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VALUATIONS AND PRUFER RINGS

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1. Introduction. The word ring is used to mean commutative ring.

Just as valuations on fields are used to study domains, so valuations on rings can be used to study rings; these rings need not have units [**12**]. We introduce slightly weaker conditions than having identity in order to get a more general theory. A Prufer ring A is one in which every finitely generated regular ideal is invertible. If we replace invertibility in the total quotient ring K, by invertibility in a ring R where $A \subseteq R \subseteq K$ we get an R-Prufer ring. These rings do occur, for example the Witt ring of a non-Pythagorean field or a ring of bounded continuous functions.

This paper is devoted to the questions: To what extent do the properties of valuations on fields and Prufer domains extend to valuations on rings? What properties on a ring ensure that valuations on it will have desirable properties?

2. Idempotents and fixing elements. A commutative ring A is called a *ring with fixing elements* if for every element x in A there exists y in A such that $xy = x$. If y can be chosen so that $y^2 = y$ then A is said to be *generated by idempotents*.

Rings with fixing elements inherit many of the properties of rings with identity; the following properties of rings with fixing elements A either may be found in Gilmer [**8**] or may be easily deduced. Each ideal is contained in a maximal ideal, and all maximal ideals are prime. If $M \neq A$ is any ideal and a is any element then there exists $y \in A\backslash M$ such that $ay = a$. If $Ax = 0$ then $x = 0$. If A is a noetherian, or contains a nonzero divisor, or is local (a ring is local if an element not in the maximal ideal generates A), then A has an identity. In particular the localization at any maximal ideal of A has an identity. Any ideal of A is completely determined by its extensions to the localization of A at its maximal ideals. Let B be any ring which is integral over A and has fixing elements in A. The "lying over" and "going up" theorems of Krull-Cohen-Seidenberg hold for the prime ideals of A and B. Any ideal which has a maximal ideal as radical is primary.

If A is a ring with fixing elements (respectively generated by idempotents, with unit) we use the expression *B is a subring of A* to mean that B contains a set of fixing elements (respectively generating idempotents, unit) of A.

We now define the analogue of the total quotient ring for a ring generated by idempotents.

Let A be a ring containing the idempotent e. $a \in A$ is called *e-regular* if

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$ae = a$, and $ax = 0$ implies $ex = 0$. Let A be a ring generated by idempotents. A ring K containing A is called a *total quotient ring of A* provided that:

(i) If $x \in K$ then there exist $e, a, b \in A$ with $ex = x$, $e^2 = e$, b e-regular and $bx = a$;

(ii) If $a \in A$ is e-regular then there exists x in K such that $ax = e$.

PROPOSITION 1. *Any ring A generated by idempotents has a total quotient ring which is unique up to an isomorphism.*

Proof. Let E be the family of idempotents of A. Order E by setting $e \leqq f$ if $ef = e$. E is a directed set since $e \leqq g, f \leqq g$ where $g = e + f - ef$. If $e \leqq f$ define $h_{f,e}: Ae \to Af$ by the inclusion map. Then

$$A = \varinjlim_{e \in E} Ae.$$

Each ring Ae has a unit and has a natural embedding in a total quotient ring K_e. There is a natural injection $K_e \to K_f$ making the diagram

$$\begin{array}{ccc} K_e & \to & K_f \\ \uparrow & & \uparrow \\ Ae & \to & Af \end{array}$$

commutative given by $a/b \to a/(b + f - e)$. Then if

$$K = \varinjlim_{e \in E} K_e$$

it is easily seen that K is a total quotient ring for A. If T is any other total quotient ring of A then the isomorphism can be established with K by noting that $K_e \cong Te$ for each idempotent e.

If A has an identity the total quotient ring defined above is identical to the usual one.

Let A, a subring of the ring generated by idempotents R, have total quotient ring K. R is called a *sub-quotient ring* of A if $A \subseteq R \subseteq K$. An element of R is called *R-e-regular* if it has an e-inverse in R. An ideal (of A or R) is called *R-e-regular* if it contains an R-e-regular element and *R-regular* if it contains an R-e-regular element for every idempotent e in R. A K-regular ideal is called *regular*.

The following results may be proved relating ideals in K and A.

(i) A prime (or primary) ideal Q is a contracted ideal if and only if it is not regular.

(ii) There is a one-to-one correspondence between the contracted ideals (respectively prime ideals, P-primary ideals) of A and the ideals (respectively prime ideals, P-primary ideals) of K. This correspondence preserves radical, intersection and quotients.

(iii) If P is a non-regular prime ideal then A_P is naturally isomorphic to K_{KP}.

3. Valuations and ideals. A *valuation* v on a commutative ring with fixing elements R is a map from R onto a totally ordered group Γ, called the value group, together with a symbol ∞ (such that $\infty > \gamma$ for all $\gamma \in \Gamma$ and $\infty + \gamma = \infty$) with the following properties: $v(ab) = v(a) + v(b)$ and $v(a + b) \geqq \min \{v(a), v(b)\}$. The ring $A_v = \{x \in R | v(x) \geqq 0\}$ is called the *valuation ring of* v; $P_v = \{x \in R | v(x) > 0\}$ is called *the prime of* v and $v^{-1}(\infty)$, which is a prime ideal of both A and R, is called the *infinite prime of* v. If the value group of v is trivial then $v^{-1}(\infty) = P_v$, $A_v = R$ and v is called a *trivial valuation*. There is a one-to-one correspondence between trivial valuations and prime ideals of R. If A_v has an identity e, then e is the identity for R. For if $y \in R \backslash A_v$ and $yx = y$, then $v(x) = 0$ so $x \in A_v$ implying $ey = exy = xy = y$. If $A_v \neq R$ then $P_v = \{x \in R | xy \in A_v \text{ for some } y \in R \backslash A_v\}$. Consequently it follows by the next proposition that a nontrivial valuation is determined by its valuation ring.

PROPOSITION 2 [**18**]. *Let R be a ring with fixing elements. Let A be a subring of R with a prime ideal P. The following conditions are equivalent:*

(1) *If B is a ring such that $A \subseteq B \subseteq R$ and M is a prime ideal of B such that $M \cap A = P$ then $A = B$.*

(2) *For all x in $R \backslash A$ there exists y in P such that $xy \in A \backslash P$.*

(3) *There is a valuation v on R such that $A = A_v$, $P = P_v$.*

Proof. This follows as in [**18**] using properties of rings with fixing elements.

Let A and B be rings with fixing elements and let $f: A \to B$ be a surjective ring homomorphism with kernel N. A valuation on B can be lifted by f to one on A, and a valuation on A with infinite prime containing N gives an image valuation on B.

