ON UNIT GROUPS OF ABSOLUTE ABELIAN NUMBER FIELDS OF DEGREE pq

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In this note, we denote by Q the rational number field, by $\mathbf{E}_\Omega$ the whole unit group of an arbitrary number field Ω of finite degree, and by r_Ω the rank of $\mathbf{E}_\Omega^*$, where generally $\mathbf{G}^*$ for an arbitrary abelian group $\mathbf{G}$ means a maximal torsion-free subgroup of $\mathbf{G}$. $(N_{K/\Omega}\mathbf{E}_K)^*$ is shortly denoted by $N_{K/\Omega}^*\mathbf{E}_K$ and $(\mathbf{G}_1 : \mathbf{G}_2)$ is, as usual, the index of a subgroup $\mathbf{G}_2$ in $\mathbf{G}_1$.

We first prove the following lemma.

Lemma. *Let $\mathbf{F}$ be a free abelian group of finite rank n, and $\mathbf{G}$ be a subgroup of $\mathbf{F}$ such that for a rational prime number l, $\mathbf{G}$ contains the group $\mathbf{F}^l$ consisting of all the l-th powers α^l of α in $\mathbf{F}$. Then, for an arbitrarily given basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of $\mathbf{F}$, $\mathbf{G}$ has the basis $(\omega_1, \ldots, \omega_n)$ of the following form:*

$$\omega_i = \begin{cases} \varepsilon_{\pi_i}^l \cdots\cdots\cdots & i = 1, \ldots, s, \ (s \geqq 0) \\ \varepsilon_{\pi_i} \prod_{j=1}^{s} \varepsilon_{\pi_j}^{a_{ij}} \cdots\cdots & i = s+1, \ldots, n, \end{cases}$$

where a_{ij} are rational integers with $0 \leqq a_{ij} < l$ and $(\pi_1, \ldots, \pi_n)$ is a suitable permutation of $(1, \ldots, n)$.

Proof. By the elementary divisor theory, there exist a basis $(f_1, \ldots, f_n)$ of $\mathbf{F}$ and a basis $(g_1, \ldots, g_n)$ of $\mathbf{G}$ such that we may write $(g_1, \ldots, g_n) = (f_1, \ldots, f_n)L$, where L is a $n \times n$ diagonal matrix with diagonal elements e_{i+1}/e_i $(i = 1, \ldots, n-1)$. By the assumption, however, all the l-th powers of the elements in $\mathbf{F}$ are contained in $\mathbf{G}$, so we have $e_1 = \cdots = e_s = l$, $e_{s+1} = \cdots = e_n = 1$ for some integer s $(0 \leqq s \leqq n)$. We express this basis $(f_1, \ldots, f_n)$ of $\mathbf{F}$ by using the basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of $\mathbf{F}$:

$$(f_1, \ldots, f_n) = (\varepsilon_1, \ldots, \varepsilon_n)U,$$

where U is an unimodular matrix of degree n.

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We now consider the $s \times s$ minor determinants which are contained in the first s rows of $V = U^{-1}$. Since V is unimodular, the greatest common divisor of these minor determinants is equal to 1. Hence in these minor determinants there exists a minor determinant which is prime to l. Let $j_1, \ldots, j_s$ be column indices of it. Namely, let the minor determinant

$$\begin{vmatrix} v_{1\,j_1}, & \ldots, & v_{1\,j_s} \\ \vdots & & \vdots \\ v_{s\,j_1}, & \ldots, & v_{s\,j_s} \end{vmatrix}$$

of $V = (v_{ij})$ be prime to l. Let

$$\begin{vmatrix} v_{1\,1}, & \ldots, & v_{1\,n} \\ \vdots & & \vdots \\ v_{s\,1}, & \ldots, & v_{s\,n} \\ lv_{s+1\,1}, & \ldots, & lv_{s+1\,n} \\ \vdots & & \vdots \\ lv_{n\,1}, & \ldots, & lv_{n\,n} \end{vmatrix} = V_1$$

and consider the $s \times s$ minor determinants which are contained in the j_1-th, $\ldots, j_s$-th columns of V_1. Then the minor determinant with row indices $(1, \ldots, s)$ is equal to the corresponding minor determinant of V and the minor determinants with other row indices are obtained from those of V by multiplying some powers of l. Since the greatest common divisor of the $s \times s$ minor determinants which are contained in the j_1-th, $\ldots, j_s$-th columns of V is equal to 1, the greatest common divisor of the corresponding minor determinants of V_1 is also equal to 1. Hence there exists a $n \times n$ unimodular matrix W such that the j_1-th, $\ldots, j_s$-th columns are equal to those of V_1.

