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ON TORSION-FREE DISCRETE SUBGROUPS OF PSL(2, C) WITH COMPACT ORBIT SPACE

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Introduction. In 1883 Poincaré [**13**] recognized that the discrete subgroups of PSL(2, C) could be extended from their natural action on the complex plane to acting on hyperbolic 3-space and he attempted to analyze these groups in an analogous manner to his classical treatment of Fuchsian groups, with fundamental polyhedra playing the role of the fundamental polygons for Fuchsian groups. This approach, however, did not lead very far, perhaps not surprisingly when one appreciates the close connection between the geometry of these groups and the topology of 3-manifolds. Since that time the state of knowledge remained essentially unchanged until 1964 when work by Ahlfors [**1**] and soon afterwards by Bers [**3**] revitalized the subject of Kleinian groups. The modern approach tends to use analytic methods, although recently Marden [**11**] has had considerable success in carrying forward Poincaré's geometric approach.

In this paper we study a particular family of discrete subgroups of PSL(2, C) which are not Kleinian, namely the groups which have compact orbit space. Any such group which is torsion-free is the fundamental group of its own orbit space, which is a compact 3-manifold. Consequently, the problem of complete enumeration of these groups is formidable, as indeed it would contain a major contribution to the classification of compact 3-manifolds which appears to be beyond the reach of topology at the present time. In this work we exhibit methods of obtaining examples of torsion-free groups and illustrate how to construct the corresponding 3-manifolds. Our main tools are an existence theorem (§ 3) by means of which the existence of groups can be established geometrically, and the Reidemeister-Schreier method (outlined in § 5). In § 6 we rediscover the hyperbolic dodecahedral space of Weber and Seifert [**15**]. This leads us to ask which groups exist having one of the regular hyperbolic solids as fundamental polyhedron and it is found, in § 7, that two such solids each admit a number of non-isomorphic torsion-free groups.

1. The group PSL(2, C). PSL(2, C) denotes the group of linear fractional transformations

$$w = \frac{az + b}{cz + d} : a, b, c, d \text{ complex}, ad - bc = 1.$$

This group acts naturally on the extended complex plane. However, the action

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of an element may be extended so that PSL(2, C) operates as a group of conformal transformations of the 3-space $R^3 \cup (\infty)$ leaving invariant the upper half space $B = \{(x_1, x_2, x_3): x_3 > 0\}$. Giving B the Riemannian metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + dx_3^2}{x_3^2},$$

it is isometric to the 3-dimensional hyperbolic space and PSL(2, C) is the entire group of orientation-preserving isometries (see [**13**]).

The elements of PSL(2, C) fall into four families according to their fixed-point set. Only the elliptic elements (rotations about hyperbolic lines) have fixed points in B, and moreover only elliptic elements may have finite order, the rotation through θ being of finite order whenever θ is a rational multiple of π.

PSL(2, C) inherits the quotient topology of the subspace

$$\{a, b, c, d): ad - bc = 1\}$$

of C^4 under the identification of points $\pm(a, b, c, d)$. A subgroup of PSL(2, C) is discrete if and only if it acts discontinuously in B. As an elliptic element generates a subgroup of the compact circle group, any elliptic element in a discrete subgroup of PSL(2, C) must have finite order. We shall be concerned only with a subfamily of the discrete subgroups, namely those with compact orbit space, and in what follows it will be understood that a discrete subgroup always has this property.

2. Fundamental polyhedra. Let Γ be a discrete subgroup of PSL(2, C). A fundamental polyhedron F for Γ is a hyperbolic polyhedron (a closed, connected subset of B whose frontier is a union of hyperbolic polygons called the faces of F) such that

(i) the faces of F are identified in pairs by elements of Γ, the pairs being distinct except in the case of a face being self-congruent under an involution of Γ,

(ii) F is a Γ-covering (i.e., $B = \bigcup \gamma F: \gamma \in \Gamma$),

(iii) F°, the interior of F, is a Γ-packing (i.e., $F^\circ \cap \gamma F^\circ = \emptyset$, all $1 \neq \gamma \in \Gamma$).

Fundamental polyhedra always exist. For example, let $p \in B$ be a point which is not fixed by any $1 \neq \gamma \in \Gamma$ and let $d(x, y)$ denote hyperbolic distance in B. Then the Dirichlet region

$$D_p = \{\xi \in B: d(\xi, p) \leqq d(\xi, \gamma p), \text{ all } \gamma \in \Gamma\}$$

is known to be fundamental for Γ (see [**8**]).

