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MOD ODD MODULAR COINVARIANTS, HOMOLOGY OPERATIONS, AND LIMIT SPACES

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ABSTRACT We compute the homology of $\lim_{n\to\infty}(G_n \wr X)$, where (G_n) is a system of subgroups of Σ_{p^n} containing a p-Sylow subgroup $(\Sigma_{p^n\,p})$ and satisfying certain properties We show that $H_*(\lim_{n\to\infty}(G_n \wr X), Z/pZ)$ is built naturally over homology operations related to (G_n) We describe this family of operations using modular coinvariants

0. **Introduction.** In this work given a connected pointed space X of finite type and a family of compatible permutation groups, $S = \{G_n/E_n \wr \cdots \wr E_1 \leq G_n \leq \Sigma_{p^n}, n = 1, 2, \ldots\}$, we construct a new space denoted $G_\infty \wr X$ and compute its mod-p homology groups. Here Σ_{p^n} is the symmetric group of all permutations of all elements of V^n, an n dimensional vector space over $\mathbf{Z}/p\mathbf{Z}$ for p a prime number; and $E_n \wr \cdots \wr E_1$ a fixed p-Sylow subgroup of Σ_{p^n}. There is an algebra of homology operations, denoted RN (see [8]), associated to each such family S and the main result of this work is that $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is an algebra generated by the free module with basis a fixed homogeneous basis of $H_*(X, \mathbf{Z}/p\mathbf{Z})$ over RN. This algebra RN called *the extended Dyer-Lashof algebra* is closely related to the rings of invariants of various parabolic subgroups of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$, [8].

It was long ago when the relation between operations in topology and the Dickson algebra (the ring of invariants of a polynomial algebra on n generators over $\mathbf{Z}/p\mathbf{Z}$ of $\mathrm{GL}_n(\mathbf{Z}/p\mathbf{Z})$) was realized. Then a natural question to ask is what about other rings of invariants. This question is answered in this work for any fixed odd prime number; namely: certain subalgebras of $\big(E(x_1, \ldots, x_n) \otimes P[y_1, \ldots, y_n]\big)^G$ can be realized as duals of coalgebras of homology operations applied to certain topological spaces associated to G so called G_n*-spaces*. Here $E(x_1, \ldots, x_n)$ is an exterior algebra on n generators and $P[y_1, \ldots, y_n]$ is a polynomial algebra on n generators over $\mathbf{Z}/p\mathbf{Z}$. They are both graded with degrees: $|x_i| = 1$ and $|y_i| = 2$ for $i = 1, \ldots, n$. G is one of the following groups: $U_n \leq B_n \leq P_n(N) \leq \mathrm{GL}_n$, (the group of upper triangular matrices with one along the main diagonal, the Borel subgroup, the parabolic subgroup associated to a sequence of positive integers, and the general linear group, respectively). We should note here that for $p = 2$ the whole ring of invariants can be realized and the theory appears to be more elegant, (see [1], [2], [7], [11]), while for odd primes a lot of technical problems arise.

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The main theorem in this work is the computation of the homology of $G_\infty \wr X$ over $\mathbf{Z}/p\mathbf{Z}$, where G_∞ is the direct limit of a sequence of permutation subgroups satisfying certain relations, and "$\wr$" stands for the wreath product between topological spaces extending the definition of the usual wreath product of permutation groups. Namely:

THEOREM 4.7. *Let X be a pointed connected space of finite type, and either $G_n = \Sigma_{p^{n_l}} \wr \cdots \wr \Sigma_{p^{n_1}}$ associated to $N = (n_1, n_2, \ldots)$ an increasing sequence of positive integers or $G_n = \Sigma_{p^n}$. Then $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a free non associative (associative, if $G_n = \Sigma_{p^n}$ for all n) commutative algebra over $\mathbf{Z}/p\mathbf{Z}$ generated by the free* RN*-module $B\big(H_*(X)\big)$ modulo the relation: $Q^s x = x^p$, if $2s = |x|$, $x \in B\big(H_*(X)\big)$.*

Moreover $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a coalgebra, where the coproduct is given by:

$$\psi Q^I x = \sum_{K+J=I, x', x''} Q^K x' \otimes Q^J x'', \text{ with } \psi x = \sum x' \otimes x''.$$

Here $B\big(H_*(X)\big)$ is a fixed homogeneous basis of $H_*(X)$ over $\mathbf{Z}/p\mathbf{Z}$.

This is a revised form of the topology chapter of my Ph.D. thesis [8]. The first chapter which deals with the algebraic structure of RN will appear in a separate work [9] because of its length and technicality. We note that this is a generalization of the work of May [4], Milgram [12], and later by Huynh [7]. There are two methods of approaching the problem: one using the topology of the space and the second using modular invariant theory. There are advantages to both; for example, the first method is natural and direct; on the other hand, the second is less abstract and overcomes the Adem phenomenon. Our method is a mixture of the above two.

The $p = 2$ case is more direct and less technical. For the families $G_n = \Sigma_{2^n}$ and $G_n = \Sigma_2 \wr \cdots \wr \Sigma_2$ the theorem above was first proved in [7] and [1] respectively using modular invariant theory, and the families $G_n = \Sigma_{2^{n_l}} \wr \cdots \wr \Sigma_{2^{n_1}}$ associated with parabolic subgroups, $\sum n_j = n$, have been considered by Campbell, McCleary, and myself in [2]. The author wishes to thank Eddy Campbell and John McCleary for suggesting the correct map associated with the direct system $\{G_n \wr X \mid n = 1, 2, \ldots\}$.

Our work is divided in to four sections. In Section 1 we recall basic elements from the literature and (G_n)-spaces are explicitly discussed as a generalization of wreath products between permutation groups as well as their properties. We also recall elements of the cohomology of symmetric subgroups and modular invariant theory in Section 2. Section 3 is devoted to the definition of homology operations and in Section 4 the theorem above is proved.

1. **Wreath product and G_n-spaces.** Let G be a subgroup of the symmetric group Σ_n on n elements and H a finite group. The wreath product or the semidirect product between G and H^n is the group denoted by $G \wr H = G \tilde{\times} H$. Here the multiplication is given by: $(g; h_1, \ldots, h_n)(g'; h'_1, \ldots, h'_n) = (gg'; h_1 h'_{g^{-1}(1)}, \ldots, h_n h'_{g^{-1}(n)})$ and $H^n = \prod_1^n H_i$, with $H_i = H$. If H is a subgroup of Σ_m, then $G \wr H$ is a subgroup of $\Sigma_{n \times m}$ as follows:

$$(g; h_1, \ldots, h_n)(s : t_1, \ldots, t_n) = \big(g(s); h_{g^{-1}(s)}(t_1), \ldots, h_{g^{-1}(s)}(t_n)\big).$$

Here the nm elements, where $\Sigma_{n\times m}$ acts, are divided into n blocks of m elements each and s denotes the number of the block while t_i the t_i-th element of the m elements where Σ_m acts. In this case we have the following inclusions: $G\times H\subset G\wr H\subset\Sigma_{n\times m}$. In particular there is an inclusion according to the discussion above: $i(\Sigma_n\wr\Sigma_m,\Sigma_{n\times m})\colon \Sigma_n\wr\Sigma_m\longrightarrow\Sigma_{n\times m}$.

This can be extended to any number of subgroups of symmetric groups and we write:

$$G_1\wr G_2\wr\cdots\wr G_m\subset\Sigma_{n_1\times\quad\times n_m}$$

Here $G_i\subset\Sigma_{n_i}$. Since there is an isomorphism from $(G_1\wr G_2)\wr G_3$ to $G_1\wr(G_2\wr G_3)$ we can omit the brackets in the wreath product.