Consider the matrix

$$U \begin{pmatrix} \overbrace{l \quad\;}^{s} & \overbrace{\quad\;}^{n-s} \\ \ddots & \\ \quad l & \\ & 1 \\ & \quad\ddots \\ & \qquad 1 \end{pmatrix} W.$$

Then the j_1-th, $\ldots, j_s$-th columns are obtained from those of UV by multiplying l. Let P be a $n \times n$ matrix corresponding to a permutation $\begin{pmatrix} 1, & \ldots, & s, & s+1, & \ldots, & n \\ j_1, & \ldots, & j_s, & *, & \ldots, & * \end{pmatrix}$. Then, since UV is the unit matrix of degree n we have

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$$P^{-1}U\begin{pmatrix} \overbrace{l \quad \ddots \quad l}^{s} & \\ & 1 \\ & & \ddots \\ & & & 1 \end{pmatrix} WP = \left(\begin{array}{ccc|c} l & & 0 & \\ & \ddots & & Y \\ 0 & & l & \\ \hline & 0 & & X \end{array}\right).$$

Taking the determinants of both sides, we have $|X| = \pm 1$, i.e. X is an unimodular matrix of degree $n-s$. Hence we have

$$P^{-1}U\left(\begin{array}{c|c} \overbrace{\begin{matrix} l & & \\ & \ddots & \\ & & l \end{matrix}}^{s} & \\ \hline & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right) WP\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{s} & 0 \\ \hline 0 & X^{-1} \end{array}\right) = \left(\begin{array}{c|c} \overbrace{\begin{matrix} l & & \\ & \ddots & \\ & & l \end{matrix}}^{s} & A \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right),$$

where $A = (a_{ij})$ is an integral $s \times (n-s)$ matrix. Moreover, let $a_{ij} = -lb_{ij} + a'_{ij}$ with the smallest non-negative residue a'_{ij} mod. l and set $B = (b_{ij})$. Then the product

$$P^{-1}U\left(\begin{array}{c|c} \overbrace{\begin{matrix} l & & \\ & \ddots & \\ & & l \end{matrix}}^{s} & 0 \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right) WP\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{s} & 0 \\ \hline 0 & X^{-1} \end{array}\right)\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{s} & B \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right)$$

is the matrix transforming the basis $(\varepsilon_1, \ldots, \varepsilon_n)P = (\varepsilon_{\pi_1}, \ldots, \varepsilon_{\pi_n})$ of **F** into the basis

$$(g_1, \ldots, g_n)WP\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{s} & 0 \\ \hline 0 & X^{-1} \end{array}\right)\left(\begin{array}{c|c} \overbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}^{s} & B \\ \hline 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right) = (\omega_1, \ldots, \omega_n)$$

of **G**, where $(\pi_1, \ldots, \pi_n)$ is a permutation of $(1, \ldots, n)$. This basis $(\omega_1, \ldots, \omega_n)$ of **G** has the required properties of our lemma.

THEOREM 1. *Let K/Q be a cyclic extension of degree l^2, where l is a prime number, and denote by Ω its subfield of degree l and by $(\varepsilon_1, \ldots, \varepsilon_{r_\Omega})$ a system of fundamental units of Ω. Then, there exists a system of fundamental units*

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$(E_1, \ldots, E_{r_K})$ of K with the following properties:

$$E_i = \begin{cases} \varepsilon_{\pi_i} \cdots\cdots\cdots & i=1,\ldots,n, \\ \sqrt[l]{\varepsilon_{\pi_i}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{a_{ij}}H_i} \cdots\cdots & i=n+1,\ldots,r_\Omega, \\ \text{Relative fundamental unit} \cdots & i=r_\Omega+1,\ldots,r_K, \end{cases}$$

where H_i are relative units, a_{ij} are rational integers with $0 \leqq a_{ij} < l$, $(\pi_1, \ldots, \pi_{r_\Omega})$ is a suitable permutation of $(1, \ldots, r_\Omega)$ and n is a rational integer with $0 \leqq n \leqq r_\Omega$ which is determined by K.