From the distribution of the faces of F into congruent pairs under Γ the edges of F are divided into distinct congruent sets called "cycles of F" (because of the cyclic ordering on the edges of each congruent set indicated in [**13**]). The sum of the dihedral angles at the edges of any cycle of F is $2\pi/k$, where k is the order of the stabilizer in Γ of each edge of the cycle (see [**13**]). If $k = 1$, we refer to the cycle as being "inessential", and otherwise "essential".

The configuration formed by the fundamental polyhedron F together with all images γF ($\gamma \in \Gamma$) is called the F-tessellation of B, each image γF being a "cell" of the tessellation. Two cells with a face in common are called "neighbours". By a standard argument it is shown that the set $E \subset \Gamma$ consisting of those elements which map F to a neighbour form a generating set for Γ. Furthermore (see [**8**; **9**]), the canonical relations obtained as follows are a complete set of defining relations for Γ in generators E. Consider an edge of F at which n cells of the tessellation intersect. By pursuing a small closed path traced around the edge we get from one cell to the next by passing from a cell to a neighbour. Consequently the n cells, encountered in order, can be expressed as

$$F, a_1F, a_1a_2F, a_1a_2a_3F, \ldots, a_1a_2a_3 \ldots a_{n-1}F,$$

where $a_i \in E$, $i = 1, 2, \ldots, n - 1$. Moreover, on completion of the path we enter cell $a_1a_2a_3 \ldots a_nF$ ($a_n \in E$) coincident with F and thereby obtain the canonical relation of the edge, $a_1a_2a_3 \ldots a_n = 1$. It is easily seen that congruent edges of the F-tessellation yield a cyclic re-ordering of the same canonical relation and hence there is one distinct relation for each cycle of F.

3. An existence theorem. We have seen in the previous section how a discrete subgroup of PSL(2, C) gives rise to a hyperbolic polyhedron from the geometric properties of which an abstract presentation of the group can be obtained. Conversely, by constructing hyperbolic polyhedra with appropriate geometric properties, the existence of discrete subgroups of PSL(2, C) can be established.

THEOREM 1. *Let F be a bounded hyperbolic polyhedron with faces (finite in number) oriented coherently, and having the following additional properties:*

(i) *The faces of F may be labelled in distinct pairs A_i and A_i^{-1}, $i \in I$, in such a way that A_i and A_i^{-1} are congruent hyperbolic polygons and for each $i \in I$ there exists $a_i \in$ PSL(2, C) which maps A_i^{-1} onto A_i preserving the orientation;*

(ii) *The congruence of pairs of faces of F induces a distribution of the edges into distinct cycles in such a way that the sum of the dihedral angles at the edges of each cycle is an integer submultiple of 2π.*

Let Γ be the subgroup of PSL(2, C) *generated by the elements $\{a_i: i \in I\}$. Then Γ is a discrete subgroup for which F is a fundamental polyhedron.*

Theorem 1 is a special case of a more general result outlined by Poincaré in [**13**], where he permitted the polyhedron to be unbounded, corresponding to the case of Kleinian groups (non-compact orbit space). The converse situation for Kleinian groups, however, is not so pleasant as the account of the previous section; for example, Greenberg [**5**] has shown that there exist finitely generated Kleinian groups which have no fundamental polyhedron with a finite number of faces. Proofs of Poincaré's theorem have recently appeared (see [**11**; **12**]). We have proved (unpublished) the case stated in Theorem 1 by means of a covering space argument. This type of argument has been developed by Macbeath [**10**] in a generalized existence theorem for groups of homeomorphisms of a simply connected space.

4. The hyperbolic tetrahedral groups. Theorem 1 enables one to uncover many families of discrete subgroups of PSL(2, C) from a geometric approach. For instance, suppose that one constructs a hyperbolic polyhedron P each of whose dihedral angles is an integer submultiple of π. Consider P^*, the union of P and its reflection in one of its faces, say A. The faces of P^* fall naturally into congruent pairs (each face and its reflection) for there is a unique orientation-preserving element which also takes each point of a face to its reflection in A. Under this congruency of faces, the cycle of an edge of P^* contains only itself and its reflection and consequently the edges of the face A (where the dihedral angle in P^* is twice that in P) form 1-cycles whilst all remaining edges belong to 2-cycles. Therefore the sum of the dihedral angles at a cycle of P^* is in all cases an integer submultiple of 2π and the hypotheses of Theorem 1 are fulfilled. The discrete subgroup Γ for which P^* is fundamental is clearly the subgroup of index 2, consisting of the orientation-preserving elements, in the group generated by reflections in the faces of P, and is itself generated by elliptic elements.