Let v be a valuation ring on a ring R, and A be any subring of R; then v is called *independent of A* if for any element γ in the group of v there exists an $a \in A$ such that $v(a) = \gamma$.

PROPOSITION 3. *Let Q be any prime ideal of the ring with fixing elements R. Let K be the quotient field of the domain R/Q. The valuations of R with infinite prime Q correspond to the valuations of K independent of R/Q.*

Proof. The proof is easily checked.

With the aid of Proposition 3 it is straightforward to construct many examples of valuation rings. (A finite direct sum of fields is the easiest situation to deal with.)

Example 1A. The following example of a valuation ring uses the notation of Gillman and Jerrison [**17**].

Let $C(X)$ be the ring of continuous functions from a completely regular (non-compact) Hausdorff space to the real numbers. Let **U** be a free ultrafilter on the set of zeros of continuous functions. (Thus **U** corresponds to some point

p in the Stone-Čech compactification of X which is not in X.) The set of all elements bounded on some elements of $\mathbf{U}$ form a valuation ring A_v. P_v consists of the functions which take values arbitrarily close to zero on each element of $\mathbf{U}$. For if $f \in C(X)\backslash A$, define $g(x) = |1/f(x)|$ on $V = \{x \mid |f(x)| \geqq 1\}$ and $g(x) = 1$ on $W = \{x \mid |f(x)| \leqq 1\}$. Since $W \notin \mathbf{U}$, $V \in \mathbf{U}$ so $fg \in A_v \backslash P_v$. P_v consists of functions which are zero on some $S \in \mathbf{U}$. The valuation is nontrivial if p does not belong to the real compactification of X.

Let A be a subring of the ring with fixing elements R. For each prime ideal P of A define

$$A_{[P]} = \{a \in R \mid da \in A \text{ for some } d \in A\backslash P\}.$$

The ideals M of A map naturally into extended ideals M^* of $A_{[P]}$ where $M^* = \{a \in R \mid da \in M \text{ for some } d \in A\backslash P\}$. It is easily checked that $M^* \supseteq MA_{[P]}$. * defines a one-to-one inclusion preserving correspondence between prime (respectively primary) ideals of A contained in P and the prime (respectively primary) ideals of $A_{[P]}$ contained in P^*.

Let (v, Γ) and (w, Λ) be two valuations of R. If there exists an order homomorphism f from (Γ, ∞) onto (Λ, ∞) such that $w = f \circ v$ then call w *coarser than* v and write $w \leqq v$. If v and w are valuations such that there is no nontrivial valuation coarser than both of them, v and w are called *independent*.

An ideal Q of a valuation pair (A_v, P_v) is called *v-closed* if a in Q and b in A_v with $v(b) \geqq v(a)$ implies $b \in Q$. A subset U of the positive elements of a totally ordered group Γ is called an *upper class* if $\alpha \in U$, $\gamma \in \Gamma$ and $\gamma > \alpha$ imply $\gamma \in U$.

PROPOSITION 4 [**18**]. *Let v be a valuation on the ring with fixing elements R. The v-closed ideals of A_v are in one-to-one order preserving correspondence with the upper classes of Γ_v. The v-closed prime ideals are $v^{-1}(\infty)$, together with those prime ideals Q such that $Q \subseteq P_v$ and $Q \nsubseteq v^{-1}(\infty)$. The nontrivial valuations coarser than v are given by the rings $A_{v[Q]}$ where Q is any prime of the latter type.*

Proof. This is essentially Propositions 3 and 4 of [**18**].

Note that if $Q \subseteq P_v$, $Q \nsubseteq v^{-1}(\infty)$ and Q is prime, then Q is v-closed. For if $x \in Q\backslash v^{-1}(\infty)$ and $v(y) > v(x)$ then if $dx \in A_v\backslash P_v$, $dy \in A_v$ so $dyx \in Q$ and since $dx \notin Q \subseteq P_v$, $y \in Q$.

This proposition implies that if w is non trivial then $v \geqq w$ if and only if $P_w \subseteq P_v$ and $A_v \subseteq A_w$. The necessity of imposing the condition $P_w \subseteq P_v$ can be seen from the following example on the ring $R = k[X, Y, Z, X^{-1}]$. Define v and w by setting $w(X^l Y^m Z^n) = l + m + n$ and $v(X^l Y^m Z^n) = l + m$, and extending to valuations of R. Then $A_v \subseteq A_w$, but $v \ngeqq w$. If $A_v \subseteq A_w$ with v any rank one valuation and w non trivial, then $P_v \subseteq P_w$, for if $a \in P_v$ and $x \notin A_w$ then $a^n x \in A_v$ for sufficiently large n so that $a^n x \in A_w$ and $a \in P_w$. However by extending v to a rank two valuation u it is easy to get $A_u \subseteq A_w$, $P_u \nsubseteq P_w$ and $P_w \nsubseteq P_u$.

Clearly the valuation can not give any information about ideals contained in $v^{-1}(\infty)$. The following proposition shows that only when R is a local ring can we get all the information about ideals not contained in $v^{-1}(\infty)$.

PROPOSITION 5. *The following conditions are equivalent for a valuation pair* (A_v, P_v) *of a ring with fixing elements* R.

(1) (A_v, P_v) *is local.*

(2) $(R, v^{-1}(\infty))$ *is local.*

(3) *All ideals of* A_v *not contained in* $v^{-1}(\infty)$ *are* v*-closed.*

(4) R *has an identity and all ideals of* A_v *not contained in* $v^{-1}(\infty)$ *are* R*-regular.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R\backslash v^{-1}(\infty)$, and $b \in R$ be such that $ab \in A_v\backslash P_v$. Then ab has an inverse in A_v and also in R.

(2) $\Rightarrow$ (3) Let Q be an ideal of A_v with $a \in Q$, and $v(b) \geqq v(a)$. Since $P_\infty \not\supseteq Q$ we may assume $v(a) < \infty$ so $ba^{-1} \in A$ and $b = ba^{-1}a \in Q$.

(3) $\Rightarrow$ (4) Let Q be an ideal of A_v with $a \in Q$, $a \notin v^{-1}(\infty)$ and $b \in R$ such that $ba \in A_v\backslash P_v$; then $(ba) = A_v$ so that the element which fixes ba is the identity and a is R-regular.

(4) $\Rightarrow$ (1) Let $a \in A_v\backslash P_v$; since $a \notin v^{-1}(\infty)$, (a), and hence a is R-regular; thus A_v has a unit and $v(a) + v(a^{-1}) = v(1) = 0$ so $a^{-1} \in A_v$.