Moreover, the unit index (Einheitenindex) Q_K[1] *of K is equal to $l^{r_\Omega - n}$ and $Q_K(\mathbf{E}_\Omega^* : N_{K/\Omega}^*\mathbf{E}_K) = l^{r_\Omega}$.*

Proof. First we suppose that K is real. Then, since the unit group $\mathbf{E}_\Omega^*$ and the norm group $N_{K/\Omega}^*\mathbf{E}_K$ satisfy the condition of lemma, there exist a basis $(\omega_1, \ldots, \omega_{r_\Omega})$ of $N_{K/\Omega}^*\mathbf{E}_K$ and a system of units $(E_1, \ldots, E_{r_\Omega})$ in $\mathbf{E}_K$ corresponding to the basis $\{\omega_i\}$ such that:

$$N_{K/\Omega}E_i = \begin{cases} \varepsilon_{\pi_i}^{l} \cdots\cdots\cdots & i=1,\ldots,n, \\ \varepsilon_{\pi_i}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{a_{ij}} \cdots\cdots & i=n+1,\ldots,r_\Omega, \end{cases}$$

for some rational integer n with $0 \leqq n < r_\Omega$ and for rational integers a_{ij} with $0 \leqq a_{ij} < l$. Here $(\pi_1, \ldots, \pi_{r_\Omega})$ means a suitable permutation of $(1, \ldots, r_\Omega)$. In particular, for $i = 1, \ldots, n$ we may take ε_{π_i} as E_i. For other $i = n+1, \ldots, r_\Omega$, $H_i = E_i^l \varepsilon_{\pi_i}^{-1}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{-a_{ij}}$ are relative units, and so we may write E_i, by using the relative units H_i, in the form $E_i = \sqrt[l]{\varepsilon_{\pi_i}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{a_{ij}}H_i}$. On the other hand, it is evident that $\{E_i\}$ forms a system of fundamental units of K together with relative fundamental units.

In case of imaginary number fields, l is equal to 2 and Ω is a real subfield. Then any fundamental unit ε of Ω is always that of Ω, and so the unit index Q_K of K is always equal to 1.

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[1] H. Hasse defined the "Einheitenindex" Q_K for imaginary number fields in his book "Über die Klassenzahl abelscher Zahlkörper" and for some real number fields in his work "Arithmetische bestimmung von Grundeinheit und Klassenzahl in zyklischen kubischen und biquadratischen Zahlkörper", Abh. Deutsch. Akad. d. Wiss. zu Berlin, Math.-Naturw. Kl., Jahrg. 1948, Nr. 2 (1950). For the real absolute abelian extension, H. W. Leopoldt defined it in his work "Über Einheitengruppe und Klassenzahl reeller abelscher Zahlkörper", Abh. Deutsch. Akad. d. Wiss. zu Berlin, Math.-Naturw. Kl., Jahrg. 1953, Nr. 2 (1954).

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THEOREM 2. *Let K/Q be a cyclic extension of degree pq (p and q are distinct rational prime numbers), and denote by Ω_p and Ω_q two subfields of relative degree $(K:\Omega_p)=p$ and $(K:\Omega_q)=q$ respectively, and by $(\varepsilon_1,\ldots,\varepsilon_{r_{\Omega_p}})$ resp. $(\eta_1,\ldots,\eta_{r_{\Omega_q}})$ a system of fundamental units of Ω_p resp. Ω_q. Then there exists a system of fundamental units $(E_1,\ldots,E_{r_K})$ of K with the following properties:*