Lannér in [**7**] enumerated the hyperbolic tetrahedra possessing the above property of having all dihedral angles equal to an integer submultiple of π. Letting π/λ_i and π/μ_i, $i = 1, 2, 3$, be the dihedral angles at opposite edges of the tetrahedron, where π/λ_i, $i = 1, 2, 3$, are the angles at the edges of a face, he showed that there are precisely nine such non-congruent tetrahedra which we list below describing each by its dihedral angles as $[\lambda_1, \lambda_2, \lambda_3 : \mu_1, \mu_2, \mu_3]$.

T1 [2, 2, 3 : 3, 5, 2],	T4 [2, 2, 5 : 2, 3, 5],	T7 [2, 3, 3 : 2, 3, 5],
T2 [2, 2, 3 : 2, 5, 3],	T5 [2, 3, 3 : 2, 3, 4],	T8 [2, 3, 4 : 2, 3, 5],
T3 [2, 2, 4 : 2, 3, 5],	T6 [2, 3, 4 : 2, 3, 4],	T9 [2, 3, 5 : 2, 3, 5].

The canonical presentation of the associated hyperbolic tetrahedral group is

$$a^{\lambda_1} = b^{\lambda_2} = c^{\lambda_3} = (bc)^{\mu_1} = (ca)^{\mu_2} = (ab)^{\mu_3} = 1.$$

5. The Reidemeister-Schreier method. In the subsequent work we will be looking for torsion-free discrete subgroups of PSL(2, C) and, to this end, we will be confronted with the following problem. Given a finitely presented group G and a homomorphism ϕ whose kernel G_0 has finite index in G, find generators and defining relations for G_0. There is a mechanical process of solution to this problem, due to Reidemeister and Schreier, which for completeness we outline here.

Suppose that G is generated by $\{g_i: i \in I\}$ and that $\{R_j = 1: j \in J\}$ is a complete set of defining relations in these generators. Choose a Schreier system $\{a_k: k \in K\}$ of right coset representatives for G_0. Let Φ be the function assigning to each element of G its Schreier coset representative of G_0. Then

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(see [**6**]) Schreier proved that G_0 is generated by the set

$$\{a_k g_i \Phi(a_k g_i)^{-1}: i \in I, k \in K\}.$$

Moreover, Reidemeister [**14**] showed that when expressed in these generators,

$$\{a_k R_j a_k^{-1} = 1: j \in J, k \in K\}$$

is a complete set of defining relations for G_0.

When G_0 is of large index in G, the calculation is somewhat cumbersome and it is often preferable to break the process up into easier stages, as follows. Suppose that H is a proper subgroup of $\phi(G)$ and that $G^* \subset G$ consists of those elements that are mapped by ϕ onto H. Replacing $\{a_k: k \in K\}$ by a Schreier system $\{b_{k'}: k' \in K'\}$ of coset representatives for G^* in G, the prescribed method gives a presentation for G^*. Now $\phi^*: G^* \to H$, the restriction of ϕ to G^*, has the same kernel as ϕ and so repetition leads to a presentation for G_0.

6. On fundamental groups of 3-manifolds. Let Γ be a discrete subgroup of PSL(2, C) and Γ_0 the normal subgroup generated by the elliptic elements in Γ. By a standard result (see for example [**2**]), Γ/Γ_0 is the fundamental group of the orbit space B/Γ. Hence, if Γ_1 and Γ_2 are non-isomorphic torsion-free groups, then the orbit spaces B/Γ_1 and B/Γ_2 are distinct compact 3-manifolds.