Let V^n be the n-dimensional vector space spanned by $(e_1,\dots,e_n)$ over $\mathbf{Z}/p\mathbf{Z}$ and Σ_{p^n} the symmetric group permuting the elements of V^n. Let E_i be a cyclic group of order p generated by the translation defined by the i-th basis vector in V^n, for $1\le i\le n$, and $A=E^n=\prod_1^n E_i$ the subgroup of Σ_{p^n} consisting of all translations of V^n. We define $\Sigma_{p^n,p}=E_n\wr\cdots\wr E_1$. Then $\Sigma_{p^n,p}$ is a p-Sylow subgroup of Σ_{p^n}. The inclusion of Σ_{p^n} into $\Sigma_{p^{n+1}}$ is given via the composition:

$$\Sigma_{p^n}\longrightarrow E_{n+1}\wr\Sigma_{p^n}\longrightarrow\Sigma_{p^{n+1}}$$

For the following ideas and notation [12] is a good reference.

We denote by BG the classifying space of G and by EG the total space of G. We can replace H by a topological space X or a chain complex C_*X in the definition of the wreath product. We study pointed connected spaces of finite type or cell complexes with finite skeleton in each dimension.

DEFINITION 1.1. $G\wr X=EG\times_G X^m$.

Here $G\le\Sigma_m$ and acts on X^m by permuting the factors. On the chain level G permutes the factors with the sign convention using the Alexander-Whitney chain equivalence. If G acts trivially on X, then let $G\circ X=BG\times X=EG\times_G X$ and the inclusion $EG\circ X\longrightarrow EG\times_G X^m$ is induced by the diagonal on X.

$H_*(G\wr X;\mathbf{Z}/p\mathbf{Z})$ is often called G-equivariant homology. See [13].

Let (G_n) be a sequence of permutation groups such that $G_0\subset G_1\subset\cdots$, where $G_n\subset\Sigma_{p^n}$ and $G_m\wr G_n\subset G_{l(m,n)}$ for some integer $l(m,n)$ depending on m and n.

DEFINITION 1.2. A pointed space $(Y,*)$ is said to be a (G_n)-*space*, if there exist maps

$$\theta_n\colon G_n\wr Y\longrightarrow Y,\quad n\ge 1$$

satisfying:

i) θ_n is homotopy equivalent to $G_n\wr Y\xrightarrow{j_{nm}}G_m\wr Y\xrightarrow{\theta_m}Y$ for $m\ge n$.
ii) If $j\colon Y\longrightarrow G_n\wr Y$ is given by $j(y)=(1;y,*,\dots,*)$, then $\theta_n\circ j\simeq id$.

NOTE. The θ_n are called the structure maps.

A map between (G_n)-spaces is required to respect the maps θ_n for $n\ge 0$, up to homotopy.

1.3. For any pointed space $(X, *)$, we define:

$$G_\infty \wr X := \lim_{n\to\infty} G_n \wr X = \bigcup_{m\geq 0} (EG_m \times_{G_m} X^{p^m})/_\sim.$$

Here $x \in EG_m \times_{G_m} X^{p^m}$ and $y \in EG_n \times_{G_n} X^{p^n}$ are equivalent iff $y = j_{n-1,n} \circ \cdots \circ j_{m,m+1}(x)$ for $n > m$, where $j_{n,n+1} \colon G_n \wr X \longrightarrow G_{n+1} \wr X$ is given by:

$$\begin{array}{ccc} EG_n \times X^{p^n} & \xrightarrow{i(G_n,G_{n+1})\times i(X^{p^n},X^{p^{n+1}})} & EG_{n+1} \times X^{p^{n+1}} \\ \text{quotient}\big\downarrow & & \text{quotient}\big\downarrow \\ G_n \wr X & \xrightarrow{j_{n,n+1}} & G_{n+1} \wr X \end{array}$$

Explicitly, $j_{n,n+1}(e, x_1, \ldots, x_{p^n}) = \big(i(G_n, G_{n+1})(e); x_1, \ldots, x_{p^n}, *, \ldots, *\big)$.

$G_\infty \wr X$ is a (G_n)-space, where the θ_n are induced from the following direct limit map:

$$\Big(\lim_{m\to\infty} G_n \wr G_m\Big) \wr X \longrightarrow \lim_{m\to\infty} G_{l(m,n)} \wr X.$$

The following (G_n) sequences are studied in this work:

a) $G_n = \Sigma_{p^n,p}$.

b) G_n associated to parabolic subgroups. Namely, let $N = (n_1, n_2, \ldots)$ be an increasing sequence of positive integers, then either let $G_0 = 1$; $G_1 = \Sigma_p$; $G_i = \Sigma_{p^i}$ if $i \leq \nu_1 = n_1$; or let $G_i = \Sigma_{p^{i-\nu_k}} \wr \Sigma_{p^{n_k}} \wr \cdots \wr \Sigma_{p^{n_1}}$ if $\nu_k < i < \nu_{k+1}$, where $\nu_k = \sum_1^k n_t$. We denote this group Σ_{N_n}.

c) $G_n = \Sigma_{p^n}$, $n \geq 0$.

For (G_n) as above, we shall define homology operations from $H_*(Y)$ to $H_*(Y)$ for Y a (G_n)-space.

Since the first case $G_n = \Sigma_{p^n,p} = E_n \wr \cdots \wr E_1$ is important in calculating the homology of $G_\infty \wr X$, we note the following:

The $(j_{n,n+1})_*$ are coalgebra monomorphisms: $(j_{n,n+1})_*(z) = 1 \otimes z \otimes 1 \otimes \cdots \otimes 1$, for $z \in H_*(E_n \wr \cdots \wr E_1 \wr X)$.

This observation implies that $H_*(G_n \wr X)$ injects into $H_*(G_{n+1} \wr X)$ and hence it is not difficult to calculate $\lim_{n\to\infty} H_*(G_n \wr X)$. It would be easy to calculate $H_*(G_\infty \wr X)$ provided we can associate $H_*(G_\infty \wr X)$ with $\lim_{n\to\infty} H_*(G_n \wr X)$. Fortunately this is true because of the following:

Let $j_n \colon G_n \wr X \longrightarrow G_\infty \wr X$ and $(j_n)_*$ the map induced in homology. These maps induce a map between $\lim_{n\to\infty} H_*(G_n \wr X)$ and $H_*(G_\infty \wr X)$ and the last map is an isomorphism:

$$\lim_{n\to\infty} H_*(G_n \wr X) \equiv H_*(G_\infty \wr X).$$

REMARK. For details see [15].

1.4. Now we discuss some properties of (G_n)-spaces:

a) Let X and Y be (G_n)-spaces, then $X \times Y$ is a (G_n)-space as follows:

$$\begin{aligned} G_n \wr (X \times Y) = EG_n \times_{G_n} (X \times Y)^{p^n} &\xrightarrow{d\times u} EG_n \times EG_n \times_{G_n} X^{P^n} \\ &\times Y^{p^n} \xrightarrow{T} EG_n \times_{G_n} X^{p^n} \times EG_n \times_{G_n} Y^{p^n} \xrightarrow{\theta_n \times \theta_n} X \times Y. \end{aligned}$$

Here d is the diagonal u the evident shuffle map, and T the interchange map. We define $\theta_n := (\theta_n \times \theta_n) \circ T \circ (d \times u)$.

b) The diagonal $d\colon X \longrightarrow X \times X$ is a map of (G_n)-spaces.

c) There is a product in X:

$$\mu\colon X \times X \longrightarrow X,$$

induced from the following composition:

$$X \times X \xrightarrow{\iota \times \iota} EG_1 \times X^p \overset{\text{quotient}}{\longrightarrow} EG_1 \times_{G_1} X^p \xrightarrow{\theta_1} X.$$

Here $i \times i$ is the obvious inclusion into the first two factors.