$$E_i=\begin{cases}\varepsilon_{\pi_i} & \cdots\cdots\cdots\ i=1,\ldots,n,\\ \sqrt[p]{\varepsilon_{\pi_i}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{a_{ij}}H_i} & \cdots\cdots\ i=n+1,\ldots,r_{\Omega_p},\\ \eta_{\pi'_{i-r_{\Omega_p}}} & \cdots\cdots\cdots\ i=r_{\Omega_p}+1,\ldots,r_{\Omega_p}+m,\\ \sqrt[q]{\eta_{\pi'_{i-r_{\Omega_p}}}\prod_{j=1}^{m}\eta_{\pi'_j}^{b_{ij}}H_i} & \cdots\cdots\ i=r_{\Omega_p}+m+1,\ldots,r_{\Omega_p}+r_{\Omega_q},\\ \text{Relative fundamental unit} & \cdots\ i=r_{\Omega_p}+r_{\Omega_q}+1,\ldots,r_K,\end{cases}$$

where H_i are relative units, a_{ij}, b_{ij} are rational integers with $0\leqq a_{ij}<p$, $0\leqq b_{ij}<q$, $(\pi_1,\ldots,\pi_{r_{\Omega_p}})$, $(\pi'_1,\ldots,\pi'_{r_{\Omega_q}})$ are permutations of $(1,\ldots,r_{\Omega_p})$, $(1,\ldots,r_{\Omega_q})$ respectively and n, m, are rational integers with $0\leqq n\leqq r_{\Omega_p}$, $0\leqq m\leqq r_{\Omega_q}$ which are determined by K.

Moreover, the unit index (Einheitenindex) Q_K of K is equal to $p^{r_{\Omega_p}-n}\cdot q^{r_{\Omega_q}-m}$ and $Q_K(\mathbf{E}^*_{\Omega_p}:N^*_{K/\Omega_p}\mathbf{E}_K)(\mathbf{E}^*_{\Omega_q}:N^*_{K/\Omega_q}\mathbf{E}_K)=p^{r_{\Omega_p}}\cdot q^{r_{\Omega_q}}$.

Proof. First we suppose that K is real. Then, since $\mathbf{E}^*_{\Omega_p}$, $N^*_{K/\Omega_p}\mathbf{E}_K$ and $\mathbf{E}^*_{\Omega_q}$, $N^*_{K/\Omega_q}\mathbf{E}_K$ satisfy respectively the condition of lemma, there exist a basis $(\bar\varepsilon_1,\ldots,\bar\varepsilon_{r_{\Omega_p}})$ of $N^*_{K/\Omega_p}\mathbf{E}_K$, a basis $(\bar\eta_1,\ldots,\bar\eta_{r_{\Omega_q}})$ of $N^*_{K/\Omega_q}\mathbf{E}_K$ and a system of units $(E_1,\ldots,E_{r_{\Omega_p}+r_{\Omega_q}})$ in $\mathbf{E}_K$ corresponding to the bases $\{\bar\varepsilon_i\}$ and $\{\bar\eta_j\}$ such that

$$N_{K/\Omega_p}E_i=\bar\varepsilon_i=\begin{cases}\varepsilon_{\pi_i}^{p} & \cdots\cdots\ i=1,\ldots,n,\\ \varepsilon_{\pi_i}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{a_{ij}} & \cdots\cdots\ i=n+1,\ldots,r_{\Omega_p},\end{cases}$$

$$N_{K/\Omega_q}E_i=\bar\eta_{i-r_{\Omega_p}}=\begin{cases}\eta_{\pi'_{i-r_{\Omega_p}}}^{q} & \cdots\cdots\ i=r_{\Omega_p}+1,\ldots,r_{\Omega_p}+m,\\ \eta_{\pi'_{i-r_{\Omega_p}}}\prod_{j=1}^{m}\eta_{\pi'_j}^{b_{ij}} & \cdots\ i=r_{\Omega_p}+m+1,\ldots,r_{\Omega_p}+r_{\Omega_q},\end{cases}$$

where a_{ij}, b_{ij} are rational integers with $0\leqq a_{ij}<p$, $0\leqq b_{ij}<q$ and $(\pi_1,\ldots,\pi_{r_{\Omega_p}})$, $(\pi'_1,\ldots,\pi'_{r_{\Omega_q}})$ are suitable permutations of $(1,\ldots,r_{\Omega_p})$, $(1,\ldots,r_{\Omega_q})$ respectively.