In the analogous 2-dimensional situation it is well known that the Fuchsian triangle group with presentation

$$x^2 = y^3 = (xy)^7 = 1$$

contains as subgroups of different finite index the fundamental groups of all compact Riemann surfaces of genus $\geqq 2$. Of course one could not hope to classify compact 3-manifolds by similar means, but nevertheless it seems most likely that a hyperbolic tetrahedral group, for instance, will have an infinite family of non-isomorphic torsion-free subgroups thereby giving rise to an infinite set of distinct 3-manifolds. The Reidemeister-Schreier method provides a way of obtaining presentations of torsion-free subgroups of the hyperbolic tetrahedral groups, as illustrated by the following typical example.

Consider the group Γ corresponding to T4:

$$a^2 = b^2 = c^5 = (bc)^2 = (ca)^3 = (ab)^5 = 1.$$

Let N be a proper normal subgroup of Γ and r, s, and t the respective images of a, b, and c under the canonical homomorphism $\phi: \Gamma \to \Gamma/N$. Then r, s, and t generate Γ/N and the relations $r^2 = s^2 = t^5 = (st)^2 = (tr)^3 = (rs)^5 = 1$ hold. Now if we had $a \in N$, it would follow that $r = 1$, whence $s = t = 1$ and $N = \Gamma$, which is a contradiction. Therefore $a \notin N$ and, being normal, N does not contain any conjugate of a. The same argument shows that N does not contain any of the elements b, c, ab, bc, and ca, nor any of their conjugates. As these are the only elliptic elements in Γ, it follows that N is torsion-free.

The smallest non-trivial group onto which Γ can be mapped homomorphically is the alternating group A_5, a homomorphism $\Psi: \Gamma \to A_5$ being given by

$$\Psi(a) = (1\,5)(3\,4), \qquad \Psi(b) = (1\,4)(2\,3), \qquad \Psi(c) = (1\,2\,3\,4\,5).$$

The kernel of Ψ is determined by the Reidemeister-Schreier method to be the group Γ_0 on six generators and defining relations

$$\begin{aligned} abcde &= 1, & cxad^{-1}e^{-1} &= 1,\\ axdb^{-1}c^{-1} &= 1, & dxbe^{-1}a^{-1} &= 1,\\ bxec^{-1}d^{-1} &= 1, & exca^{-1}b^{-1} &= 1. \end{aligned}$$

This group is in fact the fundamental group of the "hyperbolic dodecahedron" of Weber and Seifert [**15**], the manifold obtained by identifying opposite faces of a dodecahedron with a twist of $3\pi/5$. That the orbit space of Γ_0 should be this manifold will become apparent from the work in the next section.

7. Groups of the regular hyperbolic solids. All groups obtained directly by the approach of § 4 are generated by elliptic elements and so are certainly not torsion-free. In our quest for torsion-free groups, to by-pass use of the Reidemeister-Schreier method we seek polyhedra in hyperbolic space which fulfill the hypotheses of Theorem 1 in such a way that every cycle of edges is inessential. In the 2-dimensional case, all torsion-free Fuchsian groups with compact orbit space have regular hyperbolic polygons as possible fundamental domains. A natural starting point, therefore, is to examine the regular solids in hyperbolic space. There are basically five such solids, analogues of the Platonic solids of Euclidean space. However, whereas the dihedral angle of a Euclidean regular solid remains invariant under change of size of the solid, the dihedral angle of the hyperbolic solid varies, and in this sense there are infinitely many of each. But in the light of Theorem 1 we are only interested in those solids whose dihedral angle is an integer submultiple of 2π, of which there are precisely four (see [**4**]), namely the hexahedron $(2\pi/5)$, the icosahedron $(2\pi/3)$, the dodecahedron $(2\pi/5)$ and the dodecahedron $(\pi/2)$.

Barycentric subdivision cuts a regular solid into a number of copies of its "characteristic cell". If a discrete subgroup of $\mathrm{PSL}(2, C)$ has a regular solid as fundamental polyhedron, the symmetry group of the tessellation is the group generated by reflections in the faces of the characteristic cell. The characteristic cells of the forementioned regular hyperbolic solids are, respectively, the tetrahedra T3(48), T2(120), T4(120), and T3(120), the bracketed integers denoting the number of cells in the subdivision.

For the hexahedron $(2\pi/5)$ to fulfill the hypotheses of Theorem 1, we require to find a way of identifying its six faces orientably in three pairs so that the twelve edges are distributed into either 5-cycles (inessential) or 1-cycles (essential). We see immediately that a torsion-free group could not possibly ensue since any solution must contain at least two essential cycles. Indeed, it is quickly checked that there is only one solution, that is, there is a unique group (up to isomorphism) for which the hexahedron is fundamental.