REMARK. μ does not make X into an H-space. For example, if $G_n = \Sigma_{p^n}$, then $G_\infty \wr \{*\} = B\Sigma_{p^\infty}$ and the fundamental group of $B\Sigma_{p^\infty}$ is not abelian. Hence $B\Sigma_{p^\infty}$ can not be an H-space.

d) Since $H_*(G_n \wr X) = H_*(EG_n \times_{G_n} X^{p^n}) = H_*\Big(C_*(EG_n) \otimes_{G_n} \big(H_*(X)\big)^{p^n}\Big)$, we can compute the external Cartan product in $H_*(X) \otimes H_*(Y)$ using the following composition:

$$\begin{aligned} C_*(EG_n) \otimes_{G_n} (H_*(X) \otimes H_*(Y))^{p^n}) &\xrightarrow{\psi \times u} C_*(EG_n) \otimes C_*(EG_n) \otimes_{G_n} \big(H_*(X)\big)^{p^n} \\ &\otimes \big(H_*(Y)\big)^{p^n} \xrightarrow{T} C_*(EG_n) \otimes_{G_n} \big(H_*(X)\big)^{p^n} \\ &\otimes C_*(EG_n) \otimes_{G_n} \big(H_*(Y)\big)^{p^n} \xrightarrow{C_*\theta_n \otimes C_*\theta_n} H_*(X) \otimes H_*(Y). \end{aligned}$$

Here ψ is the coproduct in EG_n.

e) (G_n)-spaces are Cartan objects as defined by May in [13]. First we prove the assertion for $G_n = E_n \wr \cdots \wr E_1$. Let us denote $E_\iota \equiv \mathbf{Z}/p\mathbf{Z}$ by π. Since there is a π-equivariant chain map C: $W \longrightarrow E\pi$, we can replace $E\pi$ by the standard $\mathbf{Z}/p\mathbf{Z}$-free resolution of π.

Let us consider the following diagram:

$$\begin{tikzcd}
C_*\big(E\pi \times (X \times X)^p\big) \arrow[r, "\xi"] \arrow[d] & C_*(E\pi) \otimes C_*(X)^p \otimes C_*(X)^p \arrow[r, "C_*\Delta \otimes C_*1"] & C_*(E\pi) \otimes C_*(E\pi) \otimes C_*(X)^p \otimes C_*(X)^p \arrow[d, "T"] \\
C_*(E\pi) \otimes C_*(X \times X)^p \arrow[d, "C_*1 \otimes \xi"'] & & C_*(E\pi) \otimes C_*(X)^p \otimes C_*(E\pi) \otimes C_*(X)^p \arrow[d, "C_*\iota"] \\
C_*(E\pi) \otimes C_*\big((X \times X)\big)^p \arrow[d, "\simeq"'] & & \big(C_*(E\pi) \otimes C_*(X)^p\big)^p \arrow[d, "\simeq"] \\
W \otimes \big(C_*(X \times X)\big)^p \arrow[d, "1 \otimes C_*\iota"'] & & \Big(W \otimes \big(C_*(X)\big)^p\Big)^p \arrow[d, "C_*\iota"] \\
W \otimes \Big(W \otimes \big(C_*(X)\big)^p\Big)^p & = & W \otimes \Big(W \otimes \big(C_*(X)\big)^p\Big)^p
\end{tikzcd}$$

Let $f\colon E\pi \longrightarrow E\pi \times (E\pi)^p$ be given by $f(d) = (d, *, \ldots, *)$ and $g\colon E\pi \longrightarrow E\pi \times (E\pi)^p$ by $g(d) = (*, d, d, *, \ldots, *)$. The action of π on $E\pi \times (E\pi)^p$ for the map f is induced by the inclusion: $i_1\colon \pi \longrightarrow \pi \wr \pi$ given by $i_1(\sigma) = (\sigma, 1, \ldots, 1) = \sigma \wr 1$ and for the map g case by $i_2\colon \pi \longrightarrow \pi \wr \pi$ given by $i_2(\sigma) = (1, \sigma, \ldots, \sigma)$. Then we see that f and g are π-equivariant maps: $f(\sigma d) = (\sigma \wr 1)f(d)$ and $g(\sigma d) = i_2(\sigma)g(d)$. Since $E\pi \times (E\pi)^p$ is contractible and $\pi \wr \pi$ acts freely (hence π also acts freely), f is π-equivariant homotopic to g. This shows that the diagram above is π-equivariant and commutative and hence it is commutative in π- equivariant homology. Now it is obvious that X is a Cartan object.

The following formulas follow as in May [13] page 164:

$$\psi e_i \otimes (x \otimes y)^p = (-1)^{(p-1)|x||y|} \sum e_{i-j} x^p \otimes e_j y^p.$$
$$\psi \beta e_i \otimes (x \otimes y)^p = (-1)^{(p-1)|x||y|} \sum \beta e_{i-j} x^p \otimes e_j y^p + (-1)^{|x|} e_{i-j} x^p \otimes \beta e_j y^p.$$

The previous formula can be extended to any number of factors.

For $G_1 \equiv \Sigma_p$ consider the following commutative diagram:

$$\begin{array}{ccc} H_*(E_1) & \xrightarrow{\psi} & H_*(E_1) \otimes H_*(E_1) \\ \downarrow & & \downarrow \\ H_*(\Sigma_p) & \xrightarrow{\psi} & H_*(\Sigma_p) \otimes H_*(\Sigma_p), \end{array}$$

from which we deduce the analogous formula for Σ_p:

$$\psi(i_* e_{i(p-1)}) \otimes (x \otimes y)^p = (-1)^{(p-1)|x||y|} \sum i_*(e_{(i-j)(p-1)}) x^p \otimes i_*(e_{j(p-1)}) y^p.$$
$$\psi \beta i_*(e_{i(p-1)}) \otimes (x \otimes y)^p = (-1)^{(p-1)|x||y|} \sum i_*(\beta e_{(i-j)(p-1)}) x^p \otimes i_*(e_{j(p-1)}) y^p$$
$$+ (-1)^{|x|} i_*(e_{(i-j)(p-1)}) x^p \otimes i_*(\beta e_{j(p-1)}) y^p.$$

2. **Subgroups of the symmetric group and modular invariant theory.** For the rest of this section we recall some results concerning applications of modular invariant theory in the mod-p cohomology of p-groups and discuss the extended Dyer-Lashof algebra RN associated with $N = (n_1, n_2, \ldots)$ an increasing sequence of positive integers. This is a review from [9] where proofs will appear, although, proofs can be found in [8].

There is a well known injection $i^*: H^*(G) \longrightarrow H^*(A)^{W_G(A)}$ induced from the inclusion $i: A \longrightarrow G$, where $\Sigma_{p^n,p} \leq G \leq \Sigma_{p^n}$, $A = \prod_{i=1}^n E_i$, and $W_G(A)$ is the Weyl subgroup of A in G (see Quillen's Theorem in [6]). The image of this map has been studied by Huynh in [6]. We recall his result:

THEOREM 2.1 (HUYNH [6]). *a)* $\operatorname{Im} i^*(A, G) = \operatorname{Im} i^*(A, \Sigma_{p^n,p}) \cap H^*(A)^{W_G(A)}$

b) $\operatorname{Im} i^*(A, \Sigma_{p^n,p}) \equiv E(\eta_n W_1, \ldots, \eta_n W_n) \otimes P[\eta_n V_1, \ldots, \eta_n V_n]$

Here $W_i = M_{i,i-1} L_{i-1}^{\frac{p-3}{2}}$, $V_i = \prod_{(a_1, \ldots, a_{i-1}) \in (\mathbf{Z}/p\mathbf{Z})^{i-1}} (a_1 y_1 + \cdots + a_{i-1} y_{i-1} + y_i)$, *and* $L_i = \prod_1^i V_j$.

$$M_{i,i-1} = \begin{vmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \\ \vdots & & \\ y_1^{p^{i-2}} & \cdots & y_n^{p^{i-2}} \end{vmatrix}, \textit{ and } \eta_n = \begin{pmatrix} 0, \ldots, 0, 1 \\ \vdots \\ 1, 0, \ldots, 0 \end{pmatrix}.$$

For details see [8].