We omit the proof of the following observation:

Let P be any ideal of A_v which contains $v^{-1}(\infty)$ properly (v must be non trivial). Each of (3) and (4) above may be weakened to the corresponding conditions for ideals of A_v contained in P but not in $v^{-1}(\infty)$.

4. R-Prufer rings and valuations. Let A be a subring of the ring with fixing elements R. If $A_{[P]}$ is a valuation ring of R for every maximal ideal P of A then A is called an *R-Prufer ring*; if R is the total quotient ring of A then A is called *Prufer*. Let $A_{[P]} \neq R$ be a valuation ring corresponding to the valuation v with $Q \subseteq P$ a prime ideal of A. It is not difficult to check that $P_v \cap A = P$, $P_v = P^*$ and that $A_{[Q]}$ is a valuation ring of R. A subring of R containing A is called an *R-overring of A*.

PROPOSITION 6. *Let A be a subring of the ring with fixing elements R, such that for each $x \in R$ there is some $y \in A$ with $xy \in A, xy \neq 0$. The following conditions are equivalent*:

(1) *A is an R-Prufer ring.*

(2) *If B is any R-overring of A then for all z in B, $(A:z)B = B$.*

(3) *Every R-overring of A is integrally closed.*

Proof. This follows by generalizations of [**20**] and [**4**] or of Proposition 10 and Theorem 13 of [**10**].

Let A be a ring generated by idempotents. Let R be a subquotient ring of A. An ideal M of A is called *quasi-finitely generated* if eM is finitely generated for every idempotent e of A. An A-submodule L of R is called an *R-fractionary ideal* if for each idempotent e in A there exists an e-regular element a such that $aLe \subseteq A$. An ideal M of A is called *R-invertible* if there is an R-fractionary ideal L such that $LM = A$. It is easily seen that an R-invertible ideal must be quasi-finitely generated and R-regular.

THEOREM 7. *Let R be a subquotient ring of a ring A, with A generated by idempotents. The following conditions are equivalent:*

(1) *A is an R-Prufer ring.*

(2) *Every R-overring of A is A-flat.*

(3) *Every R-regular, quasi-finitely generated ideal of A is R-invertible.*

(4) *If L is a quasi-finitely generated R-regular ideal then $LM = LN$ implies $M = N$.*

(5) *If L, M and N are any three ideals of A, at least one of which is R-regular, then $L \cap (M + N) = L \cap M + L \cap N$.*

(6), (7), (8) *and* (9) *Each of conditions* (6), (7), (13) *and* (14) *of Theorem* 13 *of* [**10**] *holds with the word regular changed to R-regular and finitely generated changed to quasi-finitely generated.*

The proof of this theorem is omitted. The proof consists of generalizing the proof and lemmas of [**10**] or of [**16**] and using the following lemma:

LEMMA 8. *Let A be a ring generated by idempotents having a subquotient ring R. An ideal Q is R-invertible if and only if eQ is an Re-invertible ideal in eA for every idempotent e of A.*

Example 1B. As in Example 1A let $C(X)$ denote the ring of continuous functions. Let $C^*(X)$ be the ring of bounded continuous functions. Then $C^*(X)$ is a $C(X)$-Prufer ring. Note that $C(X)$ is not a total quotient ring.

The notation is in Example 1A. The maximal ideals of $C^*(X)$ are of the form $P' = P_v \cap C^*(X)$, the set of functions which take arbitrarily small values on the elements of some ultrafilter $\mathbf{U}$. We need to show that $C^*(X)_{[P']} = A_v$.

Let $f \in C^*(X)_{[P']}$; then there exists $g \in C^*(X)\backslash P'$ such that $gf \in C^*(X)$, so for some $V \in \mathbf{U}$ and some $\epsilon > 0$, $|g(x)| > \epsilon$ for all $x \in V$, and $|gf(x)| < n$ for some integer n. Thus $|f(x)| < n/\epsilon$ for $x \in V$ and $f \in A_v$.

Let $f \in A_v$; then for some $V \in \mathbf{U}$, and some integer n, $|f(x)| < n$ for all $x \in V$. Let

$$g(x) = \max\{|f(x)|, n\}^{-1}$$

and since $x \in V$ implies $g(x) = 1/n$, $g \in C^*(X)\backslash P'$. Since $|fg| \leqq 1$, $fg \in C^*(X)$; thus $f \in C^*(X)_{[P']}$.

PROPOSITION 9. *Let A be a subring of the ring with fixing elements R. Let Ω be the family of maximal ideals of A. If Q is any ideal of A,*

$$Q = \bigcap_{P \in \Omega} QA_{[P]} = \bigcap_{P \in \Omega} Q^*.$$

Proof. Since $QA_{[P]} \subseteq Q^*$,

$$Q \subseteq \bigcap_{P \in \Omega} QA_{[P]} \subseteq \bigcap_{P \in \Omega} Q^*.$$

Suppose $x \in \bigcap_{P \in \Omega} Q^*$. Let $P \in \Omega$; then $xd \in Q$ for some $d \in A \backslash P$. Thus $d \in (Q:(x))$; so $Q:(x) \not\subseteq P$. Since this holds for each maximal ideal $P \in \Omega$, $Q:(x) = A$. Let $y \in A$ fix x, then $x = yx \in Q$.

LEMMA 10. *Let A be an R-Prufer ring with a maximal ideal P such that for any other maximal ideal M, $A_{[P]} \subseteq A_{[M]}$; then A is a valuation ring.*

Proof. $A = \bigcap A_{[M]} = A_{[P]}$, where the intersection is over all maximal ideals M.

A particular case of the above lemma is when R is a subquotient ring of A and A has a unique maximal R-regular ideal.

PROPOSITION 11. *Let A be an R-Prufer ring having prime ideals $P_i, i = 1, \ldots, n$ such that*

$$A = \bigcap_{1 \leqq i \leqq n} A_{[P_i]}.$$

If M is a prime ideal contained in no P_i then $A_{[M]} = R$.