This group Λ is generated by two elliptic elements of order 5 and has defining relations

$$a^5 = b^5 = (b^{-1}a^{-1}ba)^2ba^{-1}b^{-1}aba^{-1} = 1.$$

Mapping a and b onto the same generator of the cyclic group Z_5, Λ is mapped homomorphically onto Z_5 in such a way that the kernel is a torsion-free subgroup Λ_0 of index 5 in Λ. The Reidemeister-Schreier method gives the following presentation on five generators for Λ_0:

$$rv^{-1}r(v^{-1}u)^2 = 1, \qquad ut^{-1}u(t^{-1}s)^2 = 1,$$
$$sr^{-1}s(r^{-1}v)^2 = 1, \qquad vu^{-1}v(u^{-1}t)^2 = 1.$$
$$ts^{-1}t(s^{-1}r)^2 = 1,$$

Making the group abelian, the 3-manifold for which Λ_0 is the fundamental group has first homology group $Z_{11} \oplus Z_{11}$. We can construct this manifold in the following way. Since 1, a, a^2, a^3, and a^4 are coset representatives for Λ_0 in Λ, denoting the hexahedron by F, $F^* = F \cup aF \cup a^2F \cup a^3F \cup a^4F$ is fundamental for Λ_0. F^*, sketched in Figure 1, is a convex hyperbolic polyhedron with frontier consisting of twenty congruent square faces. From the pairing of faces of F in the group Λ, the pairing of faces of F^* which results in the group Λ_0 is determined as follows:

$$ABCD \leftrightarrow J'JKK', \qquad A'B'C'D' \leftrightarrow C'D'DC,$$
$$ADEF \leftrightarrow B'BCC', \qquad A'D'E'F' \leftrightarrow E'F'FE,$$
$$AFGH \leftrightarrow D'DEE', \qquad A'F'G'H' \leftrightarrow G'H'HG,$$
$$AHIJ \leftrightarrow F'FGG', \qquad A'H'I'J' \leftrightarrow I'J'JI,$$
$$AJKB \leftrightarrow H'HII', \qquad A'J'K'B' \leftrightarrow K'B'BK.$$

The manifold is obtained from F^* by identification of these faces.

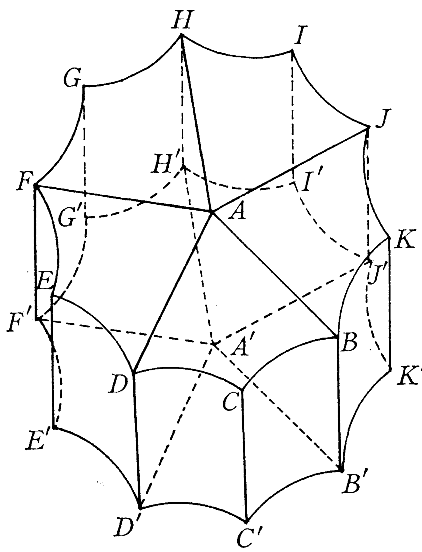

FIGURE 1

A torsion-free group with the icosahedron $(2\pi/3)$ as fundamental polyhedron is possible, for we require to pair the faces orientably so that the thirty edges fall into ten 3-cycles. There are no fewer than three different solutions, which we present in Table 1 referring to the icosahedron shown stereographically in Figure 2. The corresponding manifolds are obtained by identifying the indicated pairs of faces of the icosahedron.

Any discrete subgroup of $PSL(2, C)$ with the dodecahedron $(2\pi/5)$ as fundamental polyhedron must be a subgroup of index 120 in the group