We note here that $\operatorname{Im} i^*(A, \Sigma_{p^n,p})$ is isomorphic to a subalgebra of $H^*(A)^{U_n^t} \equiv \big(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n]\big)^{U_n^t}$ where U_n is the group consisting of the upper triangular matrices with one along the main diagonal and t stands for the transpose of a matrix.

We extend the result above to the following subgroups of GL_n:

$$W_{\Sigma_{p^n,p}}(A) = U_n \leq W_{\Sigma_p \wr \cdots \wr \Sigma_p}(A) = B_n \leq W_{\Sigma_{p^{n_l}} \wr \cdots \wr \Sigma_{p^{n_1}}}(A)$$
$$= P_n(N) \leq W_{\Sigma_{p^n}}(A) = \mathrm{GL}_n.$$

Here B_n is the Borel subgroup of GL_n and $P_n(N)$ the subgroup consisting of matrices with l blocks along the main diagonal with sizes $n_i \times n_i$, for $i = 1,\ldots,l$ and $n = \sum n_i$. Here each block is an element of GL_{n_i}, anything is allowed above the main diagonal, and zero below.

Let F be the free graded associative algebra on $\{e^i, i \geq 0\}$ and $\{\beta e^i, i > 0\}$ over $\mathbf{Z}/p\mathbf{Z}$ with $|e^i| = 2i$ and $|\beta e^i| = 2i-1$. F becomes a coalgebra equiped with coproduct $\psi: F \longrightarrow F\otimes F$ given by

$$\psi e^i = \sum e^{i-j}\otimes e^j \text{ and } \psi\beta e^i = \sum \beta e^{i-j}\otimes e^j + \sum e^{i-j}\otimes \beta e^j.$$

Elements of F are of the form $e^I = \beta^{\epsilon_1}e^{i_1}\cdots\beta^{\epsilon_n}e^{i_n}$ where $I = \big((\epsilon_1,i_1),\ldots,(\epsilon_n,i_n)\big)$ with $\epsilon_j = 0$ or 1 and i_j a non negative integer for $j = 1,\ldots,n$. Let $l(I)$ denote the length of e^I and the excess of e^I be denoted by $\operatorname{exc}(e^I) = |e^{i_1}| - \epsilon_1 - |e^{I'}|(p-1)$ where $I' = \big((\epsilon_2,i_2),\ldots,(\epsilon_n,i_n)\big)$. We define $U = F/I_e$, where I_e is the two sided ideal generated by elements of negative excess. U is a Hopf algebra and if we let $U[n]$ denote the set of all elements of U with length n, then $U[n]$ is a coalgebra. We note here that the dual Steenrod algebra acts on U via Nishida relations, (see [8]).

We extend the previous construction by restricting the degrees and imposing Adem relations. Let U' be the subalgebra of U generated by $\{e^{(p-1)i}, i\geq 0\}$ and $\{\beta e^{(p-1)i}, i > 0\}$. We denote these elements by Q^i and βQ^i and recall that $|Q^i| = 2i(p-1)$ and $|\beta Q^i| = 2i(p-1)-1$. Let B be the quotient algebra of U' by the two sided ideal generated by elements of negative excess, where $\operatorname{exc}(Q^I) = 2i_1 - \epsilon_1 - |Q^{I'}|$, with $I = \big((\epsilon_1,i_1),\ldots,(\epsilon_n,i_n)\big)$ and $I' = \big((\epsilon_2,i_2),\ldots,(\epsilon_n,i_n)\big)$.

Adem relations are as follows:

$$Q^rQ^s = \sum_i (-1)^{r+i}\binom{(p-1)(i-s)-1}{pi-r}Q^{r+s-i}Q^i, \text{ if } r > ps.$$

$$Q^r\beta Q^s = \sum_i (-1)^{r+i}\binom{(p-1)(i-s)}{pi-r}\beta Q^{r+s-i}Q^i$$
$$-\sum_i (-1)^{r+i}\binom{(p-1)(i-s)-1}{pi-r-1}Q^{r+s-i}\beta Q^i, \text{ if } r \geq ps.$$

Let $N = (n_1,n_2,\ldots)$ an increasing sequence of positive integers or $N = \emptyset$ and let I_N be the two sided ideal of B generated by allowing Adem relations everywhere except at positions described by N. We denote RN the quotient B/I_N and this quotient algebra is called *the extended Dyer-Lashof algebra*. If $N = \emptyset$, then $\mathrm{RN} = R$ the Dyer-Lashof algebra. We remark that B and R are special cases of RN. Finally, RN is a Hopf algebra and $\mathrm{RN}[n]$ is again a coalgebra. Since $\mathrm{RN}[n]$ and $U[n]$ are of finite type, they are isomorphic with their duals as vectcr spaces and these duals become algebras. Next we describe these duals in terms of modular invariants.

PROPOSITION 2.2 [8]. *a) Let T be the subalgebra of $\big(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n]\big)^{U_n}$ generated by the following elements: $\{V_\iota, W_\iota, for\ i = 1,\ldots,n\}$. Then*

$$T \cap \big(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n]\big)^{P_n(N)} = T^{P_n(N)}.$$

Here $T^{P_n(N)}$ is the subalgebra of $H_*(A)^{P_n(N)}$ dual to the extended Dyer-Lashof coalgebra of length n denoted by $\mathrm{RN}[n]$, *(see [8]).*

b) Let T' be the subalgebra of $\big(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n]\big)^{U_n^t}$ generated by the following elements: $\{\eta_n V_\iota, \eta_n W_\iota \mid i = 1,\ldots,n\}$. Then

$$T' \cap \big(E(x_1,\ldots,x_n)\otimes P[y_1,\ldots,y_n]\big)^{\eta_n P_n(N)\eta_n} = T^{P_n(N)} \text{ as algebras over the Steenrod algebra.}$$

We note here that the case $G_n = \Sigma_{p^n}$ has been studied by May in [4].

Since we are interested in homology operations, passing to the dual side we get the following coalgebra monomorphisms:

COROLLARY 2.3. *Let $i(A, G_n)$ denote the inclusion between the named subgroups, then $\big(\mathrm{Im}\, i^*(A, G_n)\big)^*$ injects into $H_*(G_n; \mathbf{Z}/p\mathbf{Z})$, where the second asterisk denotes the dual. Hence: $\big(\mathrm{Im}\, i^*(A, G_n)\big)^* \longmapsto H_*(G_n; \mathbf{Z}/p\mathbf{Z})$, implies monomorphisms:*

a) $U[n] \longmapsto H_*(\Sigma_{p^n,p}; \mathbf{Z}/p\mathbf{Z})$, *and*

b) $\mathrm{RN}[n] \longmapsto H_*(\Sigma_{N_n}; \mathbf{Z}/p\mathbf{Z})$.

REMARK. If we define $\Sigma_{p^n,p} = E_1 \wr \cdots \wr E_n$ and $G_n = \Sigma_{p^{n_1}} \wr \cdots \wr \Sigma_{p^{n_l}}$, where $n = n_1 + \cdots + n_l$, then $W_{\Sigma_{p^n,p}}(A) = U_n^t$ and $W_{G_n}(A) = P_n(N)^t$. Actually, the way the wreath products are defined indicates what subspaces of V^n are left invariant under all permutations of these subgroups of Σ_{p^n}, (see [5]).

3. **Homology operations related to parabolic subgroups.**

In this section we use modular coinvariants to define families of homology operations following mainly Huynh. The idea is based on a theorem by Steenrod, (see also May [13]).

THEOREM 3.1 [STEENROD]. $H_*(G \wr X) = H_*(G) \otimes P_G H_*(X) \oplus H_*(G; M)$.