Proof. Suppose that $A_{[M]}$ corresponds to the non trivial valuation v. Let $v^{-1}(\infty) \cap A = P$. Let $v_1, \ldots, v_m$, $m \leqq n$ be the nontrivial valuations corresponding to $P_1, \ldots, P_n$. Since $A_{[P]} = R$ and v_i is nontrivial, $P_i \not\subseteq P$, $1 \leqq i \leqq m$. Let

$$a = \prod_{0 \leqq i \leqq m} a_i,$$

where $a_i \in P_i$, $a_i \notin P$, $1 \leqq i \leqq m$ and $a_0 \in M \backslash P$. Since $M \not\subseteq P_i$, $M \not\subseteq \bigcup_{1<i<n} P_i$, so there exists $b \in M$, $b \notin P_i$, $i \leqq i \leqq m$. We may assume that $b \notin P$, for if $b \in P$ then $a + b \notin P$, and since $a + b \in M$, $a + b \notin P_i$, $i = 1, \ldots, m$, $a + b$ will serve in place of b. Thus $\infty > v(b) > 0$, and there exists $d \in R \backslash A$ with $v(db) = 0$, implying $db \in A_{[M]}$, so $cdb \in A$ for some $c \in A \backslash M$. Since $v(cd) < 0$, $cd \notin A$, so that $v_i(cd) < 0$ for some i, $1 \leqq i \leqq m$. Since $b \in A \backslash P_i$, $v_i(b) = 0$ and so $v_i(bcd) < 0$, a contradiction to $bcd \in A$.

PROPOSITION 12. *Let $A = A_v$, $P = P_v$ where v is a valuation on the ring with fixing elements R. Then A is R-Prufer if and only if for each maximal ideal M of A, $M \neq P$; all primes contained in $M \cap P$ are contained in $v^{-1}(\infty)$.*

Proof. If A is R-Prufer and $M \neq P$ is maximal, then by the previous proposition $A_{[M]} = R$. If $Q \subseteq M \cap P$ is a prime ideal, then $A_{[Q]} \supseteq A_{[M]} = R$; and $Q \subseteq v^{-1}(\infty)$ by the remark after Proposition 4.

To show A is R-Prufer it suffices to show that M maximal, $M \neq P$, implies $A_{[M]} = R$. This is trivial if $A = R$. Let $a \in R \backslash A$. Let

$$Q = \{x \in A | v(x^n) + v(a) > 0 \text{ for some positive integer } n\}.$$

Then Q is a prime ideal and since $v^{-1}(\infty) \subseteq Q \subseteq P$, $Q \not\subseteq M$. Thus there exists $d \in Q$, $d \notin M$ such that $d^n a \in A$. Since $d^n \in A \backslash M$, $a \in A_{[M]}$.

PROPOSITION 13. *Let A be an R-Prufer ring which is also the ring of a valuation v. If B is an R-overring of A, then B is R-Prufer and is the ring of a valuation coarser than v.*

Proof. We may take $B \neq R$, for if $B = R$ take $P_w = v^{-1}(\infty)$; then $w \leqq v$. Since every R-overring of B is integrally closed, B is a Prufer ring. Let Ω be the family of maximal ideals M of B such that

$$B_{[M]} \neq R \ (\Omega \neq \phi \text{ since } R \neq B = \textstyle\bigcap_{M \in \Omega} B_{[M]}).$$

Let $M \in \Omega$ and $P = M \cap A$. Since $A_{[P_v]} = A$, $P \subseteq P_v$ by Proposition 11, and $A \subseteq A_{[P]} \subseteq B_{[M]} \neq R$, so $v^{-1}(\infty) \not\subseteq P$. Consequently by Proposition 4, $A_{[P]}$ is the ring of a valuation coarser than v. Further $B \subseteq A_{[P]}$; for if $x \in B \backslash A$ there exists $y \in P_v$ such that $xy \in A \backslash P_v \subseteq A \backslash P$, and $y \notin P$, (for if $y \in P$ then $xy \in M \cap A \subseteq P_v$). Thus

$$B \subseteq \bigcap_{M \in \Omega} A_{[M \cap A]} \subseteq \bigcap_{M \in \Omega} B_{[M]} = B.$$

Let $D = \bigcup_{M \in \Omega}(M \cap A)$; since the prime ideals between $v^{-1}(\infty)$ and P_v are totally ordered by inclusion, D is a prime ideal. $A_{[D]} = \bigcap_{M \in \Omega} A_{[M \cap A]} = B$, and $A_{[D]}$ is the ring of a valuation coarser than v.

COROLLARY. *Let Q be a prime ideal of the R-Prufer ring A contained in the maximal ideal P. Then if $A_{[Q]} \neq R$ it is the ring of a valuation coarser than $A_{[P]}$.*

Proof. Suppose $A_{[Q]} \neq R$. Let * denote extensions to $A_{[P]} = B$, which is R-Prufer and the ring of a nontrivial valuation v. Since $A_{[Q]} = B_{[Q^*]} \supseteq B$ the result follows from the Proposition.

PROPOSITION 14. *Let A be a ring generated by idempotents and let R be a subquotient ring of A. Let (A, P) be the valuation pair of a nontrivial valuation v on R. The following conditions are equivalent:*

(1) *A is an R-Prufer ring.*

(2) *Each R-regular ideal of A is v-closed.*

(3) *P is maximal and the R-regular ideals of A are totally ordered by inclusion.*

(4) *P is the unique maximal R-regular ideal of A.*

Proof. (1) $\Rightarrow$ (2) Let Q be any R-regular ideal of A. Let $a \in Q$, and let $b \in A$ be such that $v(b) \geqq v(a)$. We must show $b \in Q$.

Let e be an idempotent of A such that $ea = a$ and $eb = b$. Let E be the ideal generated by all idempotents f in A such that $ef = 0$. Let r be an R-e-regular element of Q. Since the ideal (E, r, a, b) is R-regular and quasi-finitely generated it has an R-inverse D. Suppose that $e \notin (E, r, a)D = F$. It follows that F is contained in some maximal ideal M which does not contain e. Since F contains an element of value zero, $M \not\subseteq M_v$ and so by Proposition 11, $A_{[M]} = R$. This leads to a contradiction, since $r \in M$, $e \notin M$ implies $r^{-1} \in R \backslash A_{[M]}$.

Thus $(e) = eF$ and

$$(r, a) = (r, a)e(E, r, a, b)D = (E, r, a)e(r, a, b)D = (r, a, b)eF = (r, a, b).$$

So $b \in (a, r) \subseteq Q$.

(2) $\Rightarrow$ (3) P is an R-regular ideal; let e be any idempotent and let $a/b \in R \backslash A$ with b f-regular; then $e - ef + bef$ is an e-regular element of P since $(e - ef + bef)(e - ef + efb^{-1}) = e$. The result now follows from (2), since any ideal of A containing P is R-regular and the v-closed ideals are totally ordered by inclusion.

(3) $\Rightarrow$ (4) This is trivial.

(4) $\Rightarrow$ (1) $A_{[P]} = A$ and if M is any other maximal ideal, $A_{[M]} = R$.

In the last section we provide an example of a valuation ring which has a unique maximal R regular ideal but is not Prufer. In relation to (1) $\Leftrightarrow$ (4) above, and examples of valuation rings which are not Prufer, see [**2**] and [**9**].