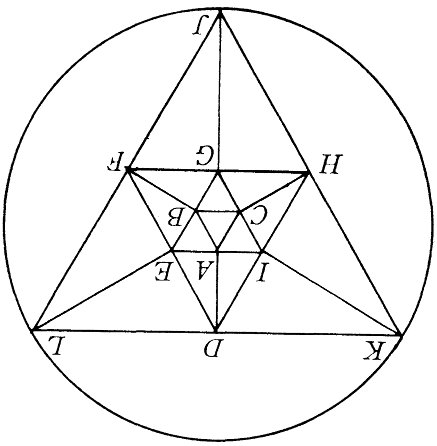

FIGURE 2

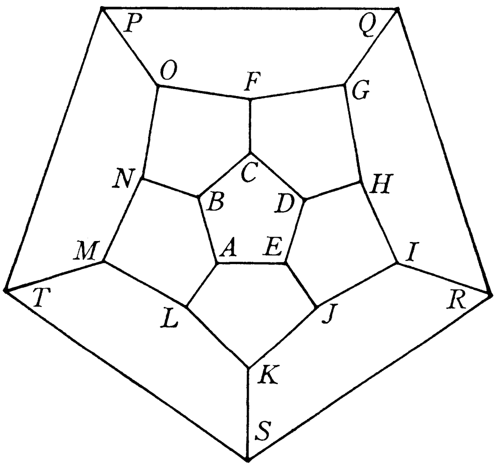

FIGURE 3

TABLE 1: *Manifolds from identifications of the icosahedron*

Identified faces	Fundamental group (Canonical presentation on ten generators)	Homology group
$ABC \leftrightarrow BEA : ACI \leftrightarrow EBF$	$b = a^2 : ek^{-1}h = 1$	
$BCG \leftrightarrow GFJ : AED \leftrightarrow FGB$	$a^{-1}dc = g^{-1}dc^{-1} = 1$	
$AID \leftrightarrow IHC : HIK \leftrightarrow FEL$	$j^{-1}fh = g^{-1}fh^{-1} = 1$	Z_{29}
$GHJ \leftrightarrow DLE : DIK \leftrightarrow KJH$	$c^{-1}jk = g^{-1}j^{-1}k = 1$	
$FJL \leftrightarrow KLD : CGH \leftrightarrow KLJ$	$d^{-1}eb = fe^{-1}b = 1$	
$ABC \leftrightarrow ICA : ABE \leftrightarrow LJF$	$a = b^2 : ah^{-1}c = 1$	
$BCG \leftrightarrow AID : CIH \leftrightarrow JLK$	$c = d^2 : fh^{-1}g = 1$	
$CGH \leftrightarrow FEB : FGJ \leftrightarrow GJH$	$e = f^2 : kh^{-1}e = 1$	Z_{35}
$BFG \leftrightarrow HKJ : ADE \leftrightarrow KIH$	$ej^{-1}a = kj^{-1}d = 1$	
$DIK \leftrightarrow EFL : DEL \leftrightarrow KLD$	$cj^{-1}g = kbg = 1$	
$ABC \leftrightarrow KDL : ACI \leftrightarrow CGB$	$ad^{-1}c = ca^{-1}b = 1$	
$ABE \leftrightarrow HJK : AID \leftrightarrow LDE$	$eb^{-1}f = bk^{-1}e = 1$	
$AED \leftrightarrow FJG : GHJ \leftrightarrow IKD$	$kj^{-1}a = jh^{-1}d = 1$	$Z_2 \oplus Z_2$
$CGH \leftrightarrow KLJ : IHK \leftrightarrow LFJ$	$eh^{-1}g = he^{-1}f = 1$	
$CHI \leftrightarrow EBF : BFG \leftrightarrow FLE$	$df^{-1}g = kg^{-1}j = 1$	

