of positive eigenvalues that lie in a neighborhood of the origin and which are not simple (their multiplicity being at least two). Furthermore, we obtain

$$
\inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^{p(x)}\, dx}{\displaystyle\int_\Omega (|u|^a + |u|^b)\, dx} = 0.
$$

4. Proof of Theorem 1.   Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. The energy functional corresponding to problem (6) is defined as $J : E \to \mathbb{R}$,

$$
J(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx - \frac{A}{a}\int_\Omega |u|^a\, dx - \frac{B}{b}\int_\Omega |u|^b\, dx.
$$

Arguments similar to those used in [11] ensure that $J \in C^1(E, \mathbb{R})$ with

$$
\langle J'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v\, dx - A\int_\Omega |u|^{a-2}uv\, dx - B\int_\Omega |u|^{b-2}uv\, dx,
$$

for all $u, v \in E$. Thus the weak solutions of (6) are exactly the critical points of J. We shall prove that the functional J possesses two distinct critical points using as main tools the Mountain Pass Theorem (see, e.g. [2]) and Ekeland's Variational Principle (see, e.g. [8]).

LEMMA 1. *The following assertions hold.*
(i) *There exist $\rho > 0$ and $\alpha > 0$ such that*

$$
J(u) \geq \alpha > 0, \quad \forall u \in E \text{ with } \|u\| = \rho.
$$

(ii) *There exists $\psi \in E$ such that*

$$
\lim_{t \to \infty} J(t\psi) = -\infty.
$$

(iii) *There exists $\varphi \in E$ such that $\varphi \geq 0$, $\varphi \neq 0$ and*

$$
J(t\varphi) < 0
$$

for $t > 0$ small enough.

Proof. (i) It is clear that E is continuously embedded in $L^a(\Omega)$ and in $L^b(\Omega)$. Thus there exist two positive constants c_1 and c_2 such that

$$\int_\Omega |u|^a \, dx \leq c_1 \|u\|^a, \quad \forall \, u \in E, \tag{8}$$

and

$$\int_\Omega |u|^b \, dx \leq c_2 \|u\|^b, \quad \forall \, u \in E. \tag{9}$$

Using relations (8), (9) and (5) we deduce that for any $u \in E$ with $\|u\| = 1$ we have

$$J(u) \geq \frac{1}{p^+} - A\frac{1}{a}c_1 - B\frac{1}{b}c_2.$$

Taking

$$\lambda = \min \left\{ \frac{a}{4p^+ c_1}, \frac{b}{4p^+ c_2} \right\} \tag{10}$$

and providing that $A, B \in (0, \lambda)$ we obtain

$$J(u) \geq \frac{1}{2p^+}, \quad \forall \, u \in E \text{ with } \|u\| = 1.$$

The first part of Lemma 1 is proved.

(ii) Let $\psi \in C_0^\infty(\Omega)$, $\psi \geq 0$, $\psi \neq 0$ and $t > 1$. We have

$$J(t\psi) = \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \psi|^{p(x)} \, dx - \frac{A}{a} t^a \int_\Omega |\psi|^a \, dx - \frac{B}{b} t^b \int_\Omega |\psi|^b \, dx$$

$$\leq \frac{t^{p^+}}{p^-} \int_\Omega |\nabla \psi|^{p(x)} \, dx - \frac{B}{b} t^b \int_\Omega |\psi|^b \, dx.$$

Since $b > p^+$ we deduce that $\lim_{t \to \infty} J(t\psi) = -\infty$ and (ii) is proved.

(iii) Let $\varphi \in C_0^\infty(\Omega)$, $\varphi \geq 0$, $\varphi \neq 0$ and $t \in (0, 1)$. We have

$$J(t\varphi) = \int_\Omega \frac{t^{p(x)}}{p(x)} |\nabla \varphi|^{p(x)} \, dx - \frac{A}{a} t^a \int_\Omega |\varphi|^a \, dx - \frac{B}{b} t^b \int_\Omega |\varphi|^b \, dx$$

$$\leq \frac{t^{p^-}}{p^-} \int_\Omega |\nabla \varphi|^{p(x)} \, dx - \frac{A}{a} t^a \int_\Omega |\varphi|^a \, dx < 0$$

for $t < \delta^{1/(p^- - a)}$ with

$$0 < \delta < \min \left\{ 1, \frac{\frac{A}{a} p^+ \int_\Omega |\varphi|^a \, dx}{\int_\Omega |\nabla \varphi|^{p(x)} \, dx} \right\}.$$

It follows that (iii) is proved.

The proof of Lemma 1 is complete.

$\square$

Proof of Theorem 1. Let $\lambda > 0$ be defined as in (10) and $A \in (0, \lambda)$, $B \in (0, \lambda)$. Using Lemma 1 (i) and (ii) and the Mountain Pass Theorem (see, e.g. [2]) we deduce

the existence of a sequence $\{u_n\}$ in E such that

$$J(u_n) \to \bar{c} > 0 \quad \text{and} \quad J'(u_n) \to 0 \text{ in } E^\star. \tag{11}$$