In particular, for $1\leqq i\leqq n$ resp. for $r_{\Omega_p}<i\leqq r_{\Omega_p}+m$ we may take ε_{π_i} resp. $\eta_{\pi'_{i-r_{\Omega_p}}}$ as E_i, and for all other i we may take E_i such that

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$$\begin{cases} N_{K/\Omega_q}E_i = \pm 1, \ N_{K/\Omega_p}E_i = \bar{\varepsilon}_i & i = n+1, \ldots, r_{\Omega_p}, \\ N_{K/\Omega_p}E_i = \pm 1, \ N_{K/\Omega_q}E_i = \bar{\eta}_{i-r_{\Omega p}} & i = r_{\Omega_p}+m+1, \ldots, r_{\Omega_p}+r_{\Omega_q}. \end{cases}$$

For, if $N_{K/\Omega_q}E_i = \prod_{j=1}^{r_{\Omega q}} \bar{\eta}_j^{x_{ij}}$ $(i = n+1, \ldots, r_{\Omega_p})$ resp.

$$N_{K/\Omega_p}E_i = \prod_{j=1}^{r_{\Omega_p}} \bar{\varepsilon}_j^{y_{ij}} \qquad (i = r_{\Omega_p}+m+1, \ldots, r_{\Omega_p}+r_{\Omega_q})$$

and $qy - px = px' - qy' = 1$ for some rational integers x_{ij}, y_{ij}, x, x', y, y', then $\overline{E}_i = E_i^{qy}\bar{\varepsilon}_i^{-x}\prod_{j=1}^{r_{\Omega q}}\bar{\eta}_j^{-x_{ij}y}$ resp. $\overline{E}_i = E_i^{px'}\bar{\eta}_i^{-y}\prod_{j=1}^{r_{\Omega p}}\bar{\varepsilon}_j^{-y_{ij}x'}$ satisfy the required conditions. For such E_i, $H_i = E_i^p \varepsilon_{\pi_i}^{-1}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{-a_{ij}}$ $(n < i \leqq r_{\Omega p})$ resp. $H_i = E_i^q \eta_{\pi'_i}^{-1}\prod_{j=1}^{m}\eta_{\pi'_j}^{-b_{ij}}$ $(r_{\Omega_p} + m < i \leqq r_{\Omega_p}+r_{\Omega_q})$ are relative units, and so they are written in the form

$$E_i = \sqrt[p]{\varepsilon_{\pi_i}\prod_{j=1}^{n}\varepsilon_{\pi_j}^{a_{ij}}H_i} \qquad \text{resp. } E_i = \sqrt[q]{\eta_{\pi'_i}\prod_{j=1}^{m}\eta_{\pi'_j}^{b_{ij}}H_i}.$$

Finally, if for any unit E of K, $N_{K/\Omega_p}E = \pm\prod_{i=1}^{r_{\Omega_p}}\bar{\varepsilon}_i^{x_i}$ and $N_{K/\Omega q}E = \pm\prod_{i=1}^{r_{\Omega q}}\bar{\eta}_i^{y_i}$ with rational integers x_i, y_i, then $H = E\prod_{i=1}^{r_{\Omega_p}}E_i^{-x_i}\prod_{j=1}^{r_{\Omega q}}E_{r_{\Omega}+j}^{-y_j}$ is a relative unit of K, and so the unit E is written, by using the relative unit H, in the form $E = \prod_{i=1}^{r_{\Omega_p}}E_i^{x_i}\prod_{j=1}^{r_{\Omega q}}E_{r_{\Omega_p}+j}^{y_j}H$. Therefore, the above obtained $\{E_i\}$ forms a system of fundamental units of K together with the relative fundamental units and it is evident that the equation

$$Q_K \cdot (\mathbf{E}_{\Omega_p}^* : N_{K/\Omega_p}^*\mathbf{E}_K)(\mathbf{E}_{\Omega_q}^* : N_{K/\Omega_q}^*\mathbf{E}_K) = p^{r_{\Omega_p}} \cdot q^{r_{\Omega_q}}$$

holds.