5. Witt rings as Prufer rings. To supplement the results summarized here the reader is referred to Lorenz [**17**].

Let H be a field of characteristic different from two. Let $A = W(H)$ be its Witt ring. Elements of $W(H)$ correspond to anisotropic quadratic forms $\langle a_1, \ldots, a_n \rangle$. The even dimensional quadratic forms form a maximal ideal M of $W(H)$. The other prime ideals of $W(H)$ correspond to the total orderings of H as follows. If P is the positive elements of an ordering define $sg_P: H \rightarrow Z$ as follows: $sg_P(0) = 0$, $sg_P(a) = 1$ if $a \in P$ and -1 otherwise. Extend sg_P to $W(H)$ by defining

$$sg_P(\phi) = \sum_{i=1}^{n} sg_P(a_i) \quad \text{where } \phi = \langle a_1 \ldots a_n \rangle.$$

For each odd prime q there is a maximal ideal $P_q = \{\phi \in W(H) | sg_P(\phi) \in (q)\}$, and there is a unique minimal prime $Q_P = \{\phi \in W(H) | sg_P(\phi) = 0\}$.

A field is called *Pythagorean* if every sum of squares is a square. Let $Z(H)$ denote the zero divisors of $W(H)$. If H is not Pythagorean or has no total ordering then $Z(H) = M$; otherwise

$$Z(H) = \bigcup_P Q_P \subsetneq M.$$

Since all primes of $W(H)$ are either maximal or minimal, $Z(H)$ is not prime if and only if H is Pythagorean with more than one ordering.

PROPOSITION 15. *$W(H)$ is not a Prufer ring if and only if H is Pythagorean with at least two orderings.*

Proof. Let K be the total quotient ring of $A = W(H)$.

Let P be the positive elements in some ordering and q be an odd prime; then $A_{[P_q]}$ is a valuation ring. By [**5**, Theorem 2.2] it suffices to check that

A_{P_q} is a valuation ring of its total quotient ring. Let $\phi \in Q_P$, so

$$\phi = \bigotimes_{i=1}^{n} \langle a_i, -b_i \rangle \quad \text{where } a_i, b_i > 0.$$

Let

$$\rho = \bigotimes_{i=1}^{n} \langle a_i, b_i \rangle.$$

Then $\phi\rho = 0$ and since the dimension of ρ is 2^n, $\rho \in A \backslash P_q$, and it follows that Q_P is the kernel of the homomorphism $h: A \to A_{P_q}$. Thus $h(A) = Z$ and $A_{P_q} = Z_{(q)}$, a valuation domain.

Thus A is a Prufer ring if and only if $A_{[M]}$ is a valuation ring of K. Since $A_{[M]} = K$ unless H is Pythagorean with at least one ordering, and $A = Z$ if H is Pythagorean with exactly one ordering, it suffices to prove that if $Z(H)$ is not prime then $A_{[M]}$ is not a valuation ring. Suppose $A_{[M]}$ is a valuation ring. Since $Z(H) \subseteq M$, $A_{[M]}$ is local, for if $a \in A_{[M]} \backslash M^*$ then for some $d \in A \backslash M$, $ad \in A \backslash M$ and has an inverse $(ad)^{-1}$ in K. Consequently $d(ad)^{-1} \in A_{[M]}$, and a has an inverse. By Proposition 5, K must be local, but this cannot be so since $Z(H)$ is not prime.

6. Arithmetical, semi-hereditary and Dedekind rings. A ring is called *arithmetical* if for any three ideals, L, M and N, $L \cap (M + N) = L \cap M + L \cap N$. The following are easy generalizations of [**14**] and [**10**].

PROPOSITION 16. *A ring A with fixing elements is arithmetical if and only if for each maximal ideal M, A_M has its ideals totally ordered by inclusion. If A is generated by idempotents, each of the following conditions is equivalent to being arithmetical*:

(1) *If M and N are ideals with $M \subseteq N$ and N quasi-finitely generated then there exists an ideal L such that $M = LN$.*

(2) *A is a Prufer ring, and some subquotient ring of A is arithmetical.*

Example 1C. We see immediately that $C^*(X)$ is arithmetical if and only if $C(X)$ is. With the notation of examples 1A and 1B, define

$$O_v = \{ f \in C(X) \mid fg = 0 \text{ for some } g \in C(X) \backslash M_v \}.$$

A ring is called *Bezout* if every finitely generated ideal is principal.

The following conditions are equivalent:

(1) *$C(X)$ is arithmetical;*

(2) *O_v is a prime ideal for all maximal ideals M_v;*

(3) *$C(X)$ is a Bezout ring.*

(1) $\Rightarrow$ (2) This follows since O_v is the intersection of all prime ideals contained in M_v [7, p. 110], and the prime ideals contained in M_v are totally ordered by inclusion.

(2) $\Rightarrow$ (3) See [**7**, p. 208].

(3) $\Rightarrow$ (1) It is easily seen that every Bezout ring is arithmetical.

A ring is called *semi-hereditary* if every finitely generated ideal is projective. It is not hard to show that a semi-hereditary ring with fixing elements is generated by idempotents and contains no nilpotents. Generalizing [**6**] and using the fact that a zero dimensional ring without nilpotents is von Neumann regular one can show:

PROPOSITION 17. *Let A be a ring generated by idempotents with total quotient ring K. The following are equivalent*:

(1) *A is semi-hereditary;*

(2) *K has dimension zero and A_M is a valuation domain for every maximal ideal M;*

(3) *K is von Neumann regular and A is Prufer.*

A ring generated by idempotents is called *r-Noetherian* if every regular ideal is quasi-finitely generated. This is equivalent to every regular ideal of Ae being finitely generated for each idempotent e of A. The following proposition generalizes a result of Maranda [**19**], and can be obtained from Proposition 17 of [**10**].

PROPOSITION 18. *The following conditions are equivalent for a ring A which is generated by idempotents*:

(1) *All regular prime ideals are invertible.*

(2) *All regular ideals are invertible.*

(3) *A is Prufer and r-Noetherian.*

(4) *For each idempotent e of A the regular ideals of Ae have a unique representation as a product of prime ideals.*

Rings satisfying these equivalent conditions are called *Dedekind rings*. We note the following additional properties:

(i) Every regular prime ideal is maximal.

(ii) If a is a regular element then it is contained in only a finite number of maximal ideals (only finitely many non zero valuations at a).

(iii) Each regular ideal can be represented as the intersection of powers of non equal prime ideals (the primes are also maximal, quasi-finite and regular) and such a representation is unique.