Here $G \leq \Sigma_m$, X is a pointed topological space or a chain complex over $\mathbf{Z}/p\mathbf{Z}$ with finite n-skeleton for each n, P_G the Steenrod map in homology associated to G, M the submodule of $\big(H_*(X)\big)^m$ generated by $\{\otimes_1^m x_{\iota_s}, x_{\iota_s} \in B\}$ such that $x_{\iota_s} \neq x_{\iota_t}$ for some s and t, where $B = \{x_\iota, i \in I\}$ is a homogeneous basis for $H_*(X)$ over $\mathbf{Z}/p\mathbf{Z}$.

Recall: $P_G x_\iota = 1 \otimes_G x_\iota^m$ and extend P_G linearly to a homomorphism of $\mathbf{Z}/p\mathbf{Z}$-vector spaces:

$$P_G\colon H_*(X) \longrightarrow H_*(G \wr X)$$

Using the direct sum decomposition in Steenrod's theorem above and Corollary 2.3, we define the following map $(d_n)_*$ which is the induced map from the composition of the inclusion i and Steenrod map P in homology:

$$(d_n)_*\colon U[n] \otimes H_*(X) \longrightarrow H_*(\Sigma_{p^n,p}) \otimes P_{\Sigma_{p^n,p}} H_*(X), \text{ or}$$
$$(d_n)_*\colon \mathrm{RN}[n] \otimes H_*(X) \longrightarrow H_*(G_n) \otimes P_{G_n} H_*(X),$$

for the appropriate subgroup G_n. Moreover, $\operatorname{Im} d_n$ is a subcoalgebra of $H_*(G_n \wr X)$.

The following theorem relates $(d_n)_*\big(U[n] \otimes H_*(X)\big)$ with a direct summand of $H_*(\Sigma_{p^n,p} \wr X)$.

THEOREM 3.2. *Let $d_n\colon E^n \circ X \longrightarrow \Sigma_{p^n,p} \wr X$ be induced by the inclusion and the diagonal, then*

$$\operatorname{Im}(d_n)^* = H_*(E_n) \otimes P_1 H_*(E_{n-1}) \otimes \cdots \otimes P_{n-1} H_*(E_1) \otimes P_n H_*(X).$$

The proof is similar to the one given by Huynh [7] for $p = 2$ and it is omitted.

On the other hand $\operatorname{Im}(d_n)^* \equiv (U[n])^* \otimes (d_n)^* P_n H^*(X)$ in cohomology and hence dually:

$$H_*(\Sigma_{p^n,p} \wr X) \equiv H_*(E_n) \otimes P_1 H_*(E_{n-1}) \otimes \cdots \otimes P_{n-1} H_*(E_1) \otimes P_n H_*(X) \oplus \ker(d_n)_*$$

Further: $U[n] \otimes P_n H_*(X) \equiv H_*(E_n) \otimes P_1 H_*(E_{n-1}) \otimes \cdots \otimes P_{n-1} H_*(E_1) \otimes P_n H_*(X)$.

We recall that $H_*(\Sigma_{p^n,p} \wr X) \equiv H_*(\Sigma_{p^n,p}) \otimes P_n H_*(X) \oplus H_*(\Sigma_{p^n,p}; M)$, where M has been defined in Steenrod's theorem and inductively:

$$H_*(\Sigma_{p^n,p}) \equiv H_*(E_n) \otimes P_1 H_*(E_{n-1}) \otimes \cdots \otimes P_{n-1} H_*(E_1) \oplus \operatorname{coker} i(E^n, \Sigma_{p^n,p})_*.$$

It is obvious now that any element of $U[n]$ or $\mathrm{RN}[n]$ can define an operation by $e^I x = d_n(e^I \otimes x)$ or $Q^I x = d_n(Q^I \otimes x)$ for $x \in H_*(X)$. See [8] for the notation.

Since we would like our operation to raise degree by $|e^I|$ or $|Q^I|$ we adjust the definition before as follows. We start again with the case $\Sigma_{p^n,p}$.

Let $e^I \in U[n]$, then $I = \sum_{\iota=1}^n (m_\iota I_{\iota,n} + k_\iota J_{\iota,n})$ uniquely, and hence $(e^I)^* = \sum_{\iota=1}^N U_\iota^{k_\iota} V_\iota^{m_\iota}$ is an element of $(E(x_1, \ldots, x_n) \otimes P[y_\iota, \ldots, y_n])^{U_n}$. (See [8] for details).

$I_{|x|}$ stands for the sequence $\big((0, \frac{p-1}{2}|x|), \ldots, (0, \frac{p-1}{2}|x|)\big)$ and $I - I_{|x|} = \big((\beta_1, i_1 - \frac{p-1}{2}|x|), \ldots, (\beta_n, i_n - \frac{p-1}{2}|x|)\big)$, where $I = \big((\beta_1, i_1, \ldots, (\beta_n, i_n)\big)$.

DEFINITION 3.3. Define e^I by

$$e^I x := (d_n)_*(e^{I - I_{|x|}} \otimes x).$$

PROPOSITION 3.4. *a) The e^I are natural monomorphisms of degree $|I|$, if $I \geq I_{|x|}$, (see [8]).*

b) If $I = \sum_{i=1}^{n} \frac{p-1}{2}|x|I_{i,n}$, then $e^I(x) = P_n x$, for $x \in H_*(X)$. Moreover,

$$e^{(0,m_1)}x = 0, \ \text{if}\ m_1 < \Big(\frac{p-1}{2}\Big)|x|$$
$$e^{(0,m_1)}x = P_1 x, \ \text{if}\ m_1 = \Big(\frac{p-1}{2}\Big)|x|.$$

c) $e^I e^J = e^{(I-I_{|J|},J)}$, where I, J, and $(I-I_{|J|},J)$ are sequences of length n, m, and $n+m$, respectively.

d) $e^I(e^J e^K) = (e^I e^J)e^K$.

PROOF. For c) we use Theorem 3.2 above and the isomorphism: $H_*(\Sigma_{p^n,p} \wr \Sigma_{p^m,p}) \equiv H_*(\Sigma_{p^{n+m},p})$. For d) we use c).

Since each e^I acts as an operator after being identified with the corresponding homology class of $H_*(\Sigma_{p^n,p})$, we note:

i) We have seen that the set of the e^I admits a coproduct: $\psi(e^I) := \sum_{J+K=I} e^J \otimes e^K$. It is obvious that if we let $e[n]$ be the set of all non trivial operations of length n, then $e[n] \equiv U[n]$ as coalgebras.

ii) The dual of the Steenrod algebra acts on this set via Nishida relations, (see [4]).

The above discussion implies that the algebraic structure of this set is the one studied in [8]. Hence:

THEOREM 3.5. *The family of operations e^I defined above is a Hopf algebra and it is isomorphic to the Hopf algebra U studied in [8]. The subset of U containing all elements with length n is a coalgebra and its dual is isomorphic to T as Steenrod algebras.*

NOTE. The action mentioned above has been discussed in [8].