We prove that $\{u_n\}$ is bounded in E. Assume the contrary. Then, passing eventually to a subsequence, still denoted by $\{u_n\}$, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Thus we may consider that $\|u_n\| > 1$, for any integer n. Relations (11) and (4) imply that for n large enough we have

$$
\begin{aligned}
\bar{c} + 1 + \|u_n\| &\geq J(u_n) - \frac{1}{b}\langle J'(u_n), u_n \rangle \\
&= \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)} \, dx - \frac{A}{a} \int_\Omega |u_n|^a \, dx - \frac{B}{b} \int_\Omega |u_n|^b \, dx \\
&\quad - \frac{1}{b} \int_\Omega |\nabla u_n|^{p(x)} \, dx + \frac{A}{b} \int_\Omega |u_n|^a \, dx + \frac{B}{b} \int_\Omega |u_n|^b \, dx \\
&\geq \left(\frac{1}{p^+} - \frac{1}{b} \right) \|u_n\|^{p^-} + A \left(\frac{1}{b} - \frac{1}{a} \right) \int_\Omega |u_n|^a \, dx.
\end{aligned}
$$

Since $a, p^+ < b$ and relation (8) holds true, we deduce the existence of a positive constant c_3 such that

$$\bar{c} + 1 + \|u_n\| \geq \left(\frac{1}{p^+} - \frac{1}{b} \right) \|u_n\|^{p^-} - c_3 \|u_n\|^a,$$

for n large enough. Dividing the above inequality by $\|u_n\|^{p^-}$ and passing to the limit as $n \to \infty$ we obtain a contradiction. It follows that $\{u_n\}$ is bounded in E. Thus, there exists $u_1 \in E$ such that passing to a subsequence, still denoted by $\{u_n\}$, it converges weakly to u_1 in E. Since $a, b < \frac{Np^-}{N-p^-}$ we deduce that E is compactly embedded in $L^a(\Omega)$ and $L^b(\Omega)$ and it follows that $\{u_n\}$ converges strongly to u_1 in $L^a(\Omega)$ and $L^b(\Omega)$. On the other hand, relation (11) yields

$$\lim_{n \to \infty} \langle J'(u_n), u_n - u_1 \rangle = 0.$$

Using the above information we find that

$$\lim_{n \to \infty} \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \nabla (u_n - u_1) \, dx = 0. \tag{12}$$

Relation (12) and the fact that $\{u_n\}$ converges weakly to u_1 in E enable us to apply Theorem 3.1 in [11] in order to obtain that $\{u_n\}$ converges strongly to u_1 in E. Then, since $J \in C^1(E, \mathbb{R})$ and relation (11) holds true we conclude that

$$J(u_1) = \bar{c} \quad \text{and} \quad J'(u_1) = 0. \tag{13}$$

It follows that u_1 is a nontrivial weak solution for problem (6).

We prove now that there exists a second weak solution $u_2 \in E$ such that $u_2 \neq u_1$. By Lemma 1 (i) it follows that on the boundary of the unit ball, centered at the origin in E and denoted by $B_1(0)$, we have

$$\inf_{\partial B_1(0)} J > 0. \tag{14}$$

On the other hand, by Lemma 1 (iii) there exists $\varphi \in E$ such that $J(t\varphi) < 0$, for all $t > 0$ small enough. Moreover, relations (8), (9) and (4) imply that, for any $u \in B_1(0)$, the inequality

$$J(u) \geq \frac{1}{p^+}\|u\|^{p^+} - \frac{A}{a}c_1\|u\|^a - \frac{B}{b}c_2\|u\|^b$$

holds and we deduce that

$$-\infty < \underline{c} := \inf_{B_1(0)} J < 0.$$

We now let $0 < \epsilon < \inf_{\partial B_1(0)} J - \inf_{B_1(0)} J$. Apply Ekeland's Variational Principle for the functional $J : \overline{B_1(0)} \to \mathbb{R}$, (see [**8**]). There exists $u_\epsilon \in \overline{B_1(0)}$ such that

$$J(u_\epsilon) < \inf_{B_1(0)} J + \epsilon,$$

$$J(u_\epsilon) < J(u) + \epsilon \cdot \|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$J(u_\epsilon) \leq \inf_{B_1(0)} J + \epsilon \leq \inf_{B_1(0)} J + \epsilon < \inf_{\partial B_1(0)} J,$$

it follows that $u_\epsilon \in B_1(0)$. Now, we define $I : \overline{B_1(0)} \to \mathbb{R}$ by $I(u) = J(u) + \epsilon \cdot \|u - u_\epsilon\|$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + t \cdot v) - I(u_\epsilon)}{t} \geq 0,$$

for a small $t > 0$ and $v \in B_1(0)$. The relation above yields

$$\frac{J(u_\epsilon + t \cdot v) - J(u_\epsilon)}{t} + \epsilon \cdot \|v\| \geq 0.$$

Letting $t \to 0$ it follows that $\langle J'(u_\epsilon), v \rangle + \epsilon \cdot \|v\| > 0$ and we infer that $\|J'(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{w_n\} \subset B_1(0)$ such that

$$J(w_n) \to \underline{c} \quad \text{and} \quad J'(w_n) \to 0. \tag{15}$$

Using the same arguments as in the case of solution u_1 we can prove that $\{w_n\}$ converges strongly to u_2 in E. Moreover, since $J \in C^1(E, \mathbb{R})$, by relation (15) it follows that

$$J(u_2) = \underline{c} \quad \text{and} \quad J'(u_2) = 0. \tag{16}$$

Thus, u_2 is also a nontrivial weak solution for problem (6).

Finally, we point out the fact that $u_1 \neq u_2$ since

$$J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2).$$

The proof of Theorem 1 is complete. $\square$

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