Next we suppose that K is imaginary. Then either p or q is equal to 2, and so if we put $q = 2$, then p is odd prime and Ω_p is imaginary quadratic and Ω_2 is real. The relative units are roots of unity and the relative norm $N_{K/\Omega_2}\zeta$ of a root of unity ζ in Ω_p generates the whole unit group $\mathbf{E}_{\Omega_p}$ except the case of $\Omega_p = Q(\sqrt{-3})$ $p = 3$.

For any basis $(\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_{r_{\Omega_2}})$ of $N_{K/\Omega_2}^*\mathbf{E}_K$, there exists a system of units $(E_1, \ldots, E_{r_{\Omega_2}})$ of K such that $N_{K/\Omega_2}E_i = \bar{\varepsilon}_i$, $N_{K/\Omega_p}E_i = 1$ $(i = 1, \ldots, r_{\Omega_2})$, and they are written in the form $E_i = \sqrt{\bar{\varepsilon}_i H_i}$, where H_i are relative units and so roots of unity. Such a system of units $\{E_i\}$ forms a system of fundamental units of K.

Example 1. If we assume in Theorem 2 that K is real and $p = 2$, $q = 3$, we

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may take ε, $\{\eta, \eta'\}$ and $\{H, H'\}$ as a system of fundamental units of Ω_3, Ω_2 and a system of relative fundamental units of K respectively, where η' resp. H' means a conjugate of η resp. H.[2] Then, we may consider the following 15 types of system of fundamental units of K:

Q_K	System of fundamental units of K
1	$\{\varepsilon, \eta, \eta', H, H'\}$
3	$\{\sqrt[3]{\varepsilon HH'}, \eta, \eta', H, H'\}$, $\{\sqrt[3]{\varepsilon H^2H'^2}, \eta, \eta', H, H'\}$
4	$\{\varepsilon, \sqrt{\eta}, \sqrt{\eta'}, H, H'\}$, $\{\varepsilon, \sqrt{\eta H}, \sqrt{\eta' H'}, H, H'\}$
	$\{\varepsilon, \sqrt{\eta H}, \sqrt{\eta' HH'}, H, H'\}$, $\{\varepsilon, \sqrt{\eta HH'}, \sqrt{\eta' H}, H, H'\}$
12	$\{\sqrt[3]{\varepsilon HH'}, \sqrt{\eta}, \sqrt{\eta'}, H, H'\}$, $\{\sqrt[3]{\varepsilon H^2H'^2}, \sqrt{\eta}, \sqrt{\eta'}, H, H'\}$
	$\{\sqrt[3]{\varepsilon HH'}, \sqrt{\eta H}, \sqrt{\eta' H'}, H, H'\}$, $\{\sqrt[3]{\varepsilon H^2H'^2}, \sqrt{\eta H}, \sqrt{\eta' H'}, H, H'\}$
	$\{\sqrt[3]{\varepsilon HH'}, \sqrt{\eta H'}, \sqrt{\eta' HH'}, H, H'\}$, $\{\sqrt[3]{\varepsilon H^2H'^2}, \sqrt{\eta H'}, \sqrt{\eta HH'}, H, H'\}$
	$\{\sqrt[3]{\varepsilon HH'}, \sqrt{\eta HH'}, \sqrt{\eta' H}, H, H'\}$, $\{\sqrt[3]{\varepsilon H^2H'^2}, \sqrt{\eta HH'}, \sqrt{\eta' H}, H, H'\}$.