We extend the definition above to operations related to

$$G_n = \Sigma_p \wr \cdots \wr \Sigma_p, \quad \Sigma_{p^{n-1}l} \wr \cdots \wr \Sigma_{p^{n_1}}, \text{ or } \Sigma_{p^n}.$$

The definition is induced by the following commutative diagrams:

$$\begin{tikzcd} H^*(G_n) \arrow[rr, "i^*(\Sigma_{p^n,p},G_n)"] \arrow[dr, "i^*(E^n,G_n)"'] & & H^*(\Sigma_{p^n,p}) \arrow[dl, "i^*(E^n,\Sigma_{p^n,p})"] \\ & H^*(E^n) & \end{tikzcd}$$

Consequently, the diagram below commutes:

$$\begin{tikzcd} H^*(G_n) \arrow[r, "i^*(\Sigma_{p^n,p},G_n)"] \arrow[d] & H^*(\Sigma_{p^n,p}) \arrow[d] \\ \operatorname{Im} i^*(E^n;G_n) \equiv \mathrm{RN}[n]^* \arrow[r] & \operatorname{Im} i^*(E^n;\Sigma_{p^n,p}) \equiv U[n]^* \end{tikzcd}$$

And hence dually:

$$\begin{tikzcd} H_*(G_n) \arrow[r, "i_*(\Sigma_{p^n,p},G_n)"] & H_*(\Sigma_{p^n,p}) \\ U[n] \arrow[u] \arrow[r] & \mathrm{RN}[n] \arrow[u] \end{tikzcd}$$

It is easy to see the relation among Steenrod maps between different subgroups.

$$\begin{tikzcd} H_*(X) \arrow[r, "P_r"] \arrow[drr, "P_{kr}"'] & H_*(\Sigma_r \wr X) \arrow[r, "P_k"] & H_*(\Sigma_k \wr \Sigma_r \wr X) \arrow[d, "\iota_*(\Sigma_k \wr \Sigma_r, \Sigma_{kr})"] \\ & & H_*(\Sigma_{kr} \wr X) \end{tikzcd}$$

The diagram above and the following one are important in our definition.

$$\begin{tikzcd} H_*(X) \arrow[r, "P_n \text{ or } e^I"] & H_*(\Sigma_{p^n,p} \wr X) \arrow[r, "\iota_*(\Sigma_{p^n\, p}, G_n)"] & H_*(G_n \wr (X) \\ & U[n] \otimes P_n H_*(X) \arrow[u] \arrow[r] & \mathrm{RN}[n] \otimes P_{G_n} H_*(X) \arrow[u] \end{tikzcd}$$

P_n is a homology operation itself and we can replace it by any other element of $U[n]$. Moreover, composing it with the appropriate inclusion we can define:

DEFINITION 3.6. Let $I = \big((\epsilon_1, i_1), \ldots, (\epsilon_n, i_n)\big)$ and $I' = \Big(\big(\epsilon_1, (p-1)i_1\big), \ldots, \big(\epsilon_n, (p-1)i_n\big)\Big)$. For each element $Q^I \in \mathrm{RN}[n]$ we define the operation

$$Q^I \colon H_q(X) \longrightarrow H_{q+|I|}(G_n \wr X);$$
$$Q^I x := (-1)^{\Sigma \epsilon_I} \nu(q)^n i_*(\Sigma_{p^n,p}; G_n) e^{I'} x.$$

Here $\nu(q) = (-1)^{q(q-1)\frac{m}{2}} (m!)^q$, with $m = \frac{(p-1)}{2}$.

Combining the definition above with the last proposition we describe the fundamental properties of this new family of operations.

PROPOSITION 3.7. *a) The Q^I are natural homomorphisms which commute with maps between spaces.*

b) $Q^i x = 0$, if $2i < |x|$. $Q^i x = x^p$, if $2i = |x|$. Moreover, if $\mathrm{exc}(I) = |x|$, then $Q^I x = (Q^{I'} x)^p$, where $I = \big((\epsilon_1 = 0, i_1), \ldots, (\epsilon_n, i_n)\big)$ and $I' = \big((\epsilon_2, i_2), \ldots, (\epsilon_n, i_n)\big)$.

c) The product of Q^I and Q^J is defined as follows: Let $l(I) = n$ and $l(J) = m$, then $Q^I Q^J = Q^{(I - I_{|J|}, J)}$, where $(I - I_{|J|}, J)$ is of length $n + m$ and $I_{|J|}$ has been defined before in Proposition 2.3.

d) $(Q^I Q^J) Q^K = Q^I (Q^J Q^K)$.

e) Let $Q[n]$ be the set of all operations of length n. Then $Q[n]$ becomes a coalgebra equipped with coproduct $\psi \colon Q[n] \longrightarrow Q[n] \otimes Q[n]$ given by: $\psi Q^I = \sum_{K+J=I} Q^J \otimes Q^K$. $Q[n] \equiv \mathrm{RN}[n]$ as coalgebras.

PROOF. a) and b) follow from the definition of the Q^I's and d) is a consequence of c).

For c) we use the following commutative diagram and proposition 2.4.

$$\begin{tikzcd} H_*(X) \arrow[r, "e^{J'}"] & H_*(\Sigma_{p^m,p} \wr X) \arrow[r, "e^{I'}"] \arrow[d, "\iota_*"] & H_*(\Sigma_{p^n,p} \wr \Sigma_{p^m,p} \wr X) \arrow[r, equal] \arrow[d, "\iota_*"] & H_*(\Sigma_{p^{n+m},p} \wr X) \arrow[d, "\iota_*"] \\ H_*(X) \arrow[r, "Q^J"] & H_*(G_m \wr X) \arrow[r, "Q^I"] & H_*\big(G_n \wr (G_m \wr X)\big) \arrow[r, equal] & H_*\big((G_n \wr G_m) \wr X\big) \end{tikzcd}$$

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As in definition above we associate I', (J') to I, (resp. J).

$$\begin{aligned} Q^I Q^J &= i_*(\Sigma_{p^n,p}, G_n)e^{I'} \circ i_*(\Sigma_{p^m,p}, G_m)e^{J'} \\ &= i_*(\Sigma_{p^{n+m},p}, G_n \wr G_m)e^{I'}e^{J'} \\ &= i_*(\Sigma_{p^{n+m},p}, G_n \wr G_m)e^{(I'-I_{|J'|},J')} \\ &= Q^{(I-I_{|J|},J)} \end{aligned}$$

Here we just used the coproduct between extended Dickson algebras, (see [4] or [8]).

REMARK. *This is exactly the way the Dyer-Lashof algebra is defined, if we concentrate on $G_n = \Sigma_{p^n}$, (see May [4] or [9]).*

If X is replaced by a (G_n)-space Y, then we do have operations from $H_*(Y)$ to $H_*(Y)$ as follows:

$$H_*(Y) \xrightarrow{Q^I} H_*(G_n \wr Y) \xrightarrow{\theta_n} H_*(Y)$$

By abuse of notation Q^I denotes $\theta_n \circ Q^I$.

We have proved the following theorem which is similar to Theorem 1.1 in May [4].

THEOREM 3.8. *Let Y be a connected (G_n)-space of finite type and let (G_n) be a sequence of permutation subgroups associated to an increasing sequence of positive integers $N = (n_1, \ldots, n_l, \ldots)$. Then there exist homomorphisms:*

$$Q^s : H_*(Y) \longrightarrow H_*(Y), \textit{ for } s \geq 0,$$

which satisfy the following properties:

1) *The Q^s are natural with respect to maps of (G_n)-spaces.*
2) *Q^s raises degree by $2s(p-1)$.*
3) *$Q^s x = 0$, if $2s < |x|$.*
4) *$Q^s x = x^p$, if $2s = |x|$.*
5) *$Q^s(x \otimes y) = \sum_{i+j=s} Q^i x \otimes Q^j y$, if $x \otimes y \in H_*(Y_1 \times Y_2)$. There is a similar formula for the internal product: $Q^s(xy) = \sum_{i+j=s} Q^i x Q^j y$, with x and $y \in H_*(Y)$.*
6) *$\psi(Q^s x) = \sum_{i+j=s,x',x''} Q^i x' \otimes Q^j x''$, if $\psi(x) = \sum x' \otimes x''$, and $x \in H_*(Y)$.*
7) *Adem relations hold everywhere except at positions $w_i = \sum_1^i n_{k-j+1}$ from the left for any element of length $n_1 + \cdots + n_k$.*

$$Q^r Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} Q^{r+s-i} Q^i, \textit{ if } r > ps.$$

$$\begin{aligned} Q^r \beta Q^s &= \sum_i (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta Q^{r+s-i} Q^i \\ &\quad - \sum_i (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} Q^{r+s-i} \beta Q^i, \textit{ if } r \geq ps. \end{aligned}$$

Here β is the mod-p *Bockstein.*

8) *The Nishida relations hold: Let P^r_* be the dual to Steenrod cohomology operation P^r. Then*

$$P^r_* Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi} Q^{s-r+i} P^i_*.$$

$$P^r_* \beta Q^s = \sum_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi} \beta Q^{s\ \ r+i} P^i_*$$

$$+ \sum_i (-1)^{r+i} \binom{(p-1)(s-r)-1}{r-pi-1} Q^{s-r+i} P^i_* \beta.$$

All coefficients are to be reduced mod-p.