7. Rings with large Jacobson radical and valuations. In this section we introduce a condition on a commutative ring which ensures good behavior of valuations. This condition also ensures that many different generalizations of valuation coincide [**11**].

The *Jacobson radical*, J, is the intersection of the maximal ideals. It is a well-known property of a ring with identity that $a \in J$ if and only if $1 + ba$ is a unit for all elements b of the ring.

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PROPOSITION 19. *Let A be a ring with identity having Jacobson radical J. The following conditions are equivalent*:

(1) *any prime ideal containing J is maximal;*

(2) *for each a in A there exists b in A such that for all d in A and for all units r, $a + rb$ and $1 + dab$ are both units;*

(3) *for each a in A there exists b in A such that $a + b$ is a unit and $ab \in J$.*

Proof. (1) $\Rightarrow$ (2) Let $\phi: A \to A/J$. Each prime of $\phi(A)$ is maximal and $\phi(A)$ has no nilpotents. It follows that there exists $b \in A$ such that $\phi(a)\phi(b) = 0$ and $\phi(a) + \phi(b)$ is a unit; see for example the corollary to Proposition 1 of [**12**]. Thus $ab \in J$ and $c(a + b) = 1 + f$, where $f \in J$; since $1 + f$ is a unit in A so is $(a + b)$. Thus exactly one element of the pair a, b belongs to each maximal ideal. Since the same is true of the pair a, rb where r is a unit, $a + rb$ is a unit. Since $ab \in J$, $1 + dab$ is a unit for all $d \in A$.

(2) $\Rightarrow$ (3) This follows immediately by the above characterization of J.

(3) $\Rightarrow$ (1) Suppose that P and M are prime ideals with $J \subseteq P \subseteq M$. Let $x \in M \backslash P$. Let y be such that $x + y$ is a unit and $xy \in J$. Since $xy \in J \subseteq P$, $y \in P$; thus $x + y \in M$, a contradiction, since $x + y$ is a unit.

Rings satisfying the conditions of Proposition 19 are said to have *large Jacobson radical.* The principal examples are rings in which every prime ideal is maximal and rings with only a finite number of maximal ideals. A Noetherian ring has large Jacobson radical if and only if it is semi-local. (For if $J = \bigcap_{1 \leqq i \leqq n} Q_i$ with $\sqrt{Q_i} = P_i$, and M is a maximal ideal, then $J \subseteq M$ so for some i, $Q_i \subseteq M$; thus $P_i \subseteq M$, and since $J \subseteq P_i \subseteq M$, $P_i = M$.) If J consists of nilpotents then the ring must have Krull dimension zero; in particular when $J = 0$ a ring has large Jacobson radical if and only if it is von Neumann regular.

Example 2. A ring with large Jacobson radical which is neither zero dimensional nor semi-local.

Let K be a field which has a non trivial valuation v with a maximal ideal P. Let R be the subring of $\prod_{i=1}^{\infty} K$ generated by the constant functions with values in K and $\bigoplus_{i=1}^{\infty} K$. Let A be the ring of the valuation obtained by trivially extending v on the first copy of K to the whole of R. It is easily checked that only maximal ideals contain the Jacobson radical which is $(P, 0, 0, \ldots)$. The maximal ideals consist of: (i) all elements which have first components in P; (ii) all elements with zero in the ith place $i > 1$; (iii) elements with only a finite number of nonzero components. The non-maximal prime ideal consists of all elements with zero in the first place.

That there are total quotient rings which are not rings with large Jacobson radical may be easily seen using the construction outlined in Section 9.

LEMMA 20. *Let A be a valuation ring of a ring with large Jacobson radical R; R is a subquotient ring of A.*

Proof. Let $x \in R\backslash A$. By Proposition 19, $(x + y)d = 1$ and $(1 + xy)b = 1$ for some $y \in R$. If $y \in A$, then $x + y \notin A$, $d \in A$ and $dx = 1 - dy \in A$. If $y \notin A$, then $1 + xy \notin A$, so $b \in A$ and $bxy = 1 - b \in A$, since $y \notin A$, $bx \in A$.

A polynomial of the form $1 + n_1 X + \ldots + n_{k-1} X^{k-1} + X^k$ with n_i an integer for $1 \leqq i \leqq k - 1$, is called *strongly integral*.

LEMMA 21. *Let v_i, $1 \leqq i \leqq n$ be valuations on the ring with identity R. Let x be any element of R. There exists a strongly integral polynomial $f(X)$ such that $v_i(f(x)) = 0$ for all those i for which $v_i(x) \geqq 0$ and $v_i(x) - v_i(f(x)) > 0$ if $v_i(x) < 0$.*

Proof. This is almost identical to § 7, Lemma 1 of [**1**].

PROPOSITION 22. *Let R be a ring with unit having large Jacobson radical. Let A_i, $1 \leqq i \leqq n$, be valuation rings of R. Then*

$$A = \bigcap_{1 \leqq i \leqq n} A_i$$

is a Prufer ring and R is a subquotient ring of A.

Proof. Let v_i, the valuation corresponding to A_i, have prime ideal M_i in A. Let P_i be the prime ideal of A at which v_i takes infinite value. Let J be the Jacobson radical of R.

(i) *If J is not contained in P_i, then there are R-regular elements in M_i at which v_i takes arbitrarily large values.* Given a in J but not in P_i and any α (which may be taken positive) in the group of v_i, there exists b in R such that $v_i(ba) < -\alpha$. By Lemma 21 there exists a strongly integral polynomial $f(X)$ such that $v_j(f(ab)) \leqq 0$ for $1 \leqq j \leqq n$ and $v_i(f(ab)) \leqq -\alpha$. Now $f(ab)$ is of the form $1 + da$, and since a is in J, $1 + da$ is a unit in R. The inverse of $1 + da$ is in A and has value greater than α.

(ii) *Given any element a in R, there exists a unit of R, b in A such that ab is in A and $v_i(b) = 0$ if $v_i(a) \geqq 0$.* Let y be an element $f(a)$ determined as in Lemma 21. Since R has large Jacobson radical there exists d in R such that $y + d$ is a unit and yd is in J. If $v_i(yd)$ is finite then J is not contained in P_i, and there exists a unit of R, d_i in A such that $v_i(y) < v_i(d_i d)$. If $v_i(yd)$ is infinite then so is $v_i(d)$. It follows that there exists a unit r such that $v_i(rd) > v_i(y)$ for $1 \leqq i \leqq n$, and that $v_i(y + rd) = v_i(y) \leqq 0$. Since $y + rd$ is unit take b to be its inverse.