THEOREM 3. *Let K/Q be a real and non-cyclic abelian extension of degree l^2, where l is a prime number. Denote by Ω_i $(i=1, \ldots, l+1)$ $l+1$ subfields of degree l and by $\{\varepsilon_{ij}\}$ $(j=1, \ldots, r_{\Omega_i})$ a system of fundamental units of Ω_i.*

Then, there exists a system of fundamental units $\{E_{ij}\}$ of K with the following properties:

$$E_{ij} = \begin{cases} \varepsilon_{i\pi_j^i} \cdots\cdots\cdots & i=1, \ldots, l+1;\ j=1, \ldots, n_i, \\ \sqrt[l]{\varepsilon_{i\pi_j^i} \prod\limits_{\substack{s=1,\ldots,l+1 \\ t=1,\ldots,n_i}} \varepsilon_{s\pi_t^s}^{a_{st}}} \cdots & i=1, \ldots, l+1;\ j=n_i+1, \ldots, r_{\Omega_i}, \end{cases}$$

where a_{st} are rational integers with $0 \leqq a_{st} < l$, $(\pi_1^i, \ldots, \pi_{r_{\Omega_i}}^i)$ are suitable permutations of $(1, \ldots, r_{\Omega_i})$ and n_i are rational integers with $0 \leqq n_i \leqq r_{\Omega_i}$ which are determined by K.

Moreover, the unit index (Einheitenindex) Q_K of K is equal to $l^{\sum_{i=1}^{l+1}(r_{\Omega_i}-n_i)}$, and so the product $Q_K \prod_{i=1}^{l+1} (\mathbf{E}_{\Omega_i}^* : N_{K/\Omega_i}^* \mathbf{E}_K)$ divedes the power $l^{\sum_{i=1}^{l+1} r_{\Omega_i}}$, but they are different in general.

Proof. For a fixed system of fundamental units $\{\varepsilon_{ij}\}$ of Ω_i, we consider the following $r_K \times r_K$ matrix $A=(a_{ij})$ with integral coefficients corresponding to a system of fundamental units $(E_1, \ldots, E_{r_K})$ of K. Namely, if the relative

[2] Cf. the latter work by H. Hasse in 1).

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norm $N_{K/\Omega_i}E_\nu$ of E_ν is $\pm\prod_{j=1}^{r_{\Omega_i}}\varepsilon_{ij}^{b_{\nu,ij}}$ with rational integers $b_{\nu,ij}$, then we put $b_{\nu,ij} = a_{\nu,(i-1)(l+1)+j}$ $(\nu=1,\ldots,r_K;\ i=1,\ldots,l+1;\ j=1,\ldots,r_{\Omega_i})$. The matrix corresponding to a second system of fundamental units $(E_1',\ldots,E_{r_K}')$, obtained from $(E_1,\ldots,E_{r_K})$ by an unimodular transformation U, is UA. Therefore, in a similar way as in lemma, we may show that there exist a system of fundamental units $\{E_{ij}\}$ of K and a system of suitably rearranged fundamental units $\{\varepsilon_{i\pi_j i}\}$ of Ω_i such that the corresponding matrix $A=(a_{st})$ is normalized in the following manner:

For a rational integer m with $0\leqq m\leqq r_K$,

$$\begin{cases} a_{ss}=\begin{cases}1 \cdots\cdots\cdots\cdots & s=1,\ldots,m,\\ l \cdots\cdots\cdots\cdots & s=m+1,\ldots,r_K,\end{cases}\\ 0\leqq a_{st}<l \cdots\cdots\cdots\cdots s=1,\ldots,m;\ t=m+1,\ldots,r_K,\\ a_{st}=0 \cdots\cdots\cdots\cdots \text{for all other pairs } (s,t).\end{cases}$$

On the other hand, since K is real, the relative units of K are only ± 1. Therefore, if the relative norm $N_{K/\Omega_i}E$ of an unit E in K is $\pm\prod_{j=1}^{r_{\Omega_i}}\varepsilon_{ij}^{b_{ij}}$, then $E^l\prod_{i,j}\varepsilon_{ij}^{-b_{ij}}=\pm 1$, and so E is written in the form $E=\pm\sqrt[l]{\prod_{i,j}\varepsilon_{ij}^{b_{ij}}}$. Hence, we may write the above system of fundamental units $\{E_{ij}\}$ of K in the form