To classify the set of homology operations properties 6), 7), and 8) must be used. But those are exactly the properties we used to determine the algebraic structure of RN in [8].

4. **The homology of $G_\infty \wr X$.** We shall compute the homology of $\lim_{n\to\infty}(\Sigma_{p^n,p} \wr X)$, for X a pointed connected space of finite type, and deduce the homology of $\lim_{n\to\infty}(G_n \wr X)$, for subgroups G_n as defined before.

THEOREM 4.1. *Let X be a pointed connected space of finite type, and $G_n = \Sigma_{p^n,p}$. Then $H_*(\Sigma_{p^\infty,p} \wr X, \mathbf{Z}/p\mathbf{Z})$ is a free non-associative, p-commutative (see fact 5 below) algebra over $\mathbf{Z}/p\mathbf{Z}$ generated by the free U-module $B\big(H_*(X)\big)$ modulo the relation: $e^s x = x^p$, if $s = (p-1)|x|/2$, $x \in B\big(H_*(X)\big)$.*

Moreover, $H_*(\Sigma_{p^\infty,p} \wr X, \mathbf{Z}/p\mathbf{Z})$ is a coalgebra where the coproduct is given by

$$\psi e^I x = \sum_{K+J=I,x',x''} e^K x' \otimes e^J x'', \text{ with } \psi x = \sum x' \otimes x''.$$

Here $B\big(H_*(X)\big)$ is a fixed homogeneous basis of $H_*(X)$ over $\mathbf{Z}/p\mathbf{Z}$.

PROOF. The proof depends on the following claims:

1) The homology of $G_n \wr X$ can be decomposed as in Steenrod's theorem.

$$H_*(G_n \wr X) \equiv H_*(G_n) \otimes PH_*(X) \oplus H_*(G_n, M).$$

2) The image of the composite: $H_*(X) \xrightarrow{J_*} H_*(X)^p \equiv H_*(1 \wr X) \xrightarrow{J(1\,E)_*} H_*(E_1 \wr X)$ is the $\mathbf{Z}/p\mathbf{Z}$-module generated by $x_i \otimes 1 \otimes \cdots \otimes 1$, where $x_i \in B\big(H_*(X)\big)$. This composition is the structure map j_0 in homology and it is a coalgebra monomorphism.

3) Inductively we have the following:

$$H_*(\Sigma_{p^n,p} \wr X) \equiv \oplus_{k=0}^n H_*(E_n) \otimes P_1 H_*(E_{n-1}) \otimes \cdots \otimes P_k H_*(E_{n-k}, M_{n-k}).$$

Here M_{n-k} is the module over $\mathbf{Z}/p\mathbf{Z}$ on products between homology classes from $H_*(E_{n-k-1} \wr X)$ and all factors are equal, and for $i = n$ the last tensor product factor is $P_n H_*(X)$.

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4) For $i = n$ in 3), $H_*(E_n) \otimes P_1 H_*(E_{n-1}) \otimes \cdots \otimes P_{n-1} H_*(E_1)$ is exactly the homology operations of length n.

5) $H_*(E_i, M_{n-i})$ is actually the coinvariants of M_{n-i}, $H_*(E_i, M_{n-i}) \equiv \mathbf{Z}/p\mathbf{Z} \otimes (M_{n-i})_{E_i}$. This observation reveals a relation between the p-th product of elements of M_{n-i}. This is what we call p-commutativity:

$$x_1 \otimes x_2 \otimes \cdots \otimes x_p \equiv x_2 \otimes x_3 \otimes \cdots \otimes x_1 \equiv x_p \otimes x_1 \otimes \cdots \otimes x_{p-1}.$$

If $p = 2$, this is exactly commutativity, or if E_i is replaced by Σ_p, since all permutations are allowed. The non-associativity comes from the same idea: Since p is odd, $p \geq 3$, (the same is true for $p = 2$). We see that $(x_1 x_2) x_3$ can not be equivalent to $x_1(x_2 x_3)$ under the group action, since for example: $1 \otimes x_1 \otimes x_2 \otimes x_3 \equiv 1 \otimes x_2 \otimes x_3 \otimes x_1 \equiv 1 \otimes x_3 \otimes x_1 \otimes x_2$, where $1 \otimes x_1 \otimes x_2 \otimes x_3 \in H_*(E_1, M) \equiv \mathbf{Z}/p\mathbf{Z} \otimes M_{E_1}$. But associativity is obtained when $G_n = \Sigma_{p^n}$, for all n.

6) The multiplication between homology classes expressed by tensor products coincides with the one induced by the product on a (G_n)-space Y:

$$\mu_* \colon H_*(Y \times Y) \longrightarrow H_*(Y), \text{ for } Y \text{ a } (G_n)\text{-space.}$$
$$\mu_*(x_1, x_2) = x_1 . x_2 = (x_1 \otimes x_2 \otimes 1 \otimes \cdots \otimes 1)_{E \equiv \mathbf{Z}/p\mathbf{Z}}.$$

Here x_1, x_2 are first injected into $H_*(G_n \wr X)$ and then their images are considered in $H_*(G_\infty \wr X)$.

$$\begin{array}{ccccc}
H_*(G_n \wr X) \otimes H_*(G_n \wr X) & \xrightarrow{(\chi_n)_*} & H_*(E_1 \wr G_n \wr X) & \xrightarrow{(j_{G_{n+1}})_*} & H_*(G_\infty \wr X) \\
\Big\downarrow {\scriptstyle (j_{G_n})_* \otimes (j_{G_n})_*} & & \Big\downarrow {\scriptstyle (j_{G_{n+1}})_*} & & \\
H_*(G_\infty \wr X) \otimes H_*(G_\infty \wr X) & \xrightarrow{(\chi)_*} & H_*(E_1 \wr G_\infty \wr X) & \xrightarrow{(\theta 1)_*} & H_*(G_\infty \wr X)
\end{array}$$

It is obvious that the diagram above commutes and $(\theta_1 \circ \chi)_* = \mu_*$. Here $(\chi_n)_*$ is a monomorphism modulo p-commutativity, and $(j_{G_n})*$ is the monomorphism induced by the inclusion. The diagram above can be generalized for G_n any other group from the family we are interested in.

7) There is a relation on this direct limit system induced by the definition of the operations, namely:

$$e^{(p-1)|x|/2} x = x^p.$$

We see that the p-th powers of homology classes appear as images of suitable operations.

We proceed by induction on n, the number of wreath factors on $G_n \wr X$. The last remark to make is that at each stage the only new elements appearing in the direct sum splitting are due to new operations of length n, namely the generators given in the statement of theorem. The rest of the homology consists of products of elements appearing at least one step before.

Before we discuss the homology of (G_n)-spaces related to parabolic subgroups, we recall some key statements from the literature.