(iii) *For each i, $A_i = A_{[M_i]}$.* Let a be any element of A_i. Let b be chosen as in (ii) above. Since b is in $A\backslash M_i$, and ab is in A, $a = ab/b$ is in $A_{[M_i]}$. Since R is a subquotient ring of A_i by Lemma 22, R is a subquotient ring of A.

(iv) *The M_i, $1 \leqq i \leqq n$, exhaust the maximal R-regular ideals of A.* Let Q be a maximal ideal of A containing a unit r. Suppose that there exists an element d in Q with $v_i(d) = 0$ for all i, $1 \leqq i \leqq n$. Since R has large Jacobson radical, there exists t in R such that $d + t$ is unit and dt is in J. By the previous

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result there exists a unit a such that $v_i(at) > 0$ for all i, $1 \leqq i \leqq n$. Since ar is a unit, $d + art$ is a unit of R, and since $v_i(d + art) = v_i(d) = 0$, $d + art$ is a unit of A. Since d and r are in Q, so is $d + art$, a contradiction. It follows that Q is contained in the union of the finite set of maximal ideals M_i, $1 \leqq i \leqq n$, and so must equal one of them.

This proposition implies in particular that valuation rings of rings with large Jacobson radical are Prufer rings. Such rank one discrete valuation rings are Dedekind rings. (The latter result was proved by Maranda [**19**] in the case of zero dimensional rings and rings with few zero divisors.)

In the case of rings with few zero divisors this proposition has been proved by Harui [**13**] and Larsen [**15**].

8. The approximation theorem. It is not hard to show that the fifth characterization of R-Prufer rings given in Theorem 7 is equivalent to the version of the Chinese Remainder Theorem given below (see [**21**, p. 279]).

C.R.T. *Let A be a ring generated by idempotents.*

Let R be a subquotient ring of the ring A. Given any finite family of ideals M_i, $1 \leqq i \leqq n$, with at most one ideal not R-regular, and elements $x_i \in A$, $1 \leqq i \leqq n$, the system of congruences $x \equiv x_i \pmod{M_i}$ admits a solution x in A if and only if $x_i \equiv x_j \pmod{(M_i + M_j)}$ for all i and j.

It is now possible to deduce a form of the approximation theorem. For simplicity we deal only with the case where all valuations are independent.

PROPOSITION 23. *Let A be an R-Prufer ring generated by idempotents where R is a subquotient ring of A. Let $M_1, \ldots, M_n$ be maximal R-regular ideals of A which have associated valuations $v_1, \ldots, v_n$ that are pairwise independent, and have groups $\Gamma_1, \ldots, \Gamma_n$. If for each i, $1 \leqq i \leqq n$, a_i is in A and γ_i is in Γ_i, then there exists a in A such that $v(a - a_i) \geqq \gamma_i$ for all i, $1 \leqq i \leqq n$.*

Proof. Clearly γ_i may be taken positive. Let

$$Q_i = \{b \in A | v_i(b) > \gamma_i\}$$

and let P_i be the radical of Q_i; then P_i is a prime ideal. Suppose that $Q_i + Q_j$ (with $i \neq j$) is contained in a maximal ideal M of A. Then $P_i + P_j \subseteq M$ so that $A_{[M]} \subseteq A_{[P_i]} \subseteq R$, and since $v_i^{-1}(\infty) \cap A \neq P_i$, $A_{[P_i]}$ is nontrivial. Thus by Proposition 13, if w_i, w_j and w correspond to P_i, P_j and M, we must have $w_i \leqq w$, $w_j \leqq w$ so either $w_i \geqq w_j$ or $w_j \geqq w_i$ and in either case v_i and v_j are not independent. Thus $Q_i + Q_j = A$, so $a_i \equiv a_j \pmod{(Q_i + Q_j)}$, and by C.R.T. there exists $a \in A$ such that $a \equiv a_i \pmod{Q_i}$, that is $v_i(a - a_i) \geqq \gamma_i$ for $1 \leqq i \leqq n$.

If $v_1, v_2, \ldots, v_n$ are a family of valuations on R such that $A = \bigcap_{1 \leqq i \leqq n} A_{v_i}$ is an R-Prufer ring and R is a subquotient ring of A, then we say that $v_1, v_2, \ldots, v_n$ is an *approximation family* on R. Proposition 22 shows that if R is a ring with

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large Jacobson radical then any finite family of valuations is an approximation family.

COROLLARY 1. *Let $v_1, v_2, \ldots, v_n$ be a pairwise independent approximation family on R, and for $1 \leqq i \leqq n$ let γ_i be any element in the group of v_i and x_i be any element in R. Then there exists z in R such that $v_i(z - x_i) \geqq \gamma_i$, for $1 \leqq i \leqq n$, and there exists y in R such that $v_i(y) = \gamma_i$ for $1 \leqq i \leqq n$.*

Proof. Let A be the intersection of the valuation rings. Let e be an idempotent such that $ex_i = x_i$ for $1 \leqq i \leqq n$. Let s be e-regular in A such that $sx_i = b_i \in A$.

Use the above proposition to choose x in A such that $v_i(x) \geqq \gamma_i + v_i(s)$; let $z = ex/s$. To prove the final statement let a_i be such that $v_i(a_i) = \gamma_i$; then choose y such that $v_i(y - a_i) > \gamma_i$.

COROLLARY 2. *Let R be a ring with large Jacobson radical. Let v_i and γ_i, $1 \leqq i \leqq n$ be as in Corollary 1. If no $\gamma_i = \infty$ then there exists a regular element t such that $v_i(t) = \gamma_i$, $1 \leqq i \leqq n$. Every finitely generated regular ideal of $A = \bigcap_{i=1}^n A_{v_i}$ is principal.*

Proof. By Corollary 1 there exists y such that $v_i(y) = \gamma_i$ for all i, $1 \leqq i \leqq n$. By Proposition 19 there exists $z \in R$ such that $yz \in J$ and $y + rz$ is a unit of R for all units $r \in R$. We can choose $r_i \in A$ such that $v_i(r_i z) > v_i(y)$ for if $v_i^{-1}(\infty) \not\supseteq J$ the existence of such an element is shown in Proposition 22, and if $v_i^{-1}(\infty) \supseteq J$, then $v_i(yz) = \infty$ so $v_i(z) = \infty$ and we can take $r_i = 1$. Let $r = \prod_{i=1}^n r_i$ and set $t = y + rz$.

$$v_i(t) = \min_{1 \leqq i \leqq n} \{v_i(y), v_i(rz)\} = \gamma_i.$$

To prove that every finitely generated regular ideal of A is principal it suffices to prove that if r is regular then (r, a) is principal. Let $b \in K$ be a regular element such that $v_i(b) = \min \{v_i(a), v_i(r)\}$. The ideal $b^{-1}(a, r)$ is a regular ideal of A contained in no maximal R regular ideal. Since A is Prufer $b^{-1}(a, r)A = A$ so $(a, r) = (b)$.

COROLLARY 3. *Let A be the intersection of a finite family of pairwise independent valuations on a von Neumann regular ring with identity R. A is Bezout and semi-hereditary.*

Proof. Let $Q = (a_1, \ldots, a_m)$ be a finitely generated ideal. Let

$$\gamma_i = \min_{1 \leqq j \leqq m} v_i(a_j).$$

Let $\alpha_i = \gamma_i$ if $\gamma_i < \infty$, and 0 otherwise. Since RQ is a finitely generated ideal of a von Neumann regular ring, $RQ = eR$ where $e^2 = e$ and $RQ \cap A = eA$. By [**12**, Lemma 12], $Q = EeA$ where $E = \{a \in R | v_i(a) \geqq \gamma_i, 1 \leqq i \leqq n\}$. Let $b \in A$ be regular such that $v_i(b) = \alpha_i$. Clearly $be \in E$ and $E \subseteq bA$; thus

$Q = EeA \subseteq bAe = be^2 \subseteq EeA = Q$, so $Q = beA$ is principal. Since A is Prufer it is semihereditary by Proposition 17.

It is easily seen that a valuation ring with identity which is Prufer has the property that every regular finitely generated ideal is principal if and only if v maps the regular elements of A_v onto the value group. It seems unlikely that for all Prufer valuation rings every finitely generated regular ideal is principal.

9. Examples. The purpose of the first half of this section is to construct a total quotient ring which has ideal structure similar to a given ring. This shows that little is gained in terms of good behaviour by restricting the study of valuations to valuations of total quotient rings, and this is used to construct examples of valuation rings which fail to be Prufer rings for various reasons.

Let A be a ring with identity. Let $\{M_h, h \in H\}$ be the family of maximal ideals of A and let $f_h: A \to A/M_h = k_h$. Let $I = H \times N$, where N denotes the natural numbers. For $i = (h, n) \in I$ define $k_i = k_h$. Define

$$K = \prod_{i \in I} k_i \text{ and } F = \bigoplus_{i \in I} k_i.$$

Let B be the image of A in K under the map f defined by $(f(x))_{(h,n)} = f_h(x)$. Let $R = B + F$. Let P_i denote the projection onto the ith component.

The following facts can be easily checked:

(i) $B \cap F = 0$.

(ii) The prime ideals of R which do not contain F are of the form $P_i = \{a \in R | p_i(a) = 0\}$ and are both minimal and maximal.

(iii) R is its own total quotient ring.

(iv) $R/F \cong A/J$ where J is the Jacobson radical of A.

(v) The valuations of R are either valuations of k_i lifted by P_i for some $i \in I$ or the valuations of A/J lifted by g, where $g: R \to R/F \cong A/J$.

(vi) If $J = 0$ and a valuation w on A lifts by g to v on R, then $A_v = f(A_w) + F$, $P_v = f(P_w) + F$, $v^{-1}(\infty) = f(w^{-1}(\infty)) + F$ and M' is a maximal regular ideal of A_v if and only if $M' = f(M) + F$ where M is maximal A-regular ideal of A_w. R is the total quotient ring of A_v if and only if A is a subquotient ring of A_w.

(vii) If A is arithmetical then so is R.

If A is an arithmetical domain but not a field, with $J = 0$, then R is a total quotient ring such that R_M is a valuation domain for every maximal ideal M, but R is not a von Neumann regular ring. The construction of such rings has been attributed to Nagata [**2**].

We call R the *total quotient ring like* A. In the following three examples R is a total quotient ring and v is a valuation on R.

Example 3. A_v has a unique maximal regular ideal, and has total quotient ring R but is not Prufer. Let

$$A = \{q(X, Y)/g(Y)X^n \in k(X, Y) | Y \nmid g(Y)\}$$

where k is a field. Define w on A by extending $w(X^n) = n$. Then

$$A_w = \{g(X, Y)/q(Y) \in k(X, Y) | Y \nmid g(Y)\}$$

and $P_w = XA_w$. It is easily checked that: $M_w = (X, Y)A_w$ is the unique maximal A-regular ideal of A_w, A is a subquotient ring of A_w and A has Jacobson radical zero. Let R be the total quotient ring like A and v the valuation corresponding to w. $f(M_w) + F$ is the unique maximal regular ideal of A_v and it properly contains $P_v = f(P_w) + F$, so A_v is not Prufer.

Example 4. P_v is a maximal regular ideal but A_v is not Prufer. Define a rank two discrete valuation on

$$A = \{q(X, Y)/r(X) \cdot s(Y) \in k(X, Y)\}$$

by extending $w(X^m Y^n) = (m, n)$.

$$A_w = \{p(X, Y)XY^{-n} + q(Y) \in k(X, Y)\}$$

and $P_w = YA_w$. P_w is maximal, but there are other maximal regular ideals in A_w, for example $(Y + 1, XY^{-1}, \ldots, XY^{-n}, \ldots)$. Extending w to v on the total quotient ring like A we get the required example. A_v cannot be Prufer since it does not have a unique maximal regular ideal.

These examples show that Proposition 14 cannot be substantially weakened. From the results on rings with large Jacobson radical one might conjecture that A_v should be a Prufer ring if $v^{-1}(\infty)$ is a maximal ideal of R and P_v is also large. The following example shows that this is not the case:

Example 5. Let $A = K[X][X_a]_{a \in K}$ where K is an algebraically closed field. Let $f: A \to K(Y)$ be the linear map given by $f(X) = Y, f(X_a) = (Y - a)^{-1}$. $f^{-1}(0) = Q$ is a maximal ideal. Let u be the valuation on $K(Y)$ given by the polynomial Y and let u lift via f to w on A. Then

$$A_w = K[X][X_k]_{k \in K, k \neq 0} + Q \quad \text{and} \quad P_w = (X) + Q.$$

Clearly P_w contains no units of A (i.e. no element of K). The image of $X_k + 1/k$ in $K(Y)$ is the same as the image of $X\ X_k\ 1/k$, and since the latter element is in P_w so is the former; thus P_w is a maximal ideal of A_w. The Jacobson radical of A is zero. Lifting to R the total quotient ring like A, we see that $v^{-1}(\infty)$ is a maximal ideal of R and P_v is a maximal ideal of A_v, but since P_v is not regular, A_v is not Prufer.

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