TABLE 2: *Manifolds from identifications of the dodecahedron*

Identified faces	Fundamental group (Canonical presentation on six generators)	Homology group
$ABCDE \leftrightarrow RSTPQ : AEJKL \leftrightarrow QGHIR$	$a^{-1}xa^{-1}eb = b^{-1}xb^{-1}ac = 1$	
$BALMN \leftrightarrow RIJKS : CBNOF \leftrightarrow SKLMT$	$c^{-1}xc^{-1}bd = d^{-1}xd^{-1}ce = 1$	$Z_5 \oplus Z_5 \oplus Z_5$
$DCFGH \leftrightarrow TMNOP : EDHIJ \leftrightarrow POFGQ$	$e^{-1}xe^{-1}da = abcde = 1$	
$ABCDE \leftrightarrow PQRST : AEJKL \leftrightarrow IRQGH$	$a^{-1}xb^{-1}ec = b^{-1}xc^{-1}ad = 1$	
$BALMN \leftrightarrow KSRIJ : CBNOF \leftrightarrow MTSKL$	$c^{-1}xd^{-1}be = d^{-1}xe^{-1}ca = 1$	$Z_5 \oplus Z_5 \oplus Z_5$
$DCFGH \leftrightarrow OPTMN : EDHIJ \leftrightarrow GQPOF$	$e^{-1}xa^{-1}db = abcde = 1$	
$ABCDE \leftrightarrow PQRST : AEJKL \leftrightarrow FGQPO$	$a^{-1}xe^{-1}cb = b^{-1}xa^{-1}ed = 1$	
$BALMN \leftrightarrow DHGFC : CBNOF \leftrightarrow JIHDE$	$c^{-1}xb^{-1}da = d^{-1}xc^{-1}be = 1$	$Z_5 \oplus Z_{15}$
$GHIRQ \leftrightarrow STMLK : IJKSR \leftrightarrow PONMT$	$e^{-1}xd^{-1}ac = abcde = 1$	
$ABCDE \leftrightarrow TPQRS : AEJKL \leftrightarrow HIRQG$	$abcde = cbedx = 1$	
$BALMN \leftrightarrow KSRIJ : CBNOF \leftrightarrow LMTSK$	$bax^{-1}cd^{-1} = xae^{-1}db^{-1} = 1$	$Z_3 \oplus Z_3$
$DCFGH \leftrightarrow TMNOP : EDHIJ \leftrightarrow POFGQ$	$acxb^{-1}e^{-1} = dcex^{-1}a^{-1} = 1$	
$ABCDE \leftrightarrow TPQRS : AEJKL \leftrightarrow IRQGH$	$abcde = bed^{-1}xc^{-1} = 1$	
$BALMN \leftrightarrow JKSRI : CBNOF \leftrightarrow SKIMT$	$xedac^{-1} = exadb^{-1} = 1$	$Z_3 \oplus Z_3$
$DCFGH \leftrightarrow TMNOP : EDHIJ \leftrightarrow FGQPO$	$caxbe^{-1} = acbxd^{-1} = 1$	
$ABCDE \leftrightarrow TPQRS : AEJKL \leftrightarrow QGHIR$	$abcde = cae^{-1}xd^{-1} = 1$	
$BALMN \leftrightarrow JKSRI : CBNOF \leftrightarrow MTSKL$	$xaebd^{-1} = axbec^{-1} = 1$	$Z_3 \oplus Z_3$
$DCFGH \leftrightarrow NOPTM : EDHIJ \leftrightarrow POFGQ$	$dbxca^{-1} = bdcxe^{-1} = 1$	
$ABCDE \leftrightarrow KJEAL : DCFGH \leftrightarrow JKSRI$	$a^2c^{-1}ab^{-1} = c^2e^{-1}da = 1$	
$BALMN \leftrightarrow KLMTS : EDHIJ \leftrightarrow QPOFG$	$e^2cex^{-1} = edbd^{-1}b = 1$	Z_{35}
$CBNOF \leftrightarrow HIRQG : MNOPT \leftrightarrow PTSRQ$	$ad^{-1}xdx = cb^{-1}x^{-1}bx = 1$	

generated by reflections in the faces of T4, its characteristic cell, and since it contains only orientation-preserving elements must therefore be a subgroup of index 60 in the group Γ of § 6. Identifying opposite pairs of faces of the dodecahedron with a twist of $3\pi/5$, the hypotheses of Theorem 1 are seen to be satisfied with six inessential cycles, and the ensuing torsion-free group is Γ_0, which we found in § 6 by means of the Reidemeister-Schreier method. There are seven further non-homeomorphic 3-manifolds obtained by identifying pairs of faces of the dodecahedron $(2\pi/5)$, which are presented in Table 2 in which we refer to Figure 3.

Finally, the dodecahedron $(\pi/2)$, like the hexahedron, cannot be fundamental for a torsion-free group, for thirty edges cannot possibly be distributed into 4-cycles (inessential). Nevertheless, there are several groups with torsion for which this dodecahedron is fundamental, and as in the case of the hexahedron these groups contain torsion-free subgroups of small index which can be determined by use of the Reidemeister-Schreier method.

For each of the icosahedron and dodecahedron there is a very large number of possible pairings of the faces and it is impracticable to investigate them all by hand. However, checking the lengths of edge cycles corresponding to a given pairing of the faces is a mechanical process, and in compiling the lists of manifolds given in Tables 1 and 2 we called upon the assistance of an electronic computer. We remark also that the Reidemeister-Schreier method is readily accessible to a computer.

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