THEOREM 4.2 NAKAOKA [14]. *$H_q(\Sigma_{p^n}, \mathbf{Z}/p\mathbf{Z}) \equiv \sum_{r\leq p^n} U_r^q\big(Q(p)\big)$, where $U_r^q\big(Q(p)\big)$ is the module over $\mathbf{Z}/p\mathbf{Z}$ generated by monomials $Q^{m_1 I_1}\cdots Q^{m_l I_l}$, where Q^{I_t} is an admissible element of the Dyer-Lashof coalgebra $R[t]$ such that $q=\sum m_t|I_t|$ and $r=\sum m_t p^t$.*

We can rewrite the isomorphism above as follows:

$$H_q(\Sigma_{p^n}, \mathbf{Z}/p\mathbf{Z}) \equiv \sum_{r\leq p^n} U_r^q\big(Q(p)\big) \equiv \sum_{r\leq p^n-1} U_r^q\big(Q(p)\big)\oplus R[n].$$

4.3. Next we consider the following decompositions:

$$\begin{array}{ll}
H_*(\Sigma_{p^n,p}\wr X) & \equiv H_*(\Sigma_{p^n,p})\otimes P_nH_*(X)\oplus H_*(\Sigma_{p^n,p};M)\\
& \equiv \oplus_{k=0}^n H_*(E_n)\otimes P_1H_*(E_{n-1})\otimes\cdots\otimes P_kH_*(E_{n-k},M_{n-k})\\
\downarrow & \\
H_*(\Sigma_p\wr\cdots\wr\Sigma_p\wr X) & \equiv H_*(\Sigma_p\wr\cdots\wr\Sigma_p\otimes P_nH_*(X)\oplus H_*(\Sigma_p\wr\cdots\wr\Sigma_p;M)\\
& \equiv \oplus_{k=0}^n H_*(\Sigma_p)\otimes P_1H_*(\Sigma_p)\otimes\cdots\otimes P_kH_*(\Sigma_p,M_{n-k})\\
\downarrow & \\
H_*(\Sigma_{N_n}\wr X) & \equiv H_*(\Sigma_{N_n})\otimes P_nH_*(X)\oplus H_*(\Sigma_{N_n};M)\\
& \equiv \mathrm{RN}[n]\otimes P_nH_*(X)\oplus\Big(\frac{H_*(\Sigma_{N_n})}{\mathrm{RN}[n]}\Big)\otimes P_nH_*(X)\oplus H_*(\Sigma_{N_n};M)\\
\downarrow & \\
H_*(\Sigma_{p^n}\wr X) & \equiv H_*(\Sigma_{p^n})\otimes P_nH_*(X)\oplus H_*(\Sigma_{p^n};M)\equiv\\
& \equiv R[n]\otimes P_nH_*(X)\oplus\sum_{r\leq p^n-1}U_r^q\big(Q(p)\big)\otimes P_nH_*(X)\\
& \qquad\qquad \oplus H_*(\Sigma_{p^n};M)
\end{array}$$

4.4. Moreover, if we restrict each of these maps to

$$H_*(E_n)\otimes P_1H_*(E_{n-1})\otimes\cdots\otimes P_{n-1}H_*(E_1),$$

we have the following epimorphisms:

$$\begin{array}{c}
U[n]\otimes P_nH_*(X)\\
\downarrow\\
B[n]\otimes P_nH_*(X)\\
\downarrow\\
\mathrm{RN}[n]\otimes P_nH_*(X)\\
\downarrow\\
R[n]\otimes P_nH_*(X)
\end{array}$$

These are the coalgebras that classify the appropriate operations of length n.

4.5. Let $G_n=\Sigma_p\wr\cdots\wr\Sigma_p$. To prove the analogous theorem for this system of groups we consider the following commutative diagram:

$$\begin{array}{ccccccccc}
H_*(X) & \to & H_*(E_1\wr X) & \to & H_*(E_2\wr E_1\wr X) & \to & \cdots & \to & H_*(\Sigma_{p^\infty,p}\wr X)\\
 & & \downarrow & & \downarrow & & & & \downarrow\\
H_*(X) & \to & H_*(\Sigma_p\wr X) & \to & H_*(\Sigma_p\wr\Sigma_p\wr X) & \to & \cdots & \to & H_*(G_\infty\wr X)
\end{array}$$

The epimorphism between $H_*(\Sigma_{p^\infty,p}\wr X)$ and $H_*(G_\infty\wr X)$ is induced by the epimorphisms between $H_*(E_n\wr\cdots\wr E_1\wr X)$ and $H_*(\Sigma_p\wr\cdots\wr\Sigma_p\wr X)$. We have seen that $H_*(\Sigma_{p^\infty,p}\wr X)$ is an associative p-commutative algebra over the extended Dyer-Lashof algebra and a coalgebra over the opposite Steenrod algebra. Since the μ_* product in homology is induced by the tensor product between homology classes of these spaces and since the epimorphisms above preserve this product, the last epimorphism is an algebra epimorphism. Finally, since the coalgebra monomorphisms are induced from the appropriate inclusions, the following theorem has been proved.

THEOREM 4.6. *Let X be a pointed connected space of finite type, and $G_n = \Sigma_p \wr \Sigma_p \wr \cdots \wr \Sigma_p$. Then $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a free non associative, commutative algebra over $\mathbf{Z}/p\mathbf{Z}$ generated by the free B-module $B\big(H_*(X)\big)$ modulo the relation: $Q^s x = x^p$, if $2s = |x|$, $x \in B\big(H_*(X)\big)$. Moreover $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a coalgebra, where the coproduct is given by:*

$$\psi Q^I x = \sum_{K+J=I, x', x''} Q^K x' \otimes Q^J x'', \text{ with } \psi x = \sum x' \otimes x''.$$

Here $B\big(H_*(X)\big)$ is a fixed homogeneous basis of $H_*(X)$ over $\mathbf{Z}/p\mathbf{Z}$.

It is obvious now that the diagram above can be extended to calculate $H_*(G_\infty \wr X; \mathbf{Z}/p\mathbf{Z})$, for $G_n = \Sigma_{p^{n_l}} \wr \cdots \wr \Sigma_{p^{n_1}}$, with $n = n_1 + \cdots + n_l$ as defined before. We only need to observe that the sequence of the positive integers $N = (n_1, \ldots)$ indicates where to expect Adem relations. Thus if $N = \emptyset$, Adem relations are carried up to the homology of the limit space, *i.e.* the Dyer-Lashof algebra R.

THEOREM 4.7. *Let X be a pointed connected space of finite type, and either $G_n = \Sigma_{p^{n_l}} \wr \cdots \wr \Sigma_{p^{n_1}}$ associated to an increasing sequence $N = (n_1, n_2, \ldots)$ of positive integers or $G_n = \Sigma_{p^n}$. Then $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a free non associative, (associative, if $G_n = \Sigma_{p^n}$ for all n), commutative algebra over $\mathbf{Z}/p\mathbf{Z}$ generated by the free* RN*-module $H_*(X)$ modulo the relation: $Q^s x = x^p$, if $2s = |x|$, $x \in B\big(H_*(X)\big)$. Moreover $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a coalgebra, where the coproduct is given by:*

$$\psi Q^I x = \sum_{K+J=I, x', x''} Q^K x' \otimes Q^J x'', \text{ with } \psi x = \sum x' \otimes x''.$$

Here $B\big(H_*(X)\big)$ is a fixed homogeneous basis of $H_*(X)$ over $\mathbf{Z}/p\mathbf{Z}$.

Let us make a few remarks before we close this work. First, $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ is a Steenrod opposite algebra via Nishida relations, admits a coproduct structure, and is an algebra over the appropriate extended Dyer-Lashof algebra (RN or R). The difference between $H_*(G_\infty \wr X, \mathbf{Z}/p\mathbf{Z})$ and $H_*(\Sigma_{p^\infty} \wr X, \mathbf{Z}/p\mathbf{Z})$ (for $N \neq \emptyset$ and $N = \emptyset$) is that the second is associative. Second, it is known that there exists an injection in homology between $\Sigma_{p^\infty} \wr X$ and QX. That is $H_*(\Sigma_{p^\infty} \wr X)$ lives inside the homology of the appropriate infinite loop space and R is an invariant in the category of QX. We are not able to observe a similarity for (G_n) a sequence of parabolas. That is "is there a category of spaces which contains $G_\infty \wr X$ and the extended Dyer-Lashof algebra as an invariant?" (See also [1].)

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