$$E_{ij}=\begin{cases}\pm\varepsilon_{i\pi_j i} \cdots\cdots\cdots\cdots i=1,\ldots,l+1;\ j=1,\ldots,n_i,\\ \pm\sqrt[l]{\varepsilon_{i\pi_j i}\prod_{\substack{s=1,\ldots,l+1\\ t=1,\ldots,n_i}}\varepsilon_{s\pi_t s}^{a_{st}}} \cdots i=1,\ldots,l+1;\ j=n_i+1,\ldots,r_{\Omega_i},\end{cases}$$

where $\sum_{i=1}^{l+1}n_i=r_K-m.$

Then the unit index Q_K of K is equal to $l^m=l^{\sum_{i=1}^{l+1}(r_{\Omega_i}-n_i)}$. The product $Q_K\cdot\prod_{i=1}^{l+1}(\mathbf{E}_{\Omega_i}^*:N_{K/\Omega_i}^*\mathbf{E}_K)$ is not necessarily equal to l^{r_K}, but it is a factor of l^{r_K}.

Example 2. In particular, we assume that in Theorem 3, $l=2$ and denote by ε_i $(i=1,2,3)$ a fundamental unit of subfield Ω_i respectively. Then, there exist following 8 possible types of normalized matrix:

(1.1) $\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$ (2.1) $\begin{pmatrix}2&&\\&2&\\&&2\end{pmatrix}$

(3.1) $\begin{pmatrix}1&&\\&1&\\&&2\end{pmatrix}$ (3.2) $\begin{pmatrix}1&&1\\&1&\\&&2\end{pmatrix}$ (3.3) $\begin{pmatrix}1&&1\\&1&1\\&&2\end{pmatrix}$

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$$(4.1)\ \begin{pmatrix}1 & & \\ & 2 & \\ & & 2\end{pmatrix} \qquad (4.2)\ \begin{pmatrix}1 & 1 & \\ & 2 & \\ & & 2\end{pmatrix} \qquad (4.3)\ \begin{pmatrix}1 & 1 & 1\\ & 2 & \\ & & 2\end{pmatrix}.$$

Here, the field of type (1.1) does not exist, but there exist infinitely many fields of any other type.[3)]

Furthermore, l^{r_K} is always equal to 2^3, and for the system of fundamental units of K, unit index Q_K, etc., we have the following tableau:

Type	System of fundamental units	Q_K	$\prod_{i=1}^{l+1}(\mathbf{E}^*_{\Omega_i} : N^*_{K/\Omega_i}\mathbf{E}_K)$	$Q_K\prod_{i=1}^{l+1}(\mathbf{E}^*_{\Omega_i} : N^*_{K/\Omega_i}\mathbf{E}_K)$
(2.1)	$\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$	1	2^3	2^3
(3.1)	$\{\sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \varepsilon_3\}$	2^2	2	2^3
(3.2)	$\{\sqrt{\varepsilon_1\varepsilon_3}, \sqrt{\varepsilon_2}, \varepsilon_3\}$	2^2	1	2^2
(3.3)	$\{\sqrt{\varepsilon_1\varepsilon_3}, \sqrt{\varepsilon_2\varepsilon_3}, \varepsilon_3\}$	2^2	1	2^2
(4.1)	$\{\sqrt{\varepsilon_1}, \varepsilon_2, \varepsilon_3\}$	2	2^2	2^3
(4.2)	$\{\sqrt{\varepsilon_1\varepsilon_2}, \varepsilon_2, \varepsilon_3\}$	2	2	2^2
(4.3)	$\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$	2	1	2

In case of imaginary number fields, l is equal to 2 and then Ω_1 is a real quadratic field and Ω_2, Ω_3 are imaginary quadratic fields. Therefore, the fundamental unit of K is either ε or $\sqrt{\zeta\varepsilon}$, where ε is a fundamental unit of Ω_1 and ζ is a root of unity in K such that $\sqrt{\zeta} \notin K$, and so the unit index Q_K of K is equal to 1 or 2.

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3) Cf. S. Kuroda, "Über den Dirichletschen Körper", J. Fac. Sci. Imp. Univ. Tokyo, Sec. I, Vol. IV, Part 5 (1